NAME (please print): ______________________________

Quiz 3 (Sample)

1. What ordinary differential equations are implied by the method of separation of variables for the partial differential equation:

$$\frac{\partial u}{\partial t} = 10\frac{\partial^2 u}{\partial x^2} - 4\frac{\partial u}{\partial x}$$

2. Consider the differential equation

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0.$$

Determine the eigenvalues λ and the corresponding eigenfunctions if ϕ satisfies the following boundary conditions:

$$\frac{d\phi}{dx}(0) = 0, \quad \frac{d\phi}{dx}(L) = 0.$$

3. Consider the heat equation $u_t = 10u_{xx}$, subject to the boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$. Solve the initial boundary value problem if the temperature is initially

$$u(x,0) = \sin\frac{3\pi x}{L}\cos\frac{\pi x}{L}.$$

Hint: Use the following trigonometric identity: $\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$ to express the initial data as the sum of two sine functions.

4. Consider $u_t = u_{xx}$, subject to $u(0,t) = 0$, $u(L,t) = 0$ and $u(x,0) = \sin\frac{\pi x}{L}$.

   (a) Find the solution $u(x,t)$.

   (b) What is the total heat energy in the rod as a function of time?

   (c) What is the flow of heat energy out of the rod at $x = 0$? at $x = L$?

QUIZ 3 (SAMPLE) ANSWER KEY

① $u(x,t) = \phi(x)G(t) \Rightarrow \quad G'(t)\phi(x) = 10\phi''G - 4\phi'G \ / \div 10G\phi$

$$\frac{G'(t)}{10\,G(t)} = \frac{\phi''(x)}{\phi(x)} - \frac{2}{5}\frac{\phi'(x)}{\phi(x)} = -\lambda$$

$$\boxed{G'(t) = -10\lambda G(t)} \quad \boxed{\phi''(x) - 2\phi'(x) + 5\lambda\phi(x) = 0}$$

② $\phi''(x) + \lambda\phi(x) = 0 \Rightarrow \phi(x) = e^{rx} \Rightarrow r^2e^{rx} + \lambda e^{rx} = 0 \ / \div e^{rx}$

$\Rightarrow r = \pm\sqrt{-\lambda} \Rightarrow \phi(x) = C_1e^{\sqrt{-\lambda}x} + C_2e^{-\sqrt{-\lambda}x}$

$$\boxed{\phi(x) = D_1\cos\sqrt{\lambda}x + D_2\sin\sqrt{\lambda}x}$$

$\phi'(x) = -D_1\sqrt{\lambda}\sin\sqrt{\lambda}x + D_2\sqrt{\lambda}\cos\sqrt{\lambda}x$

$\phi'(0) = 0 \Rightarrow D_2 = 0$. Thus: $\boxed{\phi(x) = D_1\cos\sqrt{\lambda}x}$

$\phi'(L) = -D_1\sqrt{\lambda}\sin\sqrt{\lambda}L = 0 \nearrow \sqrt{\lambda} = 0 \Rightarrow \phi(x) = D_1$ ✓

$\searrow \sqrt{\lambda}L = n\pi, \ n = 1,2,3,\dots$

$$\Rightarrow \boxed{\lambda_0 = 0, \ \lambda_n = \left(\frac{n\pi}{L}\right)^2, \ n = 1,2,3,\dots}$$

EIGENVALUES

$$\Rightarrow \boxed{\phi_0(x) = D_1, \ \text{OR} \ \phi_0(x) = 1, \quad \phi_n(x) = \cos\frac{n\pi x}{L}, \ n = 1,2,3,\dots}$$

EIGENFUNCTIONS

③ $\left\{\begin{array}{l} u_t = 10u_{xx} \\ u(0,t) = 0 \\ u(L,t) = 0 \\ u(x,0) = \sin\frac{3\pi x}{L}\cos\frac{\pi x}{L} \end{array}\right.$ $\Rightarrow$ Show your work to obtain: $\boxed{u_n(x,t) = B_n\sin\frac{n\pi x}{L}e^{-10\left(\frac{n\pi}{L}\right)^2t}}$

SEPARATED SOLUTIONS

GENERAL SOLUTION $\boxed{u(x,t) = \sum_{n=1}^{N} B_n\sin\frac{n\pi x}{L}e^{-10\left(\frac{n\pi}{L}\right)^2t}}$

INITIAL DATA $\boxed{f(x) = \sin\frac{3\pi x}{L}\cos\frac{\pi x}{L} = \frac{1}{2}\left[\sin\frac{4\pi x}{L} + \sin\frac{2\pi x}{L}\right]}$ $\longrightarrow$ $B_4 = \frac{1}{2}, B_2 = \frac{1}{2}, B_1 = B_3 = B_5 = B_6 = \dots = 0$

SOLUTION: $\boxed{u(x,t) = \frac{1}{2}\sin\frac{2\pi x}{L}e^{-10\left(\frac{2\pi}{L}\right)^2t} + \frac{1}{2}\sin\frac{4\pi x}{L}e^{-10\left(\frac{4\pi}{L}\right)^2t}}$

(4) Show your work to obtain

(a) $\boxed{u(x,t) = \sin\frac{\pi x}{L}\, e^{-\left(\frac{\pi}{L}\right)^2 t}}$

(b) $E(t) = c\rho A \int_0^L \sin\frac{\pi x}{L}\, e^{-\left(\frac{\pi}{L}\right)^2 t}\, dx =$

$$= c\rho A\, e^{-\left(\frac{\pi}{L}\right)^2 t}\left[-\cos\frac{\pi x}{L}\left(\frac{L}{\pi}\right)\right]_{x=0}^{x=L}$$

$$= c\rho A\, e^{-\left(\frac{\pi}{L}\right)^2 t}\left[\left(\frac{L}{\pi}\right)\left(-\cos\pi + \cos 0\right)\right]$$

$$= c\rho A\, e^{-\left(\frac{\pi}{L}\right)^2 t}\left(\frac{L}{\pi}\right)(1+1) = \underline{\underline{2c\rho A\frac{L}{\pi}\, e^{-\left(\frac{\pi}{L}\right)^2 t}}}$$

(c) $\underbrace{\phi(x,t)}_{\text{HEAT FLUX}} = -K_0\frac{\partial u}{\partial x} = \underbrace{-K_0\frac{\pi}{L}\cos\frac{\pi x}{L}\, e^{-\left(\frac{\pi}{L}\right)^2 t}}$

At $x=0$: $\underline{\phi(0,t) = -K_0\frac{\pi}{L}\, e^{-\left(\frac{\pi}{L}\right)^2 t}}$ HEAT FLUX AT $x=0$

At $x=L$: $\underline{\phi(L,t) = +K_0\frac{\pi}{L}\, e^{-\left(\frac{\pi}{L}\right)^2 t}}$ HEAT FLUX AT $x=L$