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Quiz 3

1. What ordinary differential equations are implied by the method of separation of variables for the partial differential equation:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \alpha\frac{\partial u}{\partial x}, \;\; k > 0, \alpha > 0.$$

2. Consider the differential equation

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0.$$

Determine the eigenvalues λ and the corresponding eigenfunctions if ϕ satisfies the following boundary conditions:

$$\phi(0) = 0, \quad \phi(L) = 0.$$

3. Consider the heat equation $u_t = u_{xx}$ defined for $x \in (0,1)$ and $t > 0$, subject to the boundary conditions $u(0,t) = 0$ and $u(1,t) = 0$. Solve the initial boundary value problem if the temperature is initially

$$u(x,0) = \sin(4\pi x)\;\cos(\pi x).$$

Hint: Use the following trigonometric identity: $\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$ to express the initial data as the sum of two sine functions.

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4. Consider $u_t = 2u_{xx}$, subject to $u(0,t) = 0$, $u(L,t) = 0$ and $u(x,0) = \sin \frac{2\pi x}{L}$.

(a) Find the solution $u(x,t)$.

(b) What is the total heat energy in the rod as a function of time?

(c) What is the flow of heat energy out of the rod at $x = 0$? at $x = L$?

QUIZ 3 (Answer key)

① $u(x,t)=\phi(x)G(t) \Rightarrow \quad \phi(x)G'(t)=k\phi''(x)G(t)-\alpha\phi'(x)G(t) \;\Big/ \div k\phi G$

$$\Rightarrow \frac{G'(t)}{kG(t)}=\frac{\phi''(x)}{\phi(x)}-\frac{\alpha}{k}\frac{\phi'(x)}{\phi(x)}=-\lambda \Rightarrow$$

$$\boxed{G'(t)=-\lambda k G(t)} \text{ and } \boxed{\phi''(x)-\frac{\alpha}{k}\phi'(x)+\lambda\phi(x)=0}$$

② $\frac{d^2\phi}{dx^2}+\lambda\phi=0,\ \lambda>0. \quad \phi(x)=e^{rx} \to r^2e^{rx}+\lambda e^{rx}=0 \;\Big/ \div e^{rx}$

$$\Rightarrow r^2+\lambda=0 \Rightarrow r=\pm\sqrt{-\lambda}$$

$$\Rightarrow \phi(x)=C_1e^{\sqrt{\lambda}x}+C_2e^{-\sqrt{\lambda}x}=D_1\cos\sqrt{\lambda}\,x+D_2\sin\sqrt{\lambda}\,x$$

$$\phi(0)=0 \Rightarrow D_1=0 \Rightarrow \phi(x)=D_2\sin\sqrt{\lambda}\,x$$

$$\phi(L)=0 \Rightarrow D_2\sin\sqrt{\lambda}L=0 \begin{cases} D_2=0 \Rightarrow \text{TRIVIAL SOLUTION} \\ \sqrt{\lambda}L=n\pi,\ n=1,2,3,\dots \end{cases}$$

$$\Rightarrow \boxed{\lambda_n=\left(\frac{n\pi}{L}\right)^2,\ n=1,2,3,\dots \quad \text{EIGENVALUES}}$$

$$\boxed{\phi_n=\sin\frac{n\pi x}{L},\ n=1,2,3,\dots \quad \text{EIGENFUNCTIONS}}$$

③ $u_n(x,t)=\sin\frac{n\pi x}{L}\,e^{-k\left(\frac{n\pi}{L}\right)^2t} \underset{\uparrow\ L=1,\ k=1}{=} \sin(n\pi x)\cdot e^{-(n\pi)^2t}$

$$u(x,0)=\sin 4\pi x\cos\pi x=\frac{1}{2}\left[\sin 5\pi x+\sin 3\pi x\right] \quad \text{(compare)}$$

$$u(x,t)=\sum_{n=1}^{\infty}B_n\sin n\pi x\,e^{-(n\pi)^2t} \Rightarrow u(x,0)=\sum B_n\sin n\pi x$$

$$n=3 \quad B_3=\frac{1}{2}, \quad n=5,\ B_5=\frac{1}{2}$$

$$\Rightarrow \boxed{u(x,t)=\frac{1}{2}\sin 3\pi x\,e^{-(3\pi)^2t}+\frac{1}{2}\sin 5\pi x\,e^{-(5\pi)^2t}}$$

④ (a) $u(x,t)=\sin\frac{2\pi x}{L}\,e^{-2\left(\frac{2\pi}{L}\right)^2t}$

(b) $$E(t)=c\rho\int_0^L u(x,t)A\,dx=c\rho A\int_0^L\sin\frac{2\pi x}{L}e^{-2\left(\frac{3\pi}{L}\right)^2t}dx=c\rho Ae^{-2\left(\frac{2\pi}{L}\right)^2t}\left[-\cos\frac{2\pi x}{L}\right]_0^L\left(\frac{L}{2\pi}\right)$$

$$=c\rho Ae^{-2\left(\frac{2\pi}{L}\right)^2t}[1+1]\left(\frac{L}{2\pi}\right)=\frac{c\rho AL}{\pi}e^{-2\left(\frac{2\pi}{L}\right)^2t}$$

(c) $$\phi(x,t)=-K_0\frac{\partial u}{\partial x}(x,t) \Rightarrow \begin{cases}\phi(0,t)=-K_0\left(\frac{2\pi}{L}\right)\cos\left(\frac{2\pi}{L}\cdot 0\right)e^{-2\left(\frac{2\pi}{L}\right)^2t}=-K_0\cdot\frac{2\pi}{L}\cdot e^{-2\left(\frac{2\pi}{L}\right)^2t} \\ \phi(L,t)=-K_0\left(\frac{2\pi}{L}\right)\cos\left(\frac{2\pi}{L}\cdot L\right)e^{-2\left(\frac{2\pi}{L}\right)^2t}=K_0\frac{2\pi}{L}e^{-2\left(\frac{2\pi}{L}\right)^2t}\end{cases}$$