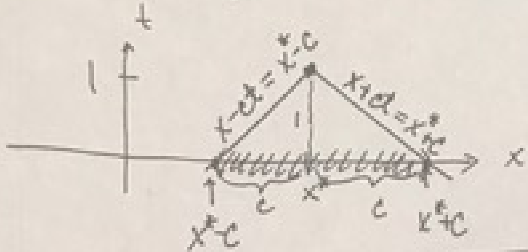
2. (a) $u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \xi^2 \, d\xi = \frac{1}{6c}\left[(x+ct)^3 - (x-ct)^3\right]$

(b)



For an arbitrary x: $\boxed{D(x,1) = (x-c,\ x+c)}$

1. $\frac{\partial u}{\partial t} = \left[F'(x-ct)\right]\cdot(-c) \Rightarrow \frac{\partial u}{\partial t}(x,0) = \left[F'(x)\right](-c) = -cF'(x)$

$\frac{\partial u}{\partial x} = \left[F'(x-ct)\right] \Rightarrow \frac{\partial u}{\partial x}(0,t) = F'(-ct)$

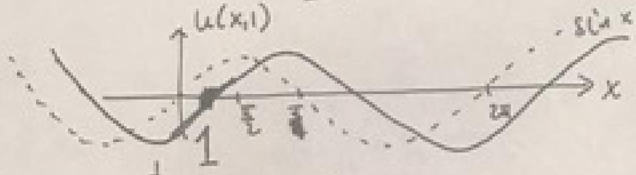
3. D'Alambert Formula:

(a) $u(x,t) = \frac{1}{2}\left[\sin(x-t) + \sin(x+t)\right] + \frac{1}{2}\int_{x-t}^{x+t} \cos\xi \, d\xi$

$= \frac{1}{2}\left[\sin(x-t) + \cancel{\sin(x+t)}\right] - \frac{1}{2}\left[\cancel{\sin(x+t)} - \sin(x-t)\right]$

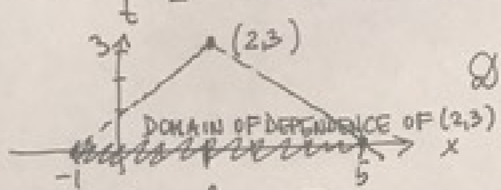
$= \sin(x-t)$

(b)

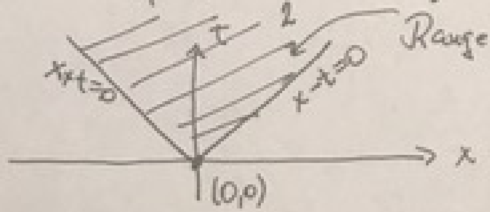


$u(x,1) = \sin(x-1)$

(c)



$\mathcal{D}(2,3) = (-1,5)$



Range of influence of (0,0)

$\mathcal{R}(0,0) = \{(x,t) \mid -t \le x \le t\}$

③

4. (a) $u_{tt} = c^2 u_{xx} \quad \Big|\cdot u_t \Big| \int_0^L dx$

calculation in class

$$\frac{d}{dt}\Big\{ \underbrace{\frac{1}{2}\int_0^L (u_t)^2 dx + \frac{c^2}{2}\int_0^L (u_x)^2 dx}_{\text{TOTAL ENERGY } E(t)} \Big\} = c^2\Big[u_x u_t \Big]_{x=0}^{x=L}$$

(b) $u(0,t)=0 \Rightarrow u_t(0,t)=0$, $u_x(L,t)=0 \Rightarrow$ $\Big\} \Rightarrow \Big[u_x u_t\Big]_{x=0}^{x=L} = 0$

$\Rightarrow \dfrac{dE(t)}{dt} = 0 \Rightarrow$ ENERGY IS CONSERVED.

(c) $E(t) = E(0)$

What is $E(0)$: $E(0) = \frac{1}{2}\int_0^L \underbrace{(u_t)^2(x,0)}_{\text{INITIAL VELOCITY}=0} dx + \frac{c^2}{2}\int_0^L (u_x)^2(x,0)\,dx$

$u(x,0) = \sin\frac{\pi x}{L}$

$u_x(x,0) = \frac{\pi}{L}\left(\cos\frac{\pi x}{L}\right)$

$$E(t) = E(0) = \frac{c^2}{2}\int_0^L \left(\frac{\pi}{L}\right)^2 \cos^2\frac{\pi x}{L}\,dx$$

$$= \frac{c^2}{2}\left(\frac{\pi}{L}\right)^2 \int_0^L \frac{1}{2}\left(1+\cos\frac{2\pi x}{L}\right)dx$$

$$= \frac{c^2}{4}\left(\frac{\pi}{L}\right)^2 \left[x + \left(\sin\frac{2\pi x}{L}\right)\cdot\frac{L}{2\pi}\right]_{x=0}^{x=L}$$

$$= \frac{c^2}{4}\left(\frac{\pi}{L}\right)^2 \Big[L + \underbrace{\left(\sin 2\pi\right)\frac{L}{2\pi} - 0 - 0}_{=0}\Big]$$

$$= \frac{c^2\pi^2}{4L}$$

5

$$\frac{\partial u}{\partial t} = 3\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \quad r \in (1,2),\ t>0$$

Separation of variables: $u(r,t) = h(t)\phi(r)$.

From PDE: $h'(t)\phi(r) = \frac{3}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}(h(t)\phi(r))\right)$

$$h'(t)\phi(r) = \frac{3}{r}h(t)\left(r\phi'(r)\right)' \quad \Big| \div h(t)\phi(r)$$

$$\underbrace{\frac{h'(t)}{h(t)}}_{\text{FUNCTION OF } t} = \underbrace{\frac{3}{r}\frac{1}{\phi(r)}\left(r\phi'(r)\right)'}_{\text{FUNCTION OF } r} = -\lambda$$

ODE FOR $h(t)$: $h'(t) = -\lambda h(t) \Rightarrow h(t) = Ce^{-\lambda t}$

ODE FOR $\phi(r)$:

$$\begin{cases} (3r\phi'(r))' + \lambda r\phi(r) = 0, & r\in(1,2) \\ \phi'(1)=0 \\ \phi'(2)=0 \end{cases}$$

THIS IS A STURM-LIOUVILLE PROBLEM WITH

$$p = 3r,\ q=0,\ \sigma = r$$

From Sturm-Liouville Theorem there are infinitely many eigenvalues: $\lambda_1 < \lambda_2 < \dots < \lambda_n < \dots$ (positive);

Each eigenvalue λ_n has an associated eigenfunction $\phi_n(r)$;

Eigenfunctions $\phi_n(r)$ and $\phi_m(r)$ are orthogonal in the sense:

$$\int_1^2 \phi_n(r)\phi_m(r)\, r\, dr = 0 \text{ for } n\neq m$$

The eigenfunctions form a complete set, i.e., $f(r) = \sum_{n=1}^{\infty} a_n\phi_n(r)$.

GENERAL SOLUTION OF PDE:

(*) $\boxed{u(r,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t}\phi_n(r)}$

DETERMINE a_n FROM INITIAL DATA: $u(r,0) = f(x)$.

FROM (*): $u(r,0) = \sum_{n=1}^{\infty} a_n\phi_n(r)$. From initial data this must be $= f(r)$.

So, we determine a_n from $\underbrace{\boxed{\sum_{n=1}^{\infty} a_n\phi_n(r) = f(r)}}_{\text{GENERALIZED F-SERIES OF } f}$

Find Fourier coefficients a_n of $f(r)$:

$$\sum_{n=1}^{\infty} a_n \phi_n(r) = f(r)$$

First multiply the above equation by $\phi_m(r)$

$$\sum_{n=1}^{\infty} a_n \phi_n(r)\, \phi_m(r) = f(r)\, \phi_m(r)$$

Multiply everything by r and integrate from $r=1$ to $r=2$:
weight

$$\sum_{n=1}^{\infty} a_n \int_1^2 \phi_n(r)\, \phi_m(r)\, r\, dr = \int_1^2 f(r)\, \phi_m(r)\, r\, dr$$

Because ϕ_n and ϕ_m are orthogonal with weight r for $n \neq m$ we have that for all $n \neq m$, $\int_1^2 \phi_n(r)\, \phi_m(r)\, r\, dr = 0$.
So, the only non-zero term is the term $n = m$:

$$a_m \int_1^2 [\phi_m(r)]^2\, r\, dr = \int_1^2 f(r)\, \phi_m(r)\, r\, dr$$

From here we get:

$$\boxed{a_m = \frac{\int_1^2 f(r)\, \phi_m(r)\, r\, dr}{\int_1^2 [\phi_m(r)]^2\, r\, dr}} \quad (**)$$

Thus, our solution $u(r,t)$ is given by the generalized Fourier series:

$$\boxed{u(r,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} \phi_n(r)}$$

where a_n are given by $(**)$.