A Class of Quasilinear Degenerate Elliptic Problems

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Proposed running head: Quasilinear Degenerate Elliptic Problems

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*Research supported by the National Science Foundation under grant DMS-9970310.
†Research supported by the National Science Foundation under grant DMS-0103823.

Abstract. We establish the existence of solutions for a class of quasilinear degenerate elliptic equations. The equations in this class satisfy a structure condition which provides ellipticity in the interior of the domain, and degeneracy only on the boundary. Equations of transonic gas dynamics, for example, satisfy this property in the region of subsonic flow and are degenerate across the sonic surface. We prove that the solution is smooth in the interior of the domain but may exhibit singular behavior at the degenerate boundary. The maximal rate of blow-up at the degenerate boundary is bounded by the "degree of degeneracy" in the principal coefficients of the quasilinear elliptic operator. Our methods and results apply to the problems recently studied by several authors which include the unsteady transonic small disturbance equation, the pressure-gradient equations of the compressible Euler equations, and the singular quasilinear anisotropic elliptic problems, and extend to the class of equations which satisfy the structure condition, such as the shallow water equation, compressible isentropic two-dimensional Euler equations, and general two-dimensional nonlinear wave equations. Our study provides a general framework to analyze degenerate elliptic problems arising in the self-similar reduction of a broad class of two-dimensional Cauchy problems.

Key words. Degenerate quasilinear elliptic equations, two-dimensional conservation laws, self-similar solutions

AMS subject classification. Primary: 35J70; Secondary: 35L65.

1 Introduction

1.1 Background

This work is motivated by the study of self-similar solutions for a class of two-dimensional hyperbolic systems of conservation laws. Systems of conservation laws in two space variables are of the form

$$\partial_t H(U) + \partial_x F(U) + \partial_y G(U) = 0, \tag{1.1}$$

with $U = (u_1, u_2, ..., u_N)$. It was shown in [6] that for Riemann data leading to shock interactions, the problem separates into two parts: a quasi-one-dimensional Riemann problem in the supersonic region and a (degenerate) elliptic equation (when $N = 2$) or an equation of mixed type (when $N > 2$) in the subsonic region. Existence, uniqueness and structure of solutions to quasi-one-dimensional Riemann problems that are posed in the supersonic region was studied in [6]. In this paper we provide a general approach to proving the existence of solutions in the subsonic region where the equations are degenerate elliptic. Coupling between the hyperbolic part and the elliptic part of the solution typically occurs through a free boundary [7, 8, 9]. We have studied free boundary problems for the steady [9] and for the unsteady, transonic small disturbance equations in [7, 8]. The basis for the existence of solutions to free-boundary problems is an analysis of solutions of the associated fixed-boundary problems. The behavior of solutions at the degenerate boundary is especially important. These issues are discussed in the present paper for a class of equations that satisfy a structure condition. Equations in this class include the equations of transonic gas dynamics, the pressure-gradient equations of the compressible Euler equations, the shallow water equations and the nonlinear wave equation.

The structure condition is a generalization of conditions first studied by Keldysh for linear degenerate equations [18]. In contrast with the Tricomi type of degeneracy [25], boundary value problems with Keldysh degeneracy require data prescribed along the degenerate boundary to guarantee well-posedness. We generalize this property to quasilinear equations.

Although there is a good deal of literature on quasilinear elliptic problems, there seems to be little work on this class of degenerate problems, even though they arise in many applications. Recent results in this field focus on two types of degenerate elliptic equations. One is the work by Zheng on the existence of solutions to the transonic pressure-gradient equations in the elliptic region [26] and the other is the work by Čanić and Keyfitz on the existence of "singular" [3] and "regular" [4] solutions for the unsteady transonic small disturbance (UTSD) equation. The approach used in this work applies to both equations. The method we use is different from both Zheng's (weak solution approach) and Čanić and Keyfitz's (monotone operator approach). Our approach is based on the Schauder fixed point theorem, and on the construction of sub- and super-solutions. The sub-solution provides strict ellipticity in the interior of the domain and the super-solution is used to show continuity up to the degenerate boundary. Our approach is closest to that of Choi and McKenna [13] where the ideas from [15] have been used in proving the existence of solutions of an anisotropic quasilinear degenerate elliptic equation of the form $\sum_i^N u^{a_i} u_{x_i x_i} + p(\mathbf{x}) = 0$ with $u = 0$ on the (smooth) boundary, where $a_1 \geq ... \geq a_N > 0$, and $p(\mathbf{x})$ strictly positive. In our problems we do not require $p(\mathbf{x})$ positive and we allow a more general form of the differential operator. See (1.3). We require that structure conditions, relating the coefficients of the operator and the boundary data, hold. The class of equations defined this way includes the problems studied by Choi and McKenna in [13], by Zheng in [26], by Čanić and Keyfitz in [3, 4], and also the nonlinear wave equation presented in Section 1.3 A.

1.2 Statement of the Problem and Summary of Results

We consider the following quasilinear Dirichlet problem

$$Qu = -f \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega, \tag{1.2}$$

where Q is given by

$$Qu \equiv \sum_{i,j}^{N} (a_{ij}(\mathbf{x}, u) u_{x_j})_{x_i} + b(\mathbf{x}, u) \cdot \nabla u, \quad a_{ij} = a_{ji}, \tag{1.3}$$

for $\mathbf{x} = (x_1, x_2, ..., x_N) \in \Omega$.

Throughout this work we will be assuming the following basic hypotheses:

H1. The domain $\Omega \subset \mathbf{R}^N$ is bounded and it satisfies a *uniform exterior cone condition* as defined by Gilbarg and Trudinger in [16]: at every point $P \in \partial\Omega$ there exists a finite right circular cone $V = V_P$ with vertex P such that $\overline{\Omega} \cap V_P = P$. Furthermore, all the cones V_P are congruent to some fixed cone V.

H2. The operator Q is *degenerate elliptic* in the sense that the coefficient matrix $[a_{ij}(\mathbf{x}, z)]$ is non-negative for all $(\mathbf{x}, z) \in \Omega \times \mathbf{R}$. In particular, this means that if $\lambda(\mathbf{x}, z)$ and $\Lambda(\mathbf{x}, z)$ denote respectively the minimum and the maximum eigenvalues of $[a_{ij}(\mathbf{x}, z)]$, then

$$0 \leq \lambda(\mathbf{x}, z)|\zeta|^2 \leq a_{ij}(\mathbf{x}, z)\zeta_i\zeta_j \leq \Lambda(\mathbf{x}, z)|\zeta|^2 \tag{1.4}$$

for all $\zeta = (\zeta_1, ..., \zeta_N) \in \mathbf{R} \setminus \{0\}$ and for all $(\mathbf{x}, z) \in \Omega \times \mathbf{R}$.

H3. The coefficients $a_{ij}(\mathbf{x}, z)$ and $b_i(\mathbf{x}, z)$ are $C^1(\overline{\Omega} \times \mathbf{R})$. The source term $f \in C^\gamma(\overline{\Omega}), 0 < \gamma < 1$ is a non-negative bounded function.

H4. The boundary data $g \in W^{1,2}(\Omega) \cap C^{0,\beta}(\overline{\Omega})$, where $0 < \beta < 1$, forces a degeneracy on $\partial\Omega$, *i.e.*, the minimal eigenvalue evaluated on $\partial\Omega$, or on a nonempty portion Σ of the boundary, vanishes.

Since Q is quasilinear, it is typically not possible to a priori say that Q satisfies hypothesis **H2** for all $(\mathbf{x}, z) \in \Omega \times \mathbf{R}$. To get around this difficulty we define a modified operator $\widetilde{Q}$ by introducing the cut-off coefficients $\tilde{a}_{ij}$ so that the modified operator $\widetilde{Q}$ becomes elliptic. Hence to obtain existence results for the problem (1.2), (1.3), we study the modified problem and show that its solution satisfies the original problem. This is done in Sections 3 and 4.

Because of the nature of the problems we study (deriving from transonic gas dynamics) we are interested in the elliptic operators that degenerate only on the boundary (or portions of the boundary) and not in the interior of the domain. To ensure *strict* ellipticity in the interior we construct an appropriate lower barrier. Sufficient conditions for the existence of such a lower barrier are listed in Section 2.2. Using the existence of a lower barrier which provides strict ellipticity in the interior we show the main result of this work: a proof of the existence of a $C^2(\Omega)$–solution of the Dirichlet boundary-value problem (1.2) with degenerate, or partially degenerate boundary data. We show (Theorem 2.7) that the boundary condition is satisfied in the sense $(u - g)^{a+1/2} \in W_0^{1,2}(\Omega)$, where a is the degree of the degeneracy, defined later in this paper.

In Section 3.2 we show that if the domain is convex and if the principal coefficients of the operator satisfy some additional assumptions, then the solution is continuous up to the degenerate boundary. We prove this by squeezing the solution between the appropriate continuous barrier functions which are equal to the boundary data at the degenerate boundary (Lemma 3.2 and Corollary 3.3). We show that the nonlinear wave equation, the shallow water equation as well as the UTSD equation have solutions continuous up to the degenerate boundary on convex domains. More information about the behavior of the solutions at the degenerate boundary is provided by Theorem 2.7. Although the solution is continuous up to the boundary, Theorem 2.7 implies that the gradient can blow up at the degenerate boundary at the maximal rate determined by the degree of the degeneracy (the rate at which the minimal eigenvalue of Q vanishes at the degenerate boundary). We mention that, to our knowledge, this is the first result showing continuity up to the degenerate boundary for the nonlinear wave equation.

To motivate the results we first present several examples that fall into the class of equations studied in this paper.

1.3 Examples

A. The nonlinear wave equation

We study the nonlinear wave equation

$$\rho_{tt} = \nabla(c^2(\rho)\nabla\rho). \tag{1.5}$$

This equation results from the compressible Euler equations assuming irrotationality and ignoring the terms quadratic in velocity (the low-speed range). It is also a low-speed reduction of the shallow water equations [14, 23]. The derivation suggested by Keyfitz [17] can be summarized as follows. Begin with the compressible Euler equations for isentropic flow

$$\begin{aligned}
\rho_t + (u\rho)_x + (v\rho)_y &= 0,\\
(u\rho)_t + (u^2\rho + p)_x + (uv\rho)_y &= 0,\\
(v\rho)_t + (uv\rho)_x + (v^2\rho + p)_y &= 0,
\end{aligned}$$

where ρ, u, v are the density, and the velocity components, respectively, and $p = p(\rho)$ is the pressure. If we assume the power-law relation $p(\rho) = \frac{1}{\gamma}\rho^\gamma$ where $\gamma > 1$ is the ratio of specific heats, then the local speed of sound $c^2 = \partial p/\partial\rho$ is equal to $c^2 = \rho^{\gamma-1}$. Assuming low velocities so that the higher-order terms in u and v can be neglected, the system becomes

$$\begin{aligned}
\rho_t + (u\rho)_x + (v\rho)_y &= 0,\\
(u\rho)_t + p(\rho)_x &= 0,\\
(v\rho)_t + p(\rho)_y &= 0.
\end{aligned} \tag{1.6}$$

We can write (1.6) as a second-order equation for the density, which is precisely the nonlinear wave equation (1.5), coupled to the linear equation $\partial_t(n_x - m_y) = 0$ where $m = u\rho$ and $n = v\rho$. Assuming that the initial data satisfies the compatibility condition $n_x = m_y$, a smooth solution will also satisfy the same condition.

In self-similar coordinates $\xi = x/t, \eta = y/t$ equation (1.5) becomes

$$\big((c^2(\rho) - \xi^2)\rho_\xi - \xi\eta\rho_\eta\big)_\xi + \big((c^2(\rho) - \eta^2)\rho_\eta - \xi\eta\rho_\xi\big)_\eta + \xi\rho_\xi + \eta\rho_\eta = 0. \tag{1.7}$$

This equation is elliptic whenever $c^2(\rho) > \xi^2 + \eta^2$ and degenerate whenever $c^2(\rho) = \xi^2 + \eta^2$.

It is of interest to look at the boundary-value problems that are degenerate at the boundary, *i.e.*, $c^2(\rho) = \xi^2 + \eta^2$ on $\partial\Omega$. They arise, for example, in the study the rarefaction wave interactions in general two-dimensional Riemann problems [26]; partially degenerate boundary data in domains with corners, covered by the results of this paper, arise in shock reflection problems, [3, 4]. In this paper we show the existence of a $C^2(\Omega)$–solution for these two types of problems (Theorem 2.7) and also continuity up to the degenerate boundary for convex domains (Section 3).

We mention two special cases included in this class of problems: the transonic pressure-gradient equation studied by Zheng in [26] which corresponds to $c^2(\rho) = e^\rho$ and the shallow equation which corresponds to $c^2(\rho) = \rho$.

B. The unsteady transonic small disturbance equation (UTSD)

The unsteady transonic small disturbance equation, also called the 2-D Burgers equation

$$\begin{aligned} u_t + uu_x + v_y &= 0, \\ v_x - u_y &= 0, \end{aligned} \tag{1.8}$$

arises in nonlinear acoustics and nonlinear geometrical optics [24], as well as in unsteady transonic flow. Brio and Hunter [1, 2] obtained this equation as an asymptotic reduction of the Euler equations to study weak shock reflection by a ramp. In the work of Morawetz [21], this equation holds in the shock interaction region. In this case, a form of (1.8) can be obtained by reduction from the transonic full potential equation.

In self-similar coordinates this system reads

$$\begin{aligned} (u-\xi)u_\xi - \eta u_\eta + v_\eta &= 0, \\ v_\xi - u_\eta &= 0. \end{aligned} \tag{1.9}$$

Linearized around a constant state (u_0, v_0) this system is hyperbolic outside the parabola $\rho = \xi + \eta^2/4 = u_0$ and elliptic inside. Notice that, in contrast with the nonlinear wave equation, the region where this system is elliptic is unbounded. After eliminating v and introducing the new variables $x = -\rho, y = \eta$ to make the degenerate boundary straight, the equation for u reads

$$\left((u+x)u_x - \frac{u}{2}\right)_x + u_{yy} = 0. \tag{1.10}$$

In terms of $w = u + x$ this can be written as

$$(ww_x)_x + w_{yy} - \frac{3}{2}w_x + \frac{1}{2} = 0. \tag{1.11}$$

We introduce the cut-off boundary to make the domain bounded, and prescribe Dirichlet data there. In Section 4 we use the techniques presented in this paper to prove the existence of a (unique) classical solution, continuous up to the degenerate boundary, for any continuous Dirichlet data which is degenerate on the boundary $x = 0$.

2 Existence of Solutions

In this section we prove the existence of a solution of the degenerate elliptic boundary value problem (1.2). We organize this section in three parts. In the first part we define the regularized problem and prove the existence of a (classical) solution u^ε to the nonlinear regularized problem. We establish a sequence of solutions of regularized problems $\{u^\varepsilon\}$ and show that the sequence is uniformly bounded (in ε) in L^∞. In the second part we construct a non-trivial lower bound assuming certain structure conditions. In the third part we use the existence of an L^∞ bound and of a non-trivial lower bound to argue, using a standard compactness argument, that there exists a subsequence of regularized solutions which converges in $C^2(\Omega)$. The limit satisfies the PDE in (1.2) in the classical sense, and the boundary condition in the sense $(u-g)^{a+1/2} \in W_0^{1,2}(\Omega)$, where a is the rate at which the minimal eigenvalue approaches zero as u approaches g at the boundary. In Sections 3 and 4, we show that the boundary condition for the the nonlinear wave equation and for the UTSD equation is satisfied in the classical sense and that the solution is continuous up to the boundary.

2.1 The Regularized Problem

For each $\varepsilon > 0$ we define

$$a_{ij}^\varepsilon(\mathbf{x}, z) \equiv \begin{cases} a_{ij}(\mathbf{x}, z) + \varepsilon & \text{if} \quad i = j, \\ a_{ij}(\mathbf{x}, z) & \text{if} \quad i \neq j, \end{cases}$$

and consider the following regularized problem:

$$Q^\varepsilon u \equiv \sum_{i,j}^{N} (a_{ij}^\varepsilon(\mathbf{x}, u) u_{x_j})_{x_i} + b(\mathbf{x}, u) \cdot \nabla u = -f(\mathbf{x}) \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega. \tag{2.1}$$

For each $\varepsilon > 0$ the operator Q^ε is strictly elliptic. Namely, there exist positive functions $\lambda^\varepsilon(\mathbf{x}, z)$ and $\Lambda^\varepsilon(\mathbf{x}, z)$ such that

$$0 < \lambda^\varepsilon(\mathbf{x}, z)|\zeta|^2 \leq a_{ij}^\varepsilon(\mathbf{x}, z)\zeta_i\zeta_j \leq \Lambda^\varepsilon(\mathbf{x}, z)|\zeta|^2, \quad \forall (\mathbf{x}, z) \in \Omega \times \mathbf{R}. \tag{2.2}$$

Since the existence result to the regularized problem (2.1) is a standard application of the Schauder Fixed Point Theorem, we outline only the main steps of the existence proof below. Similar proofs can be found for example in [13, 16, 19, 20]. If the source term f is nonzero, the following condition needs to hold.

F. There exists a constant $\lambda_0 > 0$ such that $\lambda(\mathbf{x}, z) \geq \lambda_0$ whenever $z \geq \max_{\partial\Omega} g + 1$.

THEOREM 2.1 *Suppose that hypotheses* **H1**-**H4** *are satisfied and suppose that condition* **F** *holds if* $f \neq 0$*. Then for each* $\varepsilon > 0$ $(0 < \varepsilon < 1)$*, there exists a (classical) solution* u^ε *of the regularized problem* (2.1) *and a positive number* α*,* $0 < \alpha < 1$*, such that* $u^\varepsilon \in C^{2,\alpha}(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$*. Moreover, the solution* u^ε *is uniformly bounded in* L^∞*, i.e., there exist constants* m *and* M *that depend only on* g*,* f *and* Ω*, such that* $m \leq u^\varepsilon \leq M$*.*

PROOF:Let S be a subset of $C(\overline{\Omega})$ such that

$$S \equiv \{w \in C(\overline{\Omega}) |\ m \leq w \leq M,\ w\ |_{\partial\Omega} = g\},$$

where $m \equiv \min_{\partial\Omega} g$ and M is obtained from an a priori estimate of the solution u^ε. Namely, if u^ε is a solution of (2.1) then we can apply Theorem 8.16 from Gilbarg and Trudinger [16] to the function u^ε on the set

$$\Omega^+ = \{x \in \Omega : u^\varepsilon \geq \max g + 1\},$$

to obtain $u^\varepsilon \leq M$ in Ω^+, and hence in Ω, for

$$M = \max_{\partial\Omega} g + 1 + \frac{C(N,\Omega)}{\lambda_0} ||f||_{2N},$$

where $C(N, \Omega)$ is independent of the upper ellipticity constant. The set S is closed, bounded and convex.

Define a map T on S such that for each $w \in S$, $Tw \equiv u$, where u is the generalized solution of the associated *linear* elliptic problem

$$L^\varepsilon u \equiv \sum_{i,j}^{N} (a_{ij}^\varepsilon(\mathbf{x}, w) u_{x_j})_{x_i} + b(\mathbf{x}, w) \cdot \nabla u + f(\mathbf{x}) = 0, \quad u\ |_{\partial\Omega} = g. \tag{2.3}$$

Theorem 8.30 in [16] provides the existence of a unique solution $u \in W_{loc}^{1,2}(\Omega) \cap C(\overline{\Omega})$ of (2.3).

To use the Schauder Fixed Point Theorem we need to show that the map T is into, that T is compact and continuous. The maximum principle implies that T is into. Precompactness of T is a consequence of the global Hölder estimate, Theorem 14.1 in [19]. To show continuity of T we use a standard limiting argument: for any $\{w_n\} \subset S$ such that $w_n \to w$ in S, we need to show that $Tw_n \to Tw$. This holds due to the uniform bounds in $C^\alpha(\overline{\Omega})$ and in $W^{1,2}$ and due to the uniqueness of the limit. The Schauder Fixed Point Theorem now implies that there exists a fixed point $u \in S$ such that u solves equation (2.1) and $u \in W^{1,2}(\Omega) \cap C(\overline{\Omega})$.

To show that u is a classical solution we use the standard regularity arguments: Hölder estimates [16, 19], Corollary 8.36 in [16] and the interior Schauder estimates which imply that $u \in C^{2,\alpha}(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$. Note that since $u \in S$ we have $m \leq u \leq M$. This completes the proof. ■

Remark 2.2 If Ω is of class $W^{2,q}$ where $q > N$, and $g \in W^{2,q}(\Omega)$ where $q > N$, then Theorem 15.1 in [19] implies that $u^\varepsilon \in C^{1,\alpha}(\overline{\Omega})$ where $\alpha = 1 - N/q$.

Remark 2.3 If the forcing term f is equal to zero, the assumption: "$\lambda(\mathbf{x}, z) \geq \lambda_0$ whenever $z \geq \max_{\partial\Omega} g + 1$" can be dropped. The "trick" to obtain the maximum principle for the case $f \neq 0$ was suggested to us by Lieberman.

2.2 Lower Barrier

In this section we show that if the elliptic operator (1.3) and the boundary data g satisfy the structure conditions listed below, then there exists a lower barrier for u^ε, independent of ε, which provides strict ellipticity in the interior of Ω. Using this barrier, in Section 2.3, we prove the existence of a solution of the degenerate problem (1.2).

Conditions for the existence of barrier functions in non-uniformly elliptic equations typically require information about the relationship between the coefficients of the operator and the properties of the boundary and of the boundary data. In linear problems, if one seeks a local barrier in terms of the distance function to the (degenerate) boundary, a sufficient condition for the existence of a barrier is a Fichera-type condition that needs to hold at the boundary (an inequality relating the Fichera function [22], the principal coefficients of the operator, and the principal curvatures of the boundary). See Section 6.6 in [16].

In our problem we require the following sufficient conditions for the existence of a lower barrier which provides strict ellipticity in the interior of Ω.

Structure Conditions.

S1. (Structure of the minimum eigenvalue) The minimum eigenvalue λ of $[a_{ij}]$ is such that there exists a function G satisfying the property: $\lambda(\mathbf{x}, u) = 0$ if and only if $u = G$ for $\mathbf{x} \in \overline{\Omega}$.

S2. (Generalized subsolution property) There exists a C^2–function G_1 satisfying the following two conditions:

1. $G_1 \geq G$ in $\overline{\Omega}$, $G_1 \leq g$ on $\partial\Omega$ and $G_1 = G = g$ on the degenerate boundary $\Sigma \subseteq \partial\Omega$.
2. There exist a positive constant K and a non-negative, locally bounded function $S(\mathbf{x})$ such that

$$\begin{aligned}\sum_{i,j} a_{ij}(\mathbf{x}, z) G_{1x_jx_i} + \sum_{i,j} \tfrac{\partial a_{ij}(\mathbf{x},z)}{\partial x_i} G_{1x_j} + \sum_{i,j} \tfrac{\partial a_{ij}(\mathbf{x},z)}{\partial z} G_{1x_j} G_{1x_i} \\ + b(\mathbf{x}, z)\nabla G_1 + f(\mathbf{x}) \geq K - S(\mathbf{x})(z - G_1)\end{aligned} \tag{2.4}$$

for all $\mathbf{x} \in \Omega$ and $z \in \mathbf{R}$ such that $m \leq z \leq M$, where $m = \min_{\partial\Omega} g$.

It will be shown in Sections 3 and 4 that both the nonlinear wave equation and the UTSD equation satisfy these structure conditions.

LEMMA 2.4 *Suppose that structure conditions* **S1** *and* **S2** *hold. Let G_1 be a C^2-function defined by the structure condition* **S2**. *Then for each $\varepsilon > 0$ the solution u^ε of the regularized problem* (2.1) *satisfies*

1. *$u^\varepsilon - G_1 > 0$ in Ω, and*
2. *for every function $\phi \in C_0^3(\overline{\Omega})$ positive in the interior of Ω, there exists a $\delta > 0$, independent of ε and depend on ϕ, such that $u^\varepsilon - G_1 - \delta\phi > 0$ in the interior of Ω.*

REMARK 2.5 The statement $u^\varepsilon - G_1 > 0$ in Ω implies that $\lambda(\mathbf{x}, u^\varepsilon) > 0$ in Ω. Therefore, u^ε is elliptic in the interior of the domain. However, lower barrier G_1 may be degenerate in the sense that $\lambda(\mathbf{x}, G_1) = 0$. So, the sequence (u^ε) is not uniformly elliptic in ε. To prove the existence of a solution to the degenerate problem we will need a lower barrier, say G_2, *independent* of ε, such that $u^\varepsilon - G_2 > 0$ and $\lambda(\mathbf{x}, G_2) > 0$ in Ω. This is provided by the function $\delta\phi$, whose existence is discussed in the second part of Lemma 2.4. By setting $G_2 = G_1 + \delta\phi$ we have a lower barrier G_2 which is elliptic in Ω, *i.e.*, $\lambda(\mathbf{x}, G_2) > 0$. Hence the sequence u^ε is uniformly elliptic in ε since $u^\varepsilon > G_2, \forall \varepsilon > 0$.

Proof:(of Lemma 2.4) For each $\varepsilon > 0$ define $w^\varepsilon \equiv u^\varepsilon - G_1$ in Ω. Notice that since $G_1 \le g$ on $\partial\Omega$ we have $w^\varepsilon \ge 0$ on $\partial\Omega$. We show that $w^\varepsilon > 0$ in the interior of Ω. Suppose that this is not true. Then there exists a point $\mathbf{x}_0 \in \Omega$ at which w^ε attains a non-positive local minimum. At such a point the following conditions hold

$$w^\varepsilon(\mathbf{x}_0) \le 0, \quad \nabla w^\varepsilon(\mathbf{x}_0) = 0 \quad \text{and} \quad \sum_{i,j} a^\varepsilon_{ij}(\mathbf{x}_0, u^\varepsilon) {w^\varepsilon}_{x_j x_i}(\mathbf{x}_0) \ge 0.$$

Since $u^\varepsilon = w^\varepsilon + G_1$ is a solution of (2.1) we get

$$\begin{aligned} 0 &= \sum_{i,j} \left(a^\varepsilon_{ij}(\mathbf{x}, u^\varepsilon)(w^\varepsilon + G_1)_{x_j}\right)_{x_i} + \sum_i b_i(\mathbf{x}, u^\varepsilon)(w^\varepsilon + G_1)_{x_i} + f(\mathbf{x}) \\ &= \sum_{i,j} a^\varepsilon_{ij}(\mathbf{x}, u^\varepsilon)(w^\varepsilon + G_1)_{x_j x_i} + \sum_{i,j} \left(\frac{\partial a^\varepsilon_{ij}(\mathbf{x}, u^\varepsilon)}{\partial x_i} + \frac{\partial a^\varepsilon_{ij}(\mathbf{x}, u^\varepsilon)}{\partial u} \frac{\partial u^\varepsilon}{\partial x_i} \right) (w^\varepsilon + G_1)_{x_j} \\ &\quad + \sum_i b_i(\mathbf{x}, u^\varepsilon) \left(w^\varepsilon + G_1\right)_{x_i} + f(\mathbf{x}). \end{aligned}$$

We separate the derivatives of w^ε and G_1 and write the right hand-side as

$$\begin{aligned} 0 &= \sum_{i,j} a^\varepsilon_{ij}(\mathbf{x}, u^\varepsilon) G_{1 x_j x_i} + \sum_{i,j} \frac{\partial a^\varepsilon_{ij}(\mathbf{x}, u^\varepsilon)}{\partial x_i} G_{1 x_j} + \sum_{i,j} \frac{\partial a^\varepsilon_{ij}(\mathbf{x}, u^\varepsilon)}{\partial u} G_{1 x_i} G_{1 x_j} \\ &\quad + b(\mathbf{x}, u^\varepsilon)\nabla G_1 + f(\mathbf{x}) + \sum_{i,j} a^\varepsilon_{ij}(\mathbf{x}, u^\varepsilon) {w^\varepsilon}_{x_j x_i} + \sum_j B_j(\mathbf{x}, u^\varepsilon) {w^\varepsilon}_{x_j}, \end{aligned}$$

where

$$B_j(\mathbf{x}, u^\varepsilon) \equiv \sum_i \left\{ \frac{\partial a^\varepsilon_{ij}(\mathbf{x}, u^\varepsilon)}{\partial u} {u^\varepsilon}_{x_i} + \frac{\partial a^\varepsilon_{ij}(\mathbf{x}, u^\varepsilon)}{\partial x_i} + \frac{\partial a^\varepsilon_{ij}(\mathbf{x}, u^\varepsilon)}{\partial u} G_{1 x_i} \right\} + b_j(\mathbf{x}, u^\varepsilon).$$

Here we have used the fact that $a_{ij} = a_{ji}$. By the structure condition (2.4) the first two lines can be estimated from below by $K - S(\mathbf{x})w$, where $K > 0$ and $S(\mathbf{x}) \ge 0$:

$$\begin{aligned} 0 &\ge \sum_{i,j} a^\varepsilon_{ij}(\mathbf{x}, u^\varepsilon) {w^\varepsilon}_{x_j x_i} + \sum_j B_j(\mathbf{x}, u^\varepsilon) {w^\varepsilon}_{x_j} + K - S(\mathbf{x}) w^\varepsilon \\ &> \sum_{i,j} a^\varepsilon_{ij}(\mathbf{x}, u^\varepsilon) {w^\varepsilon}_{x_j x_i} + \sum_j B_j(\mathbf{x}, u^\varepsilon) {w^\varepsilon}_{x_j} - S(\mathbf{x}) w^\varepsilon. \end{aligned}$$

If we evaluate this expression at $\mathbf{x}_0$ where $\nabla w^\varepsilon(\mathbf{x}_0) = 0$ and $w^\varepsilon(\mathbf{x}_0) \le 0$, we get

$$0 > \sum_{i,j} a^\varepsilon_{ij}(\mathbf{x}_0, u^\varepsilon) {w^\varepsilon}_{x_j x_i}(\mathbf{x}_0)$$

which contradicts $\mathbf{x}_0$ being a local minimum.

We now show that there exist a positive constant δ (independent of ε) such that for each $\phi \in C_0^3(\overline{\Omega})$, $\phi > 0$ in Ω, the following estimate holds for the solution u^ε of the regularized problem $u^\varepsilon - G_1 - \delta\phi > 0$, for all $x \in \Omega$. We again use the method of contradiction. Suppose that for a given function ϕ it is not possible to find a δ such that $u^\varepsilon - G_1 - \delta\phi = w^\varepsilon - \delta\phi > 0$ in Ω. Then, for each $\delta > 0$ there exists a point $\mathbf{x}_0$ in the interior of Ω ($\mathbf{x}_0$ depends on δ) such that $w^\varepsilon(\mathbf{x}_0) - \delta\phi(\mathbf{x}_0) \le 0$, and $\mathbf{x}_0$ is a local minimum. The following conditions hold at $\mathbf{x}_0$

$$w^\varepsilon(\mathbf{x}_0) - \delta\phi(\mathbf{x}_0) \le 0, \quad \nabla(w^\varepsilon - \delta\phi)(\mathbf{x}_0) = 0, \quad \sum_{i,j} a_{ij}^\varepsilon(\mathbf{x}_0, u^\varepsilon)(w^\varepsilon - \delta\phi)_{x_j x_i}(\mathbf{x}_0) \ge 0.$$

We will show that this contradicts the properties of the operator imposed by the structure condition (2.4). To show this we first calculate $Q^\varepsilon u^\varepsilon + f(\mathbf{x}) = 0$ at any point $\mathbf{x} \in \Omega$. In this equation we add and subtract the terms containing $\delta\phi$ to obtain

$$\begin{aligned} 0 &= \textstyle\sum_{i,j} a_{ij}^\varepsilon(\mathbf{x}, u^\varepsilon)(w^\varepsilon - \delta\phi)_{x_j x_i} + B_\delta(\mathbf{x}, u^\varepsilon)\nabla(w^\varepsilon - \delta\phi) \\ &\quad + \textstyle\sum_{i,j} a_{ij}^\varepsilon(\mathbf{x}, u^\varepsilon)(G_1 + \delta\phi)_{x_j x_i} + \sum_{i,j} \frac{\partial a_{ij}^\varepsilon(\mathbf{x}, u^\varepsilon)}{\partial x_i}(G_1 + \delta\phi)_{x_j} \\ &\quad + \textstyle\sum_{i,j} \frac{\partial a_{ij}^\varepsilon(\mathbf{x}, u^\varepsilon)}{\partial u}(G_1 + \delta\phi)_{x_j}(G_1 + \delta\phi)_{x_i} + b(\mathbf{x}, u^\varepsilon)\nabla(G_1 + \delta\phi) + f(\mathbf{x}), \end{aligned} \tag{2.5}$$

where we denote by B_δ the vector with components

$$B_{\delta j}(\mathbf{x}, u^\varepsilon) \equiv \sum_i \left\{ \frac{\partial a_{ij}^\varepsilon(\mathbf{x}, u^\varepsilon)}{\partial u} u^\varepsilon{}_{x_i} + \frac{\partial a_{ij}^\varepsilon(\mathbf{x}, u^\varepsilon)}{\partial x_i} + \frac{\partial a_{ij}^\varepsilon(\mathbf{x}, u^\varepsilon)}{\partial u}(G_1 + \delta\phi)_{x_i} \right\} + b_j(\mathbf{x}, u^\varepsilon).$$

The last two lines can be written as the left hand-side of the structure condition plus terms which are multiplied by δ. Using the structure condition we can estimate the last two lines as

$$\begin{aligned} & \sum_{i,j} a_{ij}^\varepsilon(\mathbf{x}, u^\varepsilon)(G_1 + \delta\phi)_{x_j x_i} + \sum_{i,j} \frac{\partial a_{ij}^\varepsilon(\mathbf{x}, u^\varepsilon)}{\partial x_i}(G_1 + \delta\phi)_{x_j} \\ & + \sum_{i,j} \frac{\partial a_{ij}^\varepsilon(\mathbf{x}, u^\varepsilon)}{\partial u}(G_1 + \delta\phi)_{x_j}(G_1 + \delta\phi)_{x_i} + b(\mathbf{x}, u^\varepsilon)\nabla(G_1 + \delta\phi) + f(\mathbf{x}) \\ \ge \quad & K - S(\mathbf{x})(u^\varepsilon - G_1) + \delta \Bigg(\sum_{i,j} a_{ij}^\varepsilon(\mathbf{x}, u^\varepsilon)\phi_{x_j x_i} + \sum_{i,j} \frac{\partial a_{ij}^\varepsilon(\mathbf{x}, u^\varepsilon)}{\partial x_i}\phi_{x_j} \\ & + \sum_{i,j} \frac{\partial a_{ij}^\varepsilon(\mathbf{x}, u^\varepsilon)}{\partial u}(G_{1x_i}\phi_{x_j} + G_{1x_j}\phi_{x_i} + \delta\phi_{x_j}\phi_{x_i}) + b(\mathbf{x}, u^\varepsilon)\nabla\phi \Bigg). \end{aligned}$$

Since $m \le u^\varepsilon \le M$ and $a_{ij} \in C^1$, we have bounds for the functions $a_{ij}^\varepsilon(\mathbf{x}, u^\varepsilon)$, $\partial a_{ij}^\varepsilon(\mathbf{x}, u^\varepsilon)/\partial x_j$ and $\partial a_{ij}^\varepsilon(\mathbf{x}, u^\varepsilon)/\partial u$ which depend only on m, M and Ω. Furthermore, since G_1, b and ϕ are smooth, all the terms involving these functions are bounded (independently of ε). Since $K > 0$ (by structure condition S2.2) there exists a $\delta > 0$, independent of ε, such that

$$K \quad + \quad \delta \left(\sum_{i,j} a_{ij}^{\varepsilon}(\mathbf{x}, u^{\varepsilon}) \phi_{x_j x_i} + \sum_{i,j} \frac{\partial a_{ij}^{\varepsilon}(\mathbf{x}, u^{\varepsilon})}{\partial x_i} \phi_{x_j} \right. \\ \left. + \sum_{i,j} \frac{\partial a_{ij}^{\varepsilon}(\mathbf{x}, u^{\varepsilon})}{\partial u} (G_{1 x_i} \phi_{x_j} + G_{1 x_j} \phi_{x_i} + \delta \phi_{x_i} \phi_{x_j}) + b(\mathbf{x}, u^{\varepsilon}) \nabla \phi \right) > 0.$$

Using this inequality we can estimate the right hand-side in (2.5) as

$$0 \quad > \quad \sum_{i,j} a_{ij}^{\varepsilon}(\mathbf{x}, u^{\varepsilon})(w^{\varepsilon} - \delta\phi)_{x_j x_i} + B(\mathbf{x}, u^{\varepsilon}) \nabla (w^{\varepsilon} - \delta\phi) - S(\mathbf{x})(w^{\varepsilon} - \delta\phi).$$

Notice that this estimate holds for a fixed (small) δ and is true at any point $\mathbf{x} \in \Omega$. (It is a consequence of the structure condition.) For this fixed $\delta > 0$ we evaluate the above expression at the associated local minimum $\mathbf{x}_0$ to obtain

$$\begin{aligned} 0 \quad &> \quad \sum_{i,j} a_{ij}^{\varepsilon}(\mathbf{x}_0, u^{\varepsilon})(w^{\varepsilon} - \delta\phi)_{x_j x_i}(\mathbf{x}_0) + B(\mathbf{x}_0, u^{\varepsilon}) \nabla (w^{\varepsilon} - \delta\phi)(\mathbf{x}_0) - S(\mathbf{x}_0)(w^{\varepsilon} - \delta\phi)(\mathbf{x}_0) \\ &\geq \quad \sum_{i,j} a_{ij}^{\varepsilon}(\mathbf{x}_0, u^{\varepsilon})(w^{\varepsilon} - \delta\phi)_{x_i x_j}(\mathbf{x}_0), \end{aligned}$$

which contradicts $\mathbf{x}_0$ being a point of the local minimum. This completes the proof. ∎

REMARK 2.6 If we write the right hand side of the structure condition as

$$K - S(\mathbf{x})(z - G_1)^{+}$$

with $K > 0$ and $S(\mathbf{x})$ non-negative and locally bounded, Lemma 2.4 still holds.

We mention that, although more general, the construction of the "elliptic" global lower barrier presented in this section is similar, in the spirit, to the proof of strict interior ellipticity of the pressure-gradient equations, studied in Zheng's reference [26].

2.3 EXISTENCE RESULT

In this section we prove the existence of a solution $u \in C^2(\Omega)$ to the degenerate problem (1.2) by considering the limit of the regularized solutions u^{ε}. The general hypotheses **H1-H4** are insufficient to conclude that a (smooth) solution in the interior of the domain satisfies the boundary condition in the classical sense. However, we can show that, depending on how fast the minimum eigenvalue approaches zero, in other words, depending on the "degree" of the degeneracy, a, the boundary condition will be satisfied in the sense $(u-g)^{a+1/2} \in W_0^{1,2}(\Omega)$. More precisely, by the existence of a positive lower bound that is independent of ε we now know that our operator Q is strictly elliptic in the interior of the domain and is degenerate only at the boundary where the minimum eigenvalue λ of the operator Q vanishes. If we denote by $a > 0$ the number such that $\lambda(\mathbf{x}, z) \geq C(\mathbf{x})(u-g)^a$, for some positive, bounded function $C(\mathbf{x})$, then we will show that solution u satisfies the boundary condition in the sense $(u-g)^{a+1/2} \in W_0^{1,2}(\Omega)$. This means that it is possible to have solutions to the quasilinear problem which are singular at the boundary, as was shown in [3] and [26]. In both studies [3] and [26] the minimum eigenvalue λ equals $u-g$, which implies $a=1$. Our result implies that $(u-g)^{3/2} \in W_0^{1,2}$, which is the result obtained in [3] and [26].

In Sections 3 and 4 we prove, assuming additional information about the coefficients of the operator Q and the boundary, that the solution we found is continuous up to the degenerate boundary.

To state the existence theorem we introduce the following notation for the symmetric matrix of principal coefficients $A(\mathbf{x}, u) = [a_{ij}(\mathbf{x}, u)]$.

Theorem 2.7 *Assume that hypotheses* **H1**-**H4** *are satisfied and that structure conditions* **S1** *and* **S2** *hold. Furthermore, if $f \neq 0$, assume that condition* **F** *in Section 2.1 is satisfied. Then there exists a solution $u \in C^2(\Omega)$ of (1.2). Furthermore, let $a > 0$ be such that $\lambda(\mathbf{x}, u) \geq C(\mathbf{x})(u-g)^a$ for some positive, bounded function $C(\mathbf{x})$, then $(u-g)^{a+1/2} \in W_0^{1,2}(\Omega)$.*

Proof:First we show that there exists a function $u \in C^2(\Omega)$ which satisfies the PDE in (1.2) in Ω. We use a uniform lower barrier $G_2 = G_1 + \delta\phi$, obtained in the previous section. Consider the operator Q^ε restricted to a compact subset, Ω_1, of Ω. By Remark 2.5 Q^ε is strictly elliptic in Ω_1 since $u^\varepsilon > G_2 > G$ in Ω and $\lambda(\mathbf{x}, u) = 0$ if and only if $u = G$. Furthermore, this estimate is *independent* of ε. We now treat (2.1) as a linear problem. By Theorem 8.22 in [16] the solution to this problem is locally Hölder continuous. More precisely, on any $\Omega_2 \subset\subset \Omega_1$ the following estimate holds

$$\|u^\varepsilon\|_{C^\alpha(\Omega_2)} \leq C(\Omega_2),$$

where $C(\Omega_2)$ and $0 < \alpha < 1$ are independent of ε. With this preliminary estimate of the coefficients of (2.1), we use hypothesis **H3** which provides $a_{ij} \in C^1$, the boundedness of u^ε and uniform ellipticity of the operator, and apply interior Schauder estimates, presented in Theorem 8.32 [16] and Theorem 6.2 [16], on $\Omega_4 \subset\subset \Omega_3 \subset\subset \Omega_2$ to get

$$\|u^\varepsilon\|_{C^{2+\alpha}(\Omega_4)} \leq C(\Omega_4),$$

where $C(\Omega_4)$ is independent of ε.

By the Arzela-Ascoli theorem there exists a subsequence u_n^ε such that $u_n^\varepsilon \to u$ in $C^{2,\alpha'}(\Omega_4)$, for any $\alpha' < \alpha$. Since Ω_1 is an arbitrary subset of Ω, we apply the same argument to each compact subset of Ω to extract a $C^{2,\alpha'}$-convergent subsequence. Using the diagonalization principle we obtain a subsequence of u^ε which converges in $C^2_{loc}(\Omega)$ to the limit $u \in C^2(\Omega)$ where $u > g$ and u solves the PDE in (1.2) in Ω.

We now show that if the minimum eigenvalue λ approaches zero as fast as $(u-g)^a$, then the solution u attains its boundary value g in the sense $(u-g)^{a+1/2} \in W_0^{1,2}$. To do that we show that $(u^\varepsilon - g)^{a+1/2}$ is bounded in $W_0^{1,2}$, uniformly in ε. From this it follows that there exists a subsequence with limit u such that $(u-g)^{a+1/2} \in W_0^{1,2}$, and u is the solution of the PDE found in the first part of the proof.

Denote $w^\varepsilon = u^\varepsilon - g$. Then the PDE in (2.1) expressed in terms of w^ε, reads

$$\sum_{i,j}^N \left(a_{ij}^\varepsilon(\mathbf{x}, u^\varepsilon)(w^\varepsilon + g)_{x_j}\right)_{x_i} + b(\mathbf{x}, u^\varepsilon)\nabla w^\varepsilon = -f(\mathbf{x}) - b(\mathbf{x}, u^\varepsilon)\nabla g.$$

Multiply this equation by $(w^\varepsilon)^a$ and integrate by parts to obtain

$$-a\left\langle A(\mathbf{x}, u^\varepsilon)(w^\varepsilon)^{a-1}\ \nabla(w^\varepsilon + g), \nabla w^\varepsilon\right\rangle + \langle (w^\varepsilon)^a b(\mathbf{x}, u^\varepsilon), \nabla w^\varepsilon\rangle = -\langle F(\mathbf{x}), (w^\varepsilon)^a\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the standard L^2-inner product, and $F(\mathbf{x}) = f + b(\mathbf{x}, u^\varepsilon)\nabla g$. This equation implies that

$$\begin{aligned} a\left\langle A(\mathbf{x}, u^\varepsilon)(w^\varepsilon)^{a-1}\ \nabla w^\varepsilon, \nabla w^\varepsilon\right\rangle = &- a\left\langle \sqrt{A(\mathbf{x}, u^\varepsilon)}(w^\varepsilon)^{a-1}\ \nabla g, \sqrt{A(\mathbf{x}, u^\varepsilon)}\nabla w^\varepsilon\right\rangle \\ &+ \langle (w^\varepsilon)^a b(\mathbf{x}, u^\varepsilon), \nabla w^\varepsilon\rangle + \langle F(\mathbf{x}), (w^\varepsilon)^a\rangle. \end{aligned}$$

We use the weighted Schwartz-Cauchy inequality twice to estimate the two terms on the right hand-side involving the derivatives of w^ε. Namely, since w^ε has a L^∞ bound independent of ε, and since g, f and b are smooth, we have

$$\begin{aligned} &\left\langle \sqrt{A(\mathbf{x}, u^\varepsilon)}(w^\varepsilon)^{a-1}\ \nabla g, \sqrt{A(\mathbf{x}, u^\varepsilon)}\nabla w^\varepsilon\right\rangle \\ &= \left\langle \sqrt{A(\mathbf{x}, u^\varepsilon)\ (w^\varepsilon)^{a-1}}\ \nabla g, \sqrt{A(\mathbf{x}, u^\varepsilon)\ (w^\varepsilon)^{a-1}}\nabla w^\varepsilon\right\rangle \\ &\leq C_1 + \frac{1}{\delta_1}\left\langle A(\mathbf{x}, u^\varepsilon)\ (w^\varepsilon)^{a-1}\nabla w^\varepsilon, \nabla w^\varepsilon\right\rangle, \end{aligned}$$

with $\delta_1 > 0$, where C_1 depends on g, $|w^\varepsilon|_\infty$ and Ω. Similarly we have

$$\begin{aligned}\left\langle (w^\varepsilon)^a b(\mathbf{x},u^\varepsilon), \nabla w^\varepsilon\right\rangle &= \left\langle b(\mathbf{x},u^\varepsilon)\ (w^\varepsilon)^{1/2}, (w^\varepsilon)^{a-1/2}\nabla w^\varepsilon\right\rangle \\ &\le C_2 + \frac{1}{\delta_2}\left\langle (w^\varepsilon)^{2a-1}\nabla w^\varepsilon, \nabla w^\varepsilon\right\rangle,\end{aligned}$$

with $\delta_2 > 0$, where C_2 depends on b, $|w^\varepsilon|_\infty$ and Ω. Therefore, there exists a constant D which depends on C_1, C_2 and F such that

$$\begin{aligned}\left\langle A(\mathbf{x},u^\varepsilon)(w^\varepsilon)^{a-1}\ \nabla w^\varepsilon, \nabla w^\varepsilon\right\rangle &\le D + \frac{1}{\delta_1}\left\langle A(\mathbf{x},u^\varepsilon)\ (w^\varepsilon)^{a-1}\nabla w^\varepsilon, \nabla w^\varepsilon\right\rangle \\ &+ \frac{1}{\delta_2}\left\langle (w^\varepsilon)^{2a-1}\nabla w^\varepsilon, \nabla w^\varepsilon\right\rangle.\end{aligned}$$

Hence

$$\begin{aligned}\left\langle A(\mathbf{x},u^\varepsilon)(w^\varepsilon)^{a-1}\ \nabla w^\varepsilon, \nabla w^\varepsilon\right\rangle - \frac{1}{\delta_1}\left\langle A(\mathbf{x},u^\varepsilon)(w^\varepsilon)^{a-1}\nabla w^\varepsilon, \nabla w^\varepsilon\right\rangle \\ \le \frac{1}{\delta_2}\left\langle (w^\varepsilon)^{2a-1}\nabla w^\varepsilon, \nabla w^\varepsilon\right\rangle + D.\end{aligned}$$

Choose δ_1 so that $1 - 1/\delta_1 > 0$ and estimate $A(\mathbf{x},u^\varepsilon)$ from below by its minimum eigenvalue which is assumed to behave as $\lambda(\mathbf{x},u^\varepsilon) \ge C(\mathbf{x})(w^\varepsilon)^a \ge C(w^\varepsilon)^a$. Then we have

$$\left\langle (w^\varepsilon)^{2a-1}\nabla w^\varepsilon, \nabla w^\varepsilon\right\rangle \le \frac{1}{\delta_2}\left\langle (w^\varepsilon)^{2a-1}\nabla w^\varepsilon, \nabla w^\varepsilon\right\rangle + D.$$

Now choose δ_2 so that $1 - 1/\delta_2 > 0$ to obtain $\|(w^\varepsilon)^{a-1/2}\nabla w^\varepsilon\|_{L^2} \le D_1$ which implies $\|(w^\varepsilon)^{a+1/2}\|_{W_0^{1,2}} \le D_2$ where D_2 depends only on g, f, a, δ_1, δ_2 and Ω. This completes the proof. ■

In the following sections we focus on the nonlinear wave equation and on the UTSD equation, and show that the solution we found is continuous up to the degenerate boundary.

3 The Nonlinear Wave Equation

We focus our attention on equation (1.7). To simplify the study we first rewrite equation (1.7) in terms of the new variable $u = c^2(\rho)$, where $c^2(\rho)$ is the sound speed. Under the assumption that c^2 is monotonic (which is satisfied for the power-law relation between the pressure and the density), we define the inverse $\rho = k(u)$. Then the nonlinear wave equation in self-similar coordinates, which we now denote by $\mathbf{x} = (x, y)$, can be written as $Q(u) = 0$ where

$$\begin{aligned}Q(u) = ((u-x^2)k'(u)u_x - xyk'(u)u_y)_x + ((u-y^2)k'(u)u_y - xyk'(u)u_x)_y \\ +k'(u)\left(xu_x + yu_y\right).\end{aligned} \tag{3.1}$$

Take the boundary data $g(x, y) = x^2 + y^2$, so that the operator Q is degenerate on the boundary. We study the problem

$$Q(u) = 0 \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega, \tag{3.2}$$

under the following assumptions.

A1. The boundary $\partial\Omega$ does not contain the origin.

A2. The function k is smooth and strictly increasing in the sense that $k'(z) \geq k_1 > 0$, $\forall z \in \mathbf{R}$ such that $0 < m \leq z \leq M$.

Using the results from the first part of the paper we show that there exists a $C^2(\Omega)$ solution of this boundary-value problem. This result holds on domains Ω which satisfy only the uniform exterior cone condition, as defined in hypothesis **H1**. In the second part of this section we show that if Ω is convex or satisfies an exterior plane condition (see **A3** in the following subsection), then the solution is continuous up to the degenerate boundary.

3.1 Existence Result

To work with the elliptic operator we introduce a cut-off function (mollified)

$$\tilde{u}(x, y, u) = \begin{cases} u & \text{if} \quad u \geq g(x, y), \\ g(x, y) & \text{if} \quad u < g(x, y), \end{cases} \tag{3.3}$$

and define the modified operator

$$\begin{aligned} \widetilde{Q}(u) = \big((\tilde{u} - x^2)k'(u)u_x \; &- \; xyk'(u)u_y\big)_x + \big((\tilde{u} - y^2)k'(u)u_y - xyk'(u)u_x\big)_y \\ &+ \; k'(u)\,(xu_x + yu_y)\,. \end{aligned}$$

Notice that we have introduced the cut-off only in the coefficients that influence the ellipticity of the operator. We will show that the solution of the modified problem exists and satisfies $u \geq g$ in $\overline{\Omega}$, hence it solves the original problem.

We verify the hypotheses of Theorem 2.7. We need to show that structure conditions **S1** and **S2** hold for the operator $\widetilde{Q}$ and the boundary data g. Condition **S1** is satisfied by the function $G = g = x^2 + y^2$ since the minimum eigenvalue $\lambda(x, y, u) = u - x^2 - y^2 = 0$ if and only if $u = x^2 + y^2$. To show that condition **S2** is satisfied we need to find a lower barrier G_1 satisfying the properties $G_1 \geq g$ on Ω, $G_1 = g$ on $\partial\Omega = \Sigma$ and condition (2.4). We show that a good choice for the lower barrier G_1 is the C^2 extension of the boundary data $G_1 = g = x^2 + y^2$. Indeed, we claim that structure condition (2.4) holds for g. To show that this is true we write the operator in non-divergence form and substitute the derivatives of g for the derivatives of u to verify the structure condition (2.4). For $m \leq \tilde{z} \leq M$ where $m = \min_{\partial\Omega} g > 0$, the left hand-side of the structure condition reads

$$\begin{aligned} & k'(z)\left\{(\tilde{z}-x^2)g_{xx} + (\tilde{z}-y^2)g_{yy} - (2x+y)g_x - (2y+x)g_y \right. \\ + & \left. g_x^2 + g_y^2 + xg_x + yg_y\right\} + k''(z)\left\{(\tilde{z}-x^2)g_x^2 - 2xyg_xg_y + (\tilde{z}-y^2)g_y^2\right\} \\ = & k'(z)\left\{2\tilde{z} + 2(\tilde{z}-(x^2+y^2)) + 2(x-y)^2\right\} + k''(z)\left\{4(x^2+y^2)(\tilde{z}-(x^2+y^2))\right\} \\ \geq & 2k_1\tilde{z} + \left(2k'(z) + 4k''(z)(x^2+y^2)\right)(\tilde{z}-g) \\ \geq & 2k_1 m + 4k''(z)(x^2+y^2)(\tilde{z}-g)\,. \end{aligned}$$

Now, if $k''(z) \geq 0$ for $0 < m \leq z \leq M$, where $m = \min_{\partial\Omega} g$, the entire expression is bounded from below by the constant $K = 2k_1 m = 2k_1 \min_{\partial\Omega}(x^2+y^2)$ which is strictly positive since, by assumption **A1**, $\partial\Omega$ does not contain the origin. If $k''(z) < 0$ or changes sign, then the entire expression is bounded from below by $K - S(x,y)(z-g)^+$ where $S(x,y) = 4(x^2+y^2)\max\limits_{0<m\leq z\leq M}|k''(z)|$ and the structure condition is satisfied.

We note that both the power-law pressure-density relationship, as well as $c^2(\rho) = e^\rho$ (studied in [26]) satisfy assumption **A2**. Therefore, we have the following result.

Theorem 3.1 *Let $\Omega \subset \mathbf{R}^2$ be a bounded domain which satisfies the uniform exterior cone condition and is such that $\partial\Omega$ does not contain the origin. Furthermore, let $c^2 \in C^2(\Omega)$ be an increasing function of ρ. Then the degenerate Dirichlet problem for the nonlinear wave equation in self-similar coordinates*

$$\begin{gathered} \left((c^2(\rho)-\xi^2)\rho_\xi - \xi\eta\rho_\eta\right)_\xi + \left((c^2(\rho)-\eta^2)\rho_\eta - \xi\eta\rho_\xi\right)_\eta + \xi\rho_\xi + \eta\rho_\eta = 0, \\ c^2(\rho)|_{\partial\Omega} = \xi^2 + \eta^2, \end{gathered} \tag{3.4}$$

has a solution $\rho \in C^2(\Omega)$ which satisfies the boundary condition $c^2(\rho) = g(\xi,\eta) = \xi^2+\eta^2$ in the sense $(c^2(\rho)-g)^{3/2} \in W_0^{1,2}$.

In the next section we show that if the domain is convex, the solution is continuous up to the degenerate boundary and hence satisfies the boundary condition in the classical sense.

3.2 Continuity up to the Boundary

In addition to the assumptions **A1** and **A2** listed at the beginning of Section 3 we assume the following.

A3. The domain Ω is convex.

A4. The function k is either $k(u) = u^{1/(\gamma-1)}$ with $\gamma > 1$ or $k(u) = \log u$.

The second assumption corresponds to $c^2(\rho) = \rho^{\gamma-1}$ or $c^2(\rho) = e^\rho$, respectively. Recall that solutions of the degenerate elliptic equation for $c^2(\rho) = e^\rho$ have been studied in [26]. We note that continuity up to the boundary was not obtained in [26].

We prove continuity up to the boundary by constructing an upper barrier function $\Psi^{\mathbf{x}} \in C^2(\Omega) \cap C(\overline{\Omega})$ for each point $\mathbf{x}$ on the boundary. This barrier function will satisfy $\Psi^{\mathbf{x}} > 0$ in Ω and $\Psi^{\mathbf{x}}(\mathbf{x}) = 0$ for $\mathbf{x} \in \partial\Omega$. Since u is squeezed between two continuous functions $g \geq 0$ and $g + \Psi^{\mathbf{x}}$ which are both equal to g at $\mathbf{x} \in \partial\Omega$, u must be continuous up to the point $\mathbf{x} \in \partial\Omega$ and hence satisfies the boundary condition $u(\mathbf{x}) = g(\mathbf{x})$ in the classical sense.

We construct an upper barrier by first considering the regularized problem $Q^\varepsilon(u) = -f$. For each fixed $\mathbf{x} \in \partial\Omega$ we obtain an upper barrier $\Psi^{\mathbf{x}}$ which is independent of ε and which satisfies $\Psi^{\mathbf{x}} \geq w^\varepsilon$ for each ε, where $w^\varepsilon \equiv u^\varepsilon - g$ is defined for each solution u^ε of the regularized problem. Then we let $\varepsilon \to 0$ to obtain a barrier which satisfies $\Psi^{\mathbf{x}} \geq u - g$ in Ω and $\Psi^{\mathbf{x}}(\mathbf{x}) = 0$. By repeating the construction of $\Psi^{\mathbf{x}}$ for each $\mathbf{x} \in \partial\Omega$ we obtain continuity of u for all $x \in \partial\Omega$.

We write the regularization of the operator (3.1) in non-divergence form

$$\begin{aligned} Q^\varepsilon(u) &\equiv (u - x^2 + \varepsilon)u_{xx} - 2xyu_{xy} + (u - y^2 + \varepsilon)u_{yy} \\ &+ a_1(u)(xu_x + yu_y)^2 + a_2(u_x^2 + u_y^2) - 2(xu_x + yu_y), \end{aligned} \tag{3.5}$$

where

$$a_1(u) = -\frac{k''(u)}{k'(u)} = \frac{\hat{a}}{u}, \quad \hat{a} \equiv \begin{cases} (\gamma-2)/(\gamma-1) & \text{if} \quad k(u) = u^{1/(\gamma-1)},\ \gamma > 1 \\ 1 & \text{if} \quad k(u) = \log u \end{cases}, \quad \text{and}$$

$$a_2 \equiv \begin{cases} 1/(\gamma-1) & \text{if} \quad k(u) = u^{1/(\gamma-1)},\ \gamma > 1 \\ 0 & \text{if} \quad k(u) = \log u \end{cases}.$$

Lemma 3.2 *For each $\mathbf{x} \in \partial\Omega$ there exists an upper barrier function $\Psi^{\mathbf{x}} \in C^2(\Omega) \cap C(\overline{\Omega})$ for the regularized problem $Q^\varepsilon(u) = 0$ in Ω, $u = g = x^2 + y^2$ on $\partial\Omega$, such that $\Psi^{\mathbf{x}} \geq u^\varepsilon - g$ in Ω and $\Psi^{\mathbf{x}}(\mathbf{x}) = 0$.*

Proof: We first rewrite the problem in terms of $w^\epsilon = u^\varepsilon - g$ and drop the ε superscript in w and u to simplify notation. The corresponding homogeneous Dirichlet problem reads

$$Q^\varepsilon(w) = -F^\varepsilon(\mathbf{x}, w) \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega, \tag{3.6}$$

where

$$\begin{aligned} Q^\varepsilon(w) &\equiv (w + y^2 + \varepsilon)w_{xx} - 2xyw_{xy} + (w + x^2 + \varepsilon)w_{yy} \\ &+ a_1(w+g)(xw_x + yw_y + 2g)^2 + a_2(w_x^2 + w_y^2) \\ &+ 2\,(2a_2 - 1)\,(xw_x + yw_y) \end{aligned} \tag{3.7}$$

and

$$F^{\varepsilon}(\mathbf{x}, w) \equiv 2w + 4a_2 g + 2(w-g) + 4\varepsilon. \tag{3.8}$$

Notice that since $m \leq u \leq M$ and g is bounded and strictly positive, the source term $F^{\varepsilon}(\mathbf{x}, w)$ is bounded uniformly in ε. Therefore, there exist constants, F_1 and $F_2 > 0$, depending only on m, M and Ω, such that

$$F_1 \leq F^{\varepsilon}(\mathbf{x}, w) \leq F_2.$$

We now construct an upper barrier for the homogeneous Dirichlet problem (3.6). We take a similar approach to that presented in [12, 13]. Fix a point $\mathbf{x}_0 \equiv (x_0, y_0)$ on the boundary of Ω. Since Ω is convex there is a tangent line $T_{\mathbf{x}_0}$ passing through $\mathbf{x}_0$ with Ω lying on one side of $T_{\mathbf{x}_0}$. We can also find $T_{\mathbf{x}_0'}$ parallel to $T_{\mathbf{x}_0}$ such that Ω is contained in the semi-infinite strip enclosed by $T_{\mathbf{x}_0}$ and $T_{\mathbf{x}_0'}$. Take orthogonal coordinate axes x_1 and y_1 so that the x_1-axis is perpendicular to $T_{\mathbf{x}_0}$. Let β be the angle that the x_1-axis makes with the line $y = y_0$. Then $x_1 = x\cos\beta + y\sin\beta$. Let $\psi > 0$ be the first eigenfunction of the operator $\partial^2_{x_1 x_1}$; that is, ψ satisfies $\psi_{x_1x_1} + \lambda_1\psi = 0$, and $\psi \mid_{T_{\mathbf{x}_0}} = \psi \mid_{T_{\mathbf{x}_0'}} = 0$, with $\lambda_1 > 0$. Now $\psi \in C^2(\overline{\Omega})$, $\psi > 0$, $\psi_{x_1x_1} \leq 0$ in Ω, and $\psi_{x_1} > 0$ at $\mathbf{x} \in \partial\Omega$.

Define $\Psi^{\mathbf{x}_0} \equiv K\psi^b$. We show that there exist constants $K > 0$ and $0 < b < 1$ independent of ε such that $\Psi^{\mathbf{x}_0}$ is an upper solution of (3.6) satisfying $\Psi^{\mathbf{x}_0} \geq u^{\varepsilon} - g$ and $\Psi^{\mathbf{x}_0}(\mathbf{x}_0) = 0$. We for now consider the case when $\hat{a} \geq 0$, that is $k(u)^{1/(\gamma-1)}$ with $\gamma \geq 2$ and $k(u) = \log u$, and later discuss the other case $1 < \gamma < 2$ at the end of the proof. We prove this in two steps. First we show that there exist $K > 0$ and $0 < b < 1$ such that $Q^{\varepsilon}(\Psi^{\mathbf{x}_0}) + F_2 < 0$ in Ω. Notice that since the operator Q^{ε} is quasilinear, this inequality is insufficient to conclude that $\Psi^{\mathbf{x}_0}$ is an upper barrier. In the second step we show that for such a choice of K and b we have $\Psi^{\mathbf{x}_0} \geq w^{\varepsilon}$ in Ω. Hence, since $\Psi^{\mathbf{x}_0}(\mathbf{x}_0) = w^{\varepsilon}(\mathbf{x}_0) = 0$ on $\partial\Omega$, we conclude that $\Psi^{\mathbf{x}_0}$ is an upper barrier at $\mathbf{x}_0$. For notational convenience, in the calculation that follows, we drop the superscript $\mathbf{x}_0$ in $\Psi^{\mathbf{x}_0}$.

We now show that there exist a $K > 0$ and a $0 < b < 1$ such that $Q^{\varepsilon}(\Psi) + F \leq 0$ where $F = F_2 + 1 > 0$. First notice that $\Psi_{x_1} = bK\psi^{b-1}\psi_{x_1}$ blows up at $\mathbf{x}_0 \in \partial\Omega$ at the rate $b-1$. The constant b will be determined below in such a way that it dominates the blow up of the solution of the degenerate problem at each point on the boundary. Using the fact that $\Psi_{x_1x_1} = -\lambda_1 bK\psi^b + b(b-1)K\psi^{b-2}\psi^2_{x_1} \leq 0$, we have

$$\begin{aligned}
Q^{\varepsilon}(\Psi) + F = {} & (\Psi + (x\sin\beta - y\cos\beta)^2 + \varepsilon)\Psi_{x_1x_1} + a_2\Psi^2_{x_1} \\
& + a_1(\Psi + g)(x_1\Psi_{x_1} + 2g)^2 + 2(2a_2 - 1)x_1\Psi_{x_1} + F \\
\leq {} & \Psi\Psi_{x_1x_1} + a_2\Psi^2_{x_1} + a_1(\Psi + g)(x_1\Psi_{x_1} + 2g)^2 + 2(2a_2-1)x_1\Psi_{x_1} + F.
\end{aligned}$$

By replacing the derivatives of Ψ, the above inequality becomes

$$\begin{aligned}
Q^\varepsilon(\Psi)+F &\leq K\psi^b(-\lambda_1 bK\psi^b + b(b-1)K\psi^{b-2}\psi_{x_1}^2) + a_2(bK\psi^{b-1}\psi_{x_1})^2 \\
&+ a_1(\Psi+g)(x_1 bK\psi^{b-1}\psi_{x_1}+2g)^2 + 2(2a_2-1)(x_1 bK\psi^{b-1}\psi_{x_1}) + F \\
&= -\lambda_1 bK^2\psi^{2b} + b((1+a_2)b-1)K^2\psi^{2(b-1)}\psi_{x_1}^2 \\
&+ a_1(\Psi+g)x_1^2 b^2 K^2\psi^{2(b-1)}\psi_{x_1}^2 + 4ga_1(\Psi+g)bK\psi^{b-1}x_1\psi_{x_1} \\
&+ 4g^2 a_1(\Psi+g) + 2(2a_2-1)x_1 bK\psi^{b-1}\psi_{x_1} + F.
\end{aligned}$$

We now show that the right hand-side is non-positive by first considering its behavior in the neighborhood Ω_δ of $\mathbf{x}_0 = (x_0, y_0) \in \partial\Omega$, where $\Omega_\delta \equiv \{(x,y) \in \overline{\Omega} : \operatorname{dist}((x,y),(x_0,y_0)) < \delta\}$, and then by estimating the right hand-side in the complement $\overline{\Omega}\setminus\Omega_\delta$.

The following observations will be useful. By assumption **A1** in Section 3 there exists an $m > 0$ such that $g = x^2 + y^2 \geq m$, for all $\mathbf{x} \in \partial\Omega$. Since g is continuous, there exists a $\delta_1 > 0$ such that $g \geq m/2$ for all $\mathbf{x} \in \overline{\Omega}_{\delta_1}$ and

$$a_1(\Psi+g) = \hat{a}/(\Psi+g) \leq 2\hat{a}/m \quad \text{in} \quad \overline{\Omega}_{\delta_1}.$$

Furthermore, since $\Psi_{x_1} > 0$ at $\mathbf{x}_0 = (x_0, y_0) \in \partial\Omega$, there exists a $\delta_2 > 0$ such that for a given $m_1 > \varepsilon$, $\psi_{x_1} \geq m_1 > 0, \quad \forall \mathbf{x} \in \overline{\Omega}_{\delta_2}$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Then for $\mathbf{x} \in \overline{\Omega}_\delta$ we can estimate the right hand-side of $Q^\varepsilon(\Psi)+F$ by using the upper bound on $a_1(\Psi+g)$ as follows

$$\begin{aligned}
Q^\varepsilon(\Psi)+F &\leq -\lambda_1 bK^2\psi^{2b} + b((1+a_2)b-1)K^2\psi^{2(b-1)}\psi_{x_1}^2 \\
&+ \frac{2\hat{a}}{m}b^2K^2\psi^{2(b-1)}x_1^2\psi_{x_1}^2 + 8g\frac{\hat{a}}{m}Kb\psi^{b-1}x_1\psi_{x_1} \\
&+ 8g^2\frac{\hat{a}}{m} + 2(a_2-2)x_1 bK\psi^{b-1}\psi_{x_1} + F.
\end{aligned}$$

After estimating the first term from above by zero, and after combining the second and the third term, we obtain

$$\begin{aligned}
Q^\varepsilon(\Psi)+F &\leq b\left((1+a_2+(2\hat{a}/m)x_1^2)b-1\right)K^2\psi^{2(b-1)}\psi_{x_1}^2 \\
&+ 8g\frac{\hat{a}}{m}Kb\psi^{b-1}x_1\psi_{x_1} + 8g^2\frac{\hat{a}}{m} + 2(2a_2-1)x_1 bK\psi^{b-1}\psi_{x_1} + F.
\end{aligned}$$

Since Ω is bounded, there exists an $H > 0$ such that $(2\hat{a}/m)x_1^2 \leq H$. Now choose a b^* so that

$$0 < b^* < 1/(1+a_2+H) < 1.$$

Then, since $\psi_{x_1}^2 > 0$ in Ω_δ, we have $b^*((1+a_2+H)b^*-1)K^2\psi^{2(b^*-1)}\psi_{x_1}^2 < 0$. Next we can choose $K = K_1$ big enough so that this negative term dominates other (positive) terms. Thus, for $b = b^*$ there exists a $K_1 > 1$ such that for all $\mathbf{x} \in \overline{\Omega}_\delta$ and for any $K \geq K_1$ we have

$$
\begin{aligned}
Q^\varepsilon(\Psi) + F &\leq b^*\left((1 + a_2 + H)b^* - 1\right) K^2 \psi^{2(b^*-1)} \psi_{x_1}^2 + 8g\frac{\hat{a}}{m} K b^* \psi^{b^*-1} x_1 \psi_{x_1} \\
&+ 8g^2\frac{\hat{a}}{m} + 2(2a_2 - 1)x_1 b^* K \psi^{b^*-1} \psi_{x_1} + F \leq 0.
\end{aligned}
$$

Assume that $\mathbf{x} \in \overline{\Omega} \setminus \Omega_\delta$. Since $g \geq 0$, we have $a_1(\Psi + g) = \hat{a}/(\Psi + g) \leq \hat{a}/\Psi$. Furthermore, since $\min_{\overline{\Omega}\setminus\Omega_\delta} \psi > 0$ and $0 < b^* < 1/(1 + a_2 + H) < 1/(1 + a_2) < 1$, the following estimate holds for all $\mathbf{x} \in \overline{\Omega} \setminus \Omega_\delta$

$$
\begin{aligned}
Q^\varepsilon(\Psi) + F &\leq -\lambda_1 b^* K^2 \psi^{2b^*} + b^*((1 + a_2)b^* - 1)K^2 \psi^{2(b^*-1)} \psi_{x_1}^2 \\
&+ \hat{a} x_1^2 b^{*2} K \psi^{b^*-2} \psi_{x_1}^2 + 4g\hat{a}b^* \psi^{-1} x_1 \psi_{x_1} + 4g^2 \hat{a} K^{-1} \psi^{-b^*} \\
&+ 2(2a_2 - 1)x_1 b^* K \psi^{b^*-1} \psi_{x_1} + F \\
&\leq -\lambda_1 b^* K^2 \psi^{2b^*} \\
&+ \hat{a} x_1^2 b^{*2} K \psi^{b^*-2} \psi_{x_1}^2 + 4g\hat{a}b^* \psi^{-1} x_1 \psi_{x_1} + 4g^2 \hat{a} K^{-1} \psi^{-b^*} \\
&+ 2(2a_2 - 1)x_1 b^* K \psi^{b^*-1} \psi_{x_1} + F.
\end{aligned}
$$

Therefore, there exists a $K_2 > 0$ such that for all $\mathbf{x} \in \overline{\Omega} \setminus \Omega_\delta$ the right hand-side is non-positive whenever $K \geq K_2$. Let $K^* = \max\{K_1, K_2\}$. Define $\Psi = K^* \psi^{b^*}$. Then for all $\mathbf{x} \in \overline{\Omega}$, $Q^\varepsilon(\Psi) + F \leq 0$.

We now prove that this Ψ is an upper barrier function for the solution w^ε of (3.6) by showing that $\Psi > w^\varepsilon$ for all $\mathbf{x} \in \Omega$. Recall that at the beginning of this proof we dropped the superscript ε in the notation for w^ε. This will be continued in the rest of the proof.

Suppose $\Psi - w \leq 0$ at some interior point and let $\mathbf{x}_m \equiv (x_m, y_m)$ be an interior local minimum. Then, at such a point (x_m, y_m) the following holds:

$$
\begin{gathered}
\Psi - w \leq 0, \quad (w - \Psi)_x = (w - \Psi)_y = 0 \quad \text{and} \\
(w + y^2 + \varepsilon)(\Psi - w)_{xx} - 2xy(\Psi - w)_{xy} + (w + x^2 + \varepsilon)(\Psi - w)_{yy} \geq 0.
\end{gathered}
$$

Since $Q^\varepsilon(\Psi) + F \leq 0$ and $Q^\varepsilon(w) + F^\varepsilon(\mathbf{x}, w) = 0$ we have

$$
\begin{aligned}
0 &\geq Q^\varepsilon(\Psi) + F - Q^\varepsilon(w) - F^\varepsilon(\mathbf{x}, w) \\
&= (w + y^2 + \varepsilon)(\Psi - w)_{xx} - 2xy(\Psi - w)_{xy} + (w + x^2 + \varepsilon)(\Psi - w)_{yy} \\
&+ a_2(\Psi_x + \Psi_y)^2 - a_2(w_x + w_y)^2 \\
&+ a_1(\Psi + g)(x\Psi_x + y\Psi_y + 2g)^2 - a_1(w + g)(xw_x + yw_y + 2g)^2 \\
&+ 2(2a_2 - 1)(x\Psi_x + y\Psi_y) - 2(2a_2 - 1)(xw_x + yw_y) \\
&+ (\Psi - w)\Psi_{xx} + (\Psi - w)\Psi_{yy} + F - F^\varepsilon(\mathbf{x}, w).
\end{aligned}
$$

We know that at (x_m, y_m)

$$\begin{aligned}(\Psi - w)\Psi_{xx} = (\Psi - w)\Psi_{x_1x_1}\cos^2\beta &\geq 0,\\ (\Psi - w)\Psi_{yy} = (\Psi - w)\Psi_{x_1x_1}\sin^2\beta &\geq 0,\end{aligned}$$

and

$$F - F^\varepsilon(\mathbf{x}, w) > 0. \tag{3.9}$$

Moreover, at (x_m, y_m)

$$\begin{aligned}&a_1(\Psi + g)(x\Psi_x + y\Psi_y + 2g)^2 - a_1(w+g)(xw_x + yw_y + 2g)^2\\ &\qquad = \{a_1(\Psi + g) - a_1(w+g)\}(x\Psi_x + y\Psi_y + 2g)^2 \geq 0\end{aligned}$$

since $a_1(u)$ is a decreasing function of u (hence $a_1(w+g) \leq a_1(\Psi + g)$ at the point at which $\Psi \leq w$). Furthermore, at (x_m, y_m) we have

$$\begin{aligned}a_2(\Psi_x + \Psi_y)^2 &= a_2(w_x + w_y)^2,\\ 2(2a_2 - 1)(x\Psi_x + y\Psi_y) &= 2(2a_2 - 1)(xw_x + yw_y).\end{aligned}$$

Thus at (x_m, y_m) we obtain

$$\begin{aligned}0 &\geq Q^\varepsilon(\Psi) + F - Q^\varepsilon(w) - F^\varepsilon(\mathbf{x}, w)\\ &> (w + y^2 + \varepsilon)(\Psi - w)_{xx} - 2xy(\Psi - w)_{xy} + (w + x^2 + \varepsilon)(\Psi - w)_{yy}\end{aligned}$$

which contradicts the assumption that (x_m, y_m) is a point of the local minimum. (Notice that strict inequality in this estimate is a consequence of (3.9).) Therefore, $\Psi^{\mathbf{x}_0} = K^*\psi^{b^*}$ is an upper barrier at $\mathbf{x}_0$ for $w^\varepsilon = u^\varepsilon - g$, for each $\varepsilon > 0$.

We now consider the case when $1 < \gamma < 2$. In this case since $a_1(u)$ becomes negative we can modify the operator Q^ε such that define ${Q^\varepsilon}_1(w) \equiv Q^\varepsilon(w) - a_1(w + g)(xw_x + yw_y + 2g)^2$ and show ${Q^\varepsilon}_1(\Psi^{\mathbf{x}}) + F < 0$ where $\Psi^{\mathbf{x}} = K\psi^b$ for some constants $K > 1$ and $0 < b < 1$ as we did when $\gamma \geq 2$. Thus we get

$$\begin{aligned}0 &> {Q^\varepsilon}_1(\Psi) + F - Q^\varepsilon(w) - F^\varepsilon(\mathbf{x}, w)\\ &\geq (w + y^2 + \varepsilon)(\Psi - w)_{xx} - 2xy(\Psi - w)_{xy} + (w + x^2 + \varepsilon)(\Psi - w)_{yy}\\ &+ a_2(\Psi_x + \Psi_y)^2 - a_2(w_x + w_y)^2\\ &+ 2(2a_2 - 1)(x\Psi_x + y\Psi_y) - 2(2a_2 - 1)(xw_x + yw_y)\\ &+ (\Psi - w)\Psi_{xx} + (\Psi - w)\Psi_{yy} + F - F^\varepsilon(\mathbf{x}, w),\end{aligned}$$

and hence apply the contradiction argument as we did in the case when $\gamma \geq 2$ to get the result. This completes the proof. ■

Since $\Psi^{\mathbf{x}} \geq u^{\varepsilon} - g$ in Ω for each $\varepsilon > 0$, it follows that $\Psi^{\mathbf{x}}$ is an upper barrier function for $u - g$. Therefore, we have shown that for each $\mathbf{x} \in \partial\Omega$ there exists a $C(\overline{\Omega})$-function $\Psi^{\mathbf{x}}$ such that $g < u \leq g + \Psi^{\mathbf{x}}$ in Ω, and $u(\mathbf{x}) = g(\mathbf{x})$ on $\partial\Omega$, which implies that u is continuous up to the boundary. This proves the following corollary of Lemma 3.2.

Corollary 3.3 *Suppose $c^2(\rho) = \rho^{\gamma-1}$ for some $\gamma > 1$, or $c^2(\rho) = e^{\rho}$. Then a $C^2(\Omega)$ solution ρ of the degenerate elliptic boundary-value problem* (3.4) *is continuous up to the degenerate boundary.*

In this section we have shown the existence of a $C^2(\Omega) \cap C(\overline{\Omega})$–solution for a class of quasilinear wave equations on convex domains. Theorem 3.1 implies that a solution of the quasilinear wave equation can be singular at the degenerate boundary in the sense that the derivative of solution blows up at a rate which is bounded by the degree of the degeneracy a, defined in Theorem 2.7. In particular, the solution $\rho = \rho(\xi, \eta)$ of the shallow water equation ($c^2(\rho) = \rho$), satisfies $(\rho - g)^{3/2} \in W_0^{1,2}$ which means that the solution can have the square-root singularity at the degenerate boundary.

4 The UTSD equation

We study solutions of equation (1.11) with Dirichlet boundary data

$$Q(w) \equiv (ww_x)_x + w_{yy} - \frac{3}{2}w_x = -\frac{1}{2}, \quad w\mid_{\partial\Omega} = g. \tag{4.1}$$

Linearized around a constant solution w_0 this equation is elliptic whenever $w_0 > 0$, degenerate when $w_0 = 0$, and hyperbolic otherwise. Motivated by the study of weak shock reflection and 2-D Riemann problems ([5], [10], [1, 2]) we focus our attention on the boundary value problems which have degenerate data $w = 0$ along the portion of the boundary $x = 0$, which we denote by Σ, and positive (nondegenerate) data along the rest of the boundary, contained in the half-plane $x > 0$, denoted by Γ. (This is consistent with the asymptotic analysis presented in [7].) One such domain is shown in Figure 4.1. We assume that $g \in W^{1,2}(\Omega) \cap C^{0,\beta}(\overline{\Omega})$ where $0 < \beta < 1$, $g = 0$ on Σ, and $g > 0$ on Γ, where $\partial\Omega = \Sigma \cup \Gamma$ satisfies the uniform exterior cone condition.

4.1 Existence Result

Existence of a solution to this Dirichlet problem is a consequence of Theorem 2.7. It is easy to see that hypotheses **H1, H3** and **H4** hold. Hypothesis **H2** is satisfied if

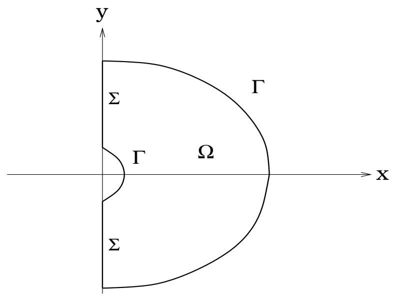

FIGURE 4.1: The figure shows a typical domain of interest arising in the study of self-similar solutions in weak shock reflection, modeled by the UTSD equation. The degenerate part of the boundary Σ corresponds to the sonic line.

we work with the modified operator

$$\widetilde{Q}(w) = (\tilde{w} w_x)_x + w_{yy} - \frac{3}{2} w_x, \quad \text{where} \quad \tilde{w}(x,y,w) = \begin{cases} w & \text{if} \quad w \geq 0 \\ 0 & \text{if} \quad w < 0. \end{cases}$$

We will show that the solution to the modified problem is non-negative, and so it satisfies the original problem.

The source term in this problem $f = 1/2 \neq 0$ and so we need to verify that condition **F**, from Section 2.1, holds. Since the minimum eigenvalue is given by $\lambda(\mathbf{x}, w) = w$, condition **F** is satisfied if we choose, for example, $\lambda_0 = 1$. Next we verify that the structure conditions **S1** and **S2** from Section 2.1 hold. Condition **S1** is satisfied with the function $G = 0$. This function is also a good choice for the lower barrier G_1 since it satisfies all the conditions listed in structure condition **S2**. Inequality (2.4) is trivially satisfied with $K = 1/2$ and $S(\mathbf{x}) = 0$.

THEOREM 4.1 *Suppose that $g \in W^{1,2}(\Omega) \cap C^{0,\beta}(\overline{\Omega})$, $0 < \beta < 1$, and let $g = 0$ on Σ and $g \geq 0$ on $\partial\Omega$. Then there exists a solution $w \in C^2(\Omega)$ of the problem (4.1). Furthermore, the solution satisfies $(w-g)^{3/2} \in W_0^{1,2}(\Omega)$.*

The proof of this theorem follows by applying Theorem 2.7 to the modified problem, and by using the lower barrier $G_1 = 0$ to conclude that the solution w in non-negative in $\overline{\Omega}$ and therefore it satisfies the original problem.

4.2 Continuity up to the boundary

To show that w is continuous up to the boundary we proceed in two steps. First, continuity up to the boundary Γ is a direct consequence of the standard local Hölder estimate. Namely, Theorem 8.29 in [16] implies that there exists an α, $0 < \alpha \leq \beta < 1$, such that for $\Omega' \subset\subset \Omega \cup \Gamma$, $\|w_\varepsilon\|_{C^\alpha(\Omega')} \leq C$, where C depends on the ellipticity ratio, $|w^\varepsilon|_\infty$, Ω', the uniform exterior cone condition, and β.

In the second step we show that w is continuous up to the degenerate boundary. We prove this by squeezing w between the lower barrier $G = 0$ and a continuous upper barrier which is equal to zero at the degenerate boundary. There are several choices for the upper barrier. As in Section 3.2, an upper barrier $\Psi^{\mathbf{x}} = K\psi^{b^*}$ can be constructed at each point $\mathbf{x} \in \Sigma$ using the positive eigenfunction ψ of the operator ∂^2_{xx}. A second choice for an upper barrier is based on the fact that the function $x + \sqrt{x}$ is a solution of the PDE in (1.11). Thus, we look for an upper barrier of the form $\Psi(\mathbf{x}) \equiv x + C_1\sqrt{x}$ where the constant C_1 (independent of ε) is such that $Q^\varepsilon(\Psi) + 1/2 \leq 0$ and $\Psi \geq g$. Using this inequality we now show that $\Psi - w^\epsilon \geq 0$ for each fixed $\varepsilon > 0$. Suppose that this is not true. Then let $\mathbf{x}_0 \in \Omega$ be such that $\Psi(\mathbf{x}_0) - w^\epsilon(\mathbf{x}_0) < 0$ and suppose that $\mathbf{x}_0$ is a point of local minimum. Then the following inequality holds:

$$
\begin{aligned}
0 \geq Q^\varepsilon(\Psi) - Q^\varepsilon(w^\epsilon) = (w^\epsilon + \varepsilon)(\Psi - w^\epsilon)_{xx} + (\Psi - w^\epsilon)_{yy} \\
+ \left(w^\epsilon_x + \Psi_x - \frac{3}{2}\right)(\Psi - w^\epsilon)_x + \Psi_{xx}(\Psi - w^\epsilon).
\end{aligned}
$$

Since at a local minimum $\mathbf{x}_0$ we have $\Psi_{xx}(\mathbf{x}_0) < 0$ and $(\Psi - w^\epsilon)_x(\mathbf{x}_0) = 0$, the above inequality, evaluated at $\mathbf{x}_0$, becomes $0 > (w^\epsilon + \Psi)(\Psi - w^\epsilon)_{xx} + (\Psi - w^\epsilon)_{yy}$. This is in contradiction with the convexity condition that holds at a local minimum $\mathbf{x}_0$. Thus, we have $\Psi \geq w^\epsilon$ for each fixed $\varepsilon > 0$. Therefore $\Psi \geq w > 0$ and Ψ is an upper barrier function for w.

Theorem 4.2 *Suppose that $\partial\Omega$ satisfies the exterior cone condition. Then the $C^2(\Omega)$ solution found in Theorem 4.1 is continuous up to the boundary, i.e., $u \in C^2(\Omega) \cap C(\overline{\Omega})$.*

Remark 4.3 Existence of a solution of (4.1) where the boundary data exhibit the square-root singularity at the degenerate boundary was proved in [3]. In this case it was shown that the solution must have the same type of singularity at Σ.

Remark 4.4 In contrast with the results in [3] and [4] we can handle data which is not of fixed sign. Furthermore, singular solutions (square-root) are permitted even if the data do not exhibit singular behavior. On the other hand, we cannot obtain higher smoothness (C^1) up to the degenerate boundary as was obtained in [4].

Remark 4.5 Existence and continuity up to the boundary for (4.1) can be obtained using Choi, Laser and McKenna's techniques presented in [12, 13]. Their approach can be used because the forcing term $f(x, y) = \frac{1}{2}$ is strictly positive, and because the degenerate boundary Σ is convex.

5 Acknowledgments

The authors would like to thank Gary Lieberman for reading the manuscript and for making several useful suggestions. Lieberman remarked to us that our Theorem 2.1 holds even in the case when the forcing term is different from zero, after assuming condition **F**. He also pointed out to us that condition (2.9) is similar to a local barrier condition for linear problems, discussed in Section 6.6 of Gilbarg and Trudinger's [16]. Finally, we thank Lieberman for bringing to our attention reference [20].

The authors would like to thank Barbara L. Keyfitz for sharing her derivation of the nonlinear wave equation from the compressible Euler equations, for her careful reading of the manuscript and for making several suggestions which improved the exposition of the work. We also thank Barbara for her support and encouragement in writing this manuscript.

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