A Proof of Existence of Perturbed Steady Transonic Shocks via a Free Boundary Problem

Sunčica Čanić *　　Barbara Lee Keyfitz †
Gary M. Lieberman ‡

December 14, 1998

ABSTRACT. We prove the existence of a solution of a free boundary problem for the transonic small disturbance equation. The free boundary is the position of a transonic shock dividing two regions of smooth flow. Assuming inviscid, irrotational flow, as modeled by the transonic small disturbance equation, the equation is hyperbolic upstream where the flow is supersonic, and elliptic in the downstream subsonic region. To study the stability of a uniform planar transonic shock, we consider perturbation by a steady $C^{1+\epsilon}$ upstream disturbance. If the upstream disturbance is small in a C^1 sense, then there is a steady solution in which the downstream flow and the transonic shock are Hölder continuous perturbations of the uniform configuration. This result provides a new use of inviscid perturbation techniques to demonstrate, in two dimensions, the stability of transonic shock waves of the type that appear, for example, over the wing of an airplane, along an airfoil, or as bow shocks in a flow with a supersonic free-stream velocity.

KEY WORDS. Transonic small disturbance equation, free boundary problem, oblique derivative problem, shock stability.

AMS SUBJECT CLASSIFICATION. Primary: 35R35; Secondary: 35J65, 35L65, 75H05

CONTENTS

* Department of Mathematics, Iowa State University, Ames, Iowa 50011. Research supported by the National Science Foundation, grant DMS-9625831.

† Department of Mathematics, University of Houston, Houston, Texas 77204-3476. Research supported by the Department of Energy, grant DE-FG-03-94-ER25222. Support from the Energy Laboratory at the University of Houston is also acknowledged.

‡ Department of Mathematics, Iowa State University, Ames, Iowa 50011.

1 INTRODUCTION

This paper uses tools of quasilinear elliptic equations to solve a theoretical problem in compressible fluid dynamics. Estimates on solutions of oblique derivative and mixed boundary problems are used to solve an interesting free boundary problem.

The transonic small disturbance (TSD) equation is a simplified model for steady potential flow. It is the first term of an asymptotic expansion valid for flows around slender bodies at free-stream speeds close to sonic speed. The perturbation potential, for the deviation of the flow from sonic speed, satisfies

$$\phi_x\phi_{xx} + \phi_{yy} = 0\,,$$

which can be written as a system of conservation laws for $(u, v) = \nabla\phi$:

$$\begin{aligned} (u^2/2)_x + v_y &= 0\,, \\ v_x - u_y &= 0\,. \end{aligned}$$

This is the system we consider in this paper. The equations are hyperbolic if $u < 0$, corresponding to supersonic velocities in the flow, and subsonic (elliptic) if $u > 0$; states with $u = 0$ correspond to sonic flow.

A smooth shock is a curve, $x = \xi(y)$, dividing two regions in which $U = (u, v)$ is a classical solution, while across the shock the Rankine-Hugoniot conditions hold, so that the entire flow forms a weak solution. In the upstream region the flow vector points toward the shock; denoting the upstream flow by U_∞ and the downstream flow by U_1, the Rankine-Hugoniot conditions, for the TSD equation, are

$$\xi' = \frac{1}{2}\,\frac{u_\infty^2 - u_1^2}{v_\infty - v_1} = -\frac{v_\infty - v_1}{u_\infty - u_1}\,.$$

In a *uniform* shock configuration, U_∞ and U_1 are constant and the shock angle $\xi' = s$ is also fixed. The roles of U_∞ and U_1 can be reversed for such a shock, yielding another weak solution, not physically realizable because it is not compressive. For the small disturbance equation, the condition for a physical transonic shock is $u_\infty < 0 < u_1$; this inequality plays the same role in steady transonic flows that the Lax entropy condition plays in time-dependent hyperbolic conservation laws: it is a convenient geometric condition for determining shock admissibility.

There are a number of ways of relating a geometric condition to physical and mathematical stability. The new result, which we establish in this paper, is determining the stability of transonic shocks by a steady perturbation of the upstream flow.

A further motivation for studying this problem arises in work by Čanić and Keyfitz [4] on self-similar solutions of two-dimensional conservation laws. Here, the quasi-steady flow on the downstream side of a shock may contain a subsonic region. Although the shock may be uniform where the downstream flow is supersonic, the transonic portion of the shock is nonuniform because of the influence of other boundary conditions. To handle the quasi-steady problem, we would need to extend the results of the present paper to include degenerate elliptic equations. Work on this problem is in progress.

In this paper, we establish the following theorem.

Theorem 1.1 *Let U_∞ and U_1 be two constant states, supersonic and subsonic respectively, separated by a uniform shock with speed $s \neq 0$. Then there exist positive constants a and b such that for any C^2-perturbation $U_0(x,y)$ of the constant state U_∞, satisfying*

$$|U_0(x,y) - U_\infty|_0 < a, \quad |DU_0(x,y)|_0 < b,$$

there exist a perturbed shock satisfying the Rankine-Hugoniot conditions, and a continuously differentiable solution (u,v) of the transonic small disturbance equation downstream from the shock, which remain close to the uniform solution in any bounded region downstream.

A more precise statement of the closeness condition is given in Theorem 2.1.

When the upstream flow is known, the downstream flow, augmented by the Rankine-Hugoniot conditions, becomes a free boundary problem. We solve the free boundary problem by a fixed point technique.

A comparison of our approach with other methods for solving free boundary problems is given in Section 5.1.

2 Description of the Problem

We study nonlinear stability of transonic shocks for the TSD equation

$$\begin{aligned} uu_x + v_y &= 0, \\ v_x - u_y &= 0. \end{aligned} \tag{2.1}$$

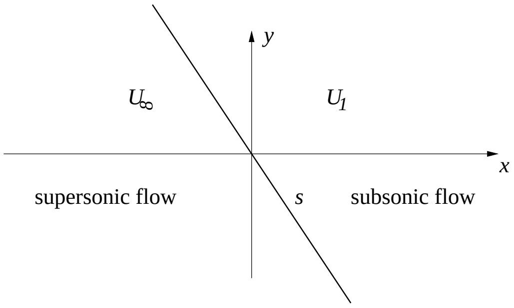

Figure 2.1: Unperturbed Shock

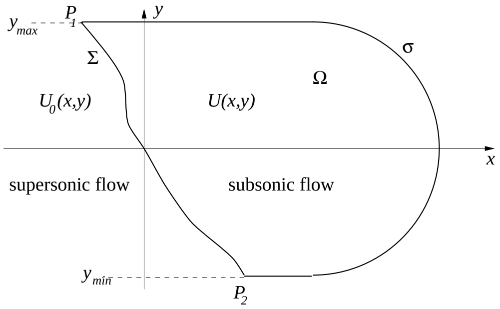

Figure 2.2: Perturbed Shock

The unperturbed configuration consists of a uniform shock separating two regions of uniform flow, satisfying the Rankine-Hugoniot relation. Denoting by U_∞ and U_1 the uniform states, and by s the uniform shock speed, the uniform solution is

$$\begin{aligned} U(x,y) &= U_\infty, \quad x < sy, \\ U(x,y) &= U_1, \quad x > sy. \end{aligned} \tag{2.2}$$

We cannot handle the case of a normal shock ($s = 0$) by the method of this paper. We assume, without further loss of generality, that $s < 0$ as shown in Figure 2.1. A shock is called *transonic* if the state on one side is supersonic ($u > 0$) and on the other subsonic ($u < 0$). When the upstream state U_∞ is supersonic and the downstream state U_1 is subsonic, this corresponds to a compressive, physically realizable shock. See Cole and Cook [5].

We denote by $x = \xi(y)$ the position of the perturbed shock. We consider an upstream perturbation of the constant state U_∞ into a steady, smooth, nonuniform supersonic flow $U_0(x,y)$. The function $x = \xi(y)$ and the downstream flow $U(x,y)$ solve a free boundary problem, which we formulate as follows. In the subsonic region, equation (2.1) is a quasilinear elliptic system. To use existing theory of oblique derivative problems for second-order elliptic equations, we eliminate v and replace (2.1) by

$$Q(u) \equiv (uu_x)_x + u_{yy} = 0\,. \tag{2.3}$$

To avoid technical difficulties which arise in solving elliptic problems in unbounded regions, we introduce a smooth and convex cutoff boundary, σ, as shown in Figure 2.2, which coincides with the lines $y = y_{\min}$ and $y = y_{\max}$ in a neighborhood of the unperturbed shock. Thus, we solve (2.3) in the domain Ω bounded by the free boundary, Σ, and the cutoff boundary, σ. The boundary data along Σ are determined by the Rankine-Hugoniot conditions (the notation $'$ means d/dy):

$$\xi' = -\frac{[v]}{[u]} = \frac{\overline{u}[u]}{[v]} \quad \text{on} \quad \Sigma, \tag{2.4}$$

where [ ] denotes the jump across the discontinuity, and $\overline{u}$ is the arithmetic mean of the first velocity component between the two states across the shock. Because of translation invariance, we fix one point on the free boundary by imposing the condition

$$\xi(0) = 0.$$

On the cutoff boundary σ we set $u(x, y) = u_1$, the first component of the unperturbed downstream flow. We eliminate v from the boundary condition by using a differentiated form of (2.4), and then we specify v at a single point, say at the corner P_1 with $y = y_{\max}$, by imposing (2.4) there. Therefore, we have

$$u = u_1 \text{ on } \sigma \quad \text{and} \quad v \text{ given at } P_1. \tag{2.5}$$

The main theorem of this paper is the following. It implies Theorem 1.1.

THEOREM 2.1 *Let U_∞ and U_1 be two constant states, supersonic and subsonic respectively, separated by a uniform shock with speed $s \neq 0$. There exist positive constants a and b such that if U_0 is a $C^{1+\delta}$ perturbation of the constant state U_∞ for some $\delta > 0$, satisfying*

$$|U_0 - U_\infty|_0 < a, \quad |DU_0|_0 < b,$$

then there exist exponents α_* and γ, $0 < \alpha_* < \min(\gamma, \delta)$, and a solution $(U, \Sigma) \in H^{(-\gamma)}_{1+\alpha_*} \times H_{1+\gamma}$ of the free boundary problem

$$\begin{array}{rcll} \left.\begin{array}{rcl} uu_x + v_y &=& 0 \\ v_x - u_y &=& 0 \end{array}\right\} & & \textit{in} & \Omega, \\ \xi' = -\dfrac{[v]}{[u]} = -\dfrac{\overline{u}[u]}{[v]} & & \textit{on} & \Sigma \equiv \{x = \xi(y)\}, \\ u = u_1 & & \textit{on} & \sigma, \\ v & & \textit{given at} & P_1, \\ \xi(0) = 0. & & & \end{array}$$

The function spaces are defined in Section 2.2.

We use the Rankine-Hugoniot conditions to define iterations for the position of the free boundary, after rewriting (2.4) in the following way. We obtain one equation by eliminating $[v]$ from (2.4):

$$\xi' = -\sqrt{-\overline{u}}\,, \tag{2.6}$$

choosing the negative root because the shock speed is negative. A second condition is obtained by eliminating ξ' from (2.4):

$$v = v_0 + (u - u_0)\sqrt{-\frac{u + u_0}{2}}. \tag{2.7}$$

We derive the boundary condition for (2.3) along Σ by eliminating v from (2.7) in such as way as to express the equation as an oblique derivative boundary condition for u on Σ. We differentiate v along Σ (where $' = \xi'\partial_x + \partial_y = d/dy$ along Σ):

$$v' = v_0' + (u' - u_0')\sqrt{-\frac{u+u_0}{2}} - (u-u_0)\frac{(u'+u_0')}{4\sqrt{-\frac{u+u_0}{2}}}.$$

We express u' as $u_x\xi' + u_y$ and use the differential equation (2.1) to write

$$v' = v_x\xi' + v_y = u_y\xi' - uu_x\,;$$

substituting this for v' and collecting terms in u_x and u_y, we obtain

$$N(x,y,u,Du) \equiv \beta^1(u;u_0,\xi')u_x + \beta^2(u;u_0,\xi')u_y = \psi(u;u_0,U_0'), \tag{2.8}$$

where

$$\begin{aligned}
\beta^1(u;u_0,\xi') &= u + \xi'\left(\sqrt{-\frac{u+u_0}{2}} - \frac{u-u_0}{4\sqrt{-\frac{u+u_0}{2}}}\right), \\
\beta^2(u;u_0,\xi') &= -\xi' + \sqrt{-\frac{u+u_0}{2}} - \frac{u-u_0}{4\sqrt{-\frac{u+u_0}{2}}}, \\
\psi(u;u_0,U_0') &= -v_0' + u_0'\left(\sqrt{-\frac{u+u_0}{2}} - \frac{u-u_0}{4\sqrt{-\frac{u+u_0}{2}}}\right).
\end{aligned} \tag{2.9}$$

Conditions (2.6) and (2.8) are the form of the Rankine-Hugoniot conditions that will be used in the rest of this paper. By satisfying one and the other alternately, we define an iterative procedure to solve the free boundary problem for the position of the transonic shock.

2.1 Bounds on the Size of the Perturbation

We allow perturbations $U_0(x,y)$ of the constant state U_∞ which are uniformly bounded in such a way that the partial differential equation (2.1) remains hyperbolic upstream. We choose the perturbation so that (2.3) is strictly elliptic downstream, and so that we obtain a compact nonlinear operator, which we will use to prove the existence of the perturbed shock. For this, we require that U_0 give rise to a solution U downstream in which the function u is positive and bounded away from zero, and such that $\sup u + \sup u_0 < 0$. In fact, we shall show that if U_0 is in a C^0 neighborhood (ball of radius a_0) of U_∞, then our approximate solution U is in a neighborhood of U_1. The following proposition tells us how small we must take the second neighborhood to be, and we verify, later in this paper, that we obtain this bound.

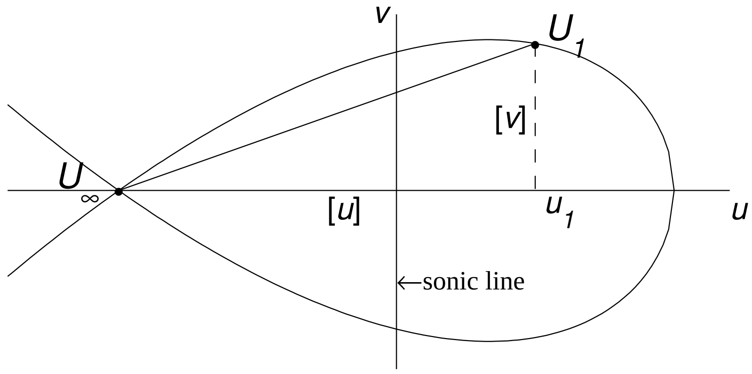

Figure 2.3: The Shock Polar

Proposition 2.2 *Let U_∞ and U_1 be given states with $0 < u_1 < -u_\infty$. Let a_0 be any number satisfying*

$$0 < a_0 < -(u_\infty + u_1)\,, \tag{2.10}$$

and let A satisfy

$$0 < A < \min\left(\frac{u_1}{2}, -\frac{1}{2}(u_\infty + u_1 + a_0)\right)\,. \tag{2.11}$$

Then for any perturbed solution of the problem which satisfies $U_0 \in B_{a_0}(U_\infty)$ upstream and $U \in B_{2A}(U_1)$ downstream, U_0 is a uniformly hyperbolic and U a uniformly elliptic state, and the perturbed shock angle is bounded by

$$S_1 \le \xi' \le S_2 < 0\,, \tag{2.12}$$

where $s^2 = -(u_\infty + u_1)/2$ and

$$S_1 = -\sqrt{s^2 + \left(\frac{a_0}{2} + A\right)} \quad \text{and} \quad S_2 = -\sqrt{s^2 - \left(\frac{a_0}{2} + A\right)}\,. \tag{2.13}$$

Proof: Without loss of generality we assume that v_∞ is zero. The shock polar corresponding to the unperturbed shock is shown in Figure 2.3. The state U_0 is hyperbolic if $u_0 < 0$, and this is guaranteed by (2.10), for if $U_0 \in B_{a_0}(U_\infty)$ then

$$u_0 < u_\infty + a_0 < -u_1\,. \tag{2.14}$$

The state U is elliptic if $u > 0$, and this follows from (2.11):

$$u > u_1 - 2A > 0\,. \tag{2.15}$$

The shock angle satisfies $(\xi')^2 + (u + u_0)/2 = 0$, from the Rankine-Hugoniot condition (2.6), and we verify that

$$u + u_0 < u_1 + 2A + u_\infty + a_0 < 0\,, \tag{2.16}$$

by (2.11). Since $u+u_0$ is never zero, ξ' remains the negative square root throughout. Since, by (2.16),

$$-\frac{u+u_0}{2} > -\frac{u_1+u_\infty}{2} - \left(A+\frac{a_0}{2}\right) = s^2 - \left(A+\frac{a_0}{2}\right) > 0$$

and, using the inequalities $u_0 > u_\infty - a_0$ and (2.15),

$$-\frac{u+u_0}{2} < -\frac{u_1}{2} + A - \frac{u_\infty}{2} + \frac{a_0}{2} = s^2 + \left(A+\frac{a_0}{2}\right),$$

we obtain (2.12). ∎

From this result, coupled with the requirement that $\xi(0) = 0$, we see that the shock is confined to a cone-shaped neighborhood of its unperturbed position.

We also require a C^1 bound on the perturbed flow U_0. A solution to (2.1) in the hyperbolic region, with Cauchy data $U_0(x_0+sy, y)$, say, remains smooth for only a finite distance x downstream from x_0. As for any genuinely nonlinear, strictly hyperbolic equation, one can estimate the distance to shock formation and the growth rate of the C^1 norm of the solution [8]. On the basis of this theory, we can impose a bound on the upstream solution of the form $|Du| \le b$ in a neighborhood of the shock, which is what we do in Theorem 2.1.

2.2 Weighted Hölder Norms

The basic analysis in this paper involves a mixed boundary problem for a quasilinear elliptic operator; a Dirichlet boundary condition is given on part of the boundary, σ, which is smooth, and an oblique derivative condition on the remaining part, Σ, which has minimal regularity. There are two corners, P_1 and P_2, at which σ and Σ meet, at angles which are very close to fixed values θ_1 and θ_2. The theory of classical solutions needed to treat a linear version of this problem was worked out by Lieberman in a series of papers, [11, 14, 12]. In the present paper we apply the theory to a quasilinear equation. The current problem is specialized to an operator in the plane, and is in divergence form. We cannot give a self-contained presentation of the theory, but we give references where details can be found.

To handle loss of regularity at the corners, we use weighted Hölder norms; these are the same as the partially interior norms defined in standard texts, for example Gilbarg and Trudinger [7, page 96ff], which treat differently the different parts of the boundary. The notation that will be used in this paper is as follows.

Let $X = (x, y)$ denote a point in $\mathbf{R}^2$. We use $D = (D_1, D_2)$ to denote partial derivatives, and $D^k u$ for the set of k-th order derivatives.

For functions defined on an open set S in $\mathbf{R}^2$, we have the usual supremum norm and Hölder semi-norms:

$$|u|_{0;S} = \sup_{X\in S} |u(X)| \quad \text{and} \quad [u]_{\alpha;S} = \sup_{X,Y\in S} \frac{|u(X)-u(Y)|}{|X-Y|^\alpha}$$

for $0 < \alpha \leq 1$; we omit the S when there is no danger of confusion. We define Hölder norms of any order:

$$|u|_{\alpha;S} = |u|_{0;S} + [u]_{\alpha;S} \quad \text{for} \quad 0 < \alpha \leq 1\,,$$

and, for $a = k + \alpha$ where k is an integer and $0 < \alpha \leq 1$,

$$|u|_{a;S} = \sum_{j<k} |D^j u|_{0;S} + |D^k u|_{\alpha;S}\,.$$

The notation $C^{k,\alpha}$ or $H_{k+\alpha}$ denotes the space of functions whose $(k+\alpha)$-Hölder norm is finite.

We describe the regularity of Σ by means of the function $\xi(y)$ which defines the curve; since

$$\Sigma = \{(\xi(y), y) \mid y_{\min} < y < y_{\max}\}\,,$$

we say $\Sigma \in C^{1,\alpha}$ or $\Sigma \in H_{1+\alpha}$ if $\xi \in C^{1,\alpha}([y_{\min}, y_{\max}])$, for $0 < \alpha \leq 1$. The notation $|\Sigma|_{1+\alpha}$ denotes the $H_{1+\alpha}$ norm of ξ.

We let V denote the set of corners of Ω, $V = \{P_1, P_2\}$, and let

$$d_V(X) \equiv \min\{|X - P_1|, |X - P_2|\}$$

denote the distance from a point X in Ω to the corner points.

To define our basic spaces, we use weighted or partially interior seminorms, defined as follows. For a subset S of $\partial\Omega$, define

$$\Omega_{\delta;S} = \{X \in \Omega \mid \operatorname{dist}(X, S) > \delta\}\,.$$

For any $a > 0$ and $a + b \geq 0$, the weighted norms are defined to be (S^c is the complement of S in $\partial\Omega$)

$$|u|^{(b)}_{a;\Omega\cup S^c} = \sup_{\delta>0} \delta^{a+b} |u|_{a;\Omega_{\delta;S}}\,.$$

The set of functions on Ω with finite norm $|u|^{(b)}_{a;\Omega\cup S^c}$ is denoted $H^{(b)}_{a;\Omega\cup S^c}$; the spaces corresponding to $b = 0$ are also denoted H'_a. Although Lieberman's previous theorems for mixed problems have used $S = \sigma$, the portion of the boundary on which Dirichlet conditions are posed, in this paper, we find the choice of $S = \Sigma \cup \sigma = \partial\Omega - V$ more useful. In this paper, we define

$$H^{(b)}_a = H^{(b)}_{a;\Omega\cup\Sigma\cup\sigma}\,.$$

These spaces have a compactness property (see [6, 9]): for $0 < b' < b$, $0 < a' < a$, $a \geq b$ and $a' \geq b'$, a bounded sequence in $H^{(-b)}_a$ is precompact in $H^{(-b')}_{a'}$.

2.3 A First Look at the Procedure

To prove existence of a solution of the free boundary problem (2.3), (2.4), (2.5), we make an approximation to the free boundary and solve a nonlinear boundary value problem with this fixed boundary, on which the data satisfy one Rankine-Hugoniot condition, or, what is equivalent, condition (2.8). Then, we update the position of the free boundary by using the second Rankine-Hugoniot condition (2.6), and this becomes a new approximation to the boundary. We show that this iterative procedure has a fixed point, which is then a solution of the free boundary problem.

The following are the main steps in the proof.

Step 1

Define the problem. We begin with a function $x = \xi(y)$, defining an approximate boundary Σ. Here, ξ belongs to a closed, convex subset $\mathcal{K}_\eta$ of the Banach space $H_{1+\alpha_\Sigma}$ with Hölder exponent $\alpha_\Sigma \in (0,1)$. Based on the bounds of the previous section, we define $\mathcal{K}_\eta$ to consist of functions ξ in $H_{1+\alpha_\Sigma}$ defined on $[y_{\min}, y_{\max}]$, satisfying the properties

(K1) $\xi(0) = 0$,

(K2) $|\xi' - s| \le \eta$.

where η is small enough that (2.12) is satisfied. The Hölder exponent α_Σ is specified in Section 4.1, equation (4.7).

This set is clearly closed and convex.

Step 2

Solve the fixed boundary problem

$$\begin{array}{rcll} Q(u) \;\equiv\; (uu_x)_x + u_{yy} & = & 0 & \text{in } \Omega\,, \\ N(X,u,Du) & = & \psi(u;u_0,U_0') & \text{on } \Sigma\,, \\ u & = & u_1 & \text{on } \sigma, \end{array} \tag{2.17}$$

where $N(X,u,Du)$ is defined by (2.8) and (2.9). This is a boundary value problem for the first component of a subsonic flow in the region Ω shown in Figure 2.2. If the perturbed solution $U(x,y)$ satisfies the bounds presented in the previous section, $Q(u)$ is uniformly elliptic in Ω. However, we do not have an a priori bound, and so we proceed by modifying Q outside $B_A(U_1)$ to obtain a uniformly elliptic operator; then, using sub- and super-solutions, we show that $U \in B_A(U_1)$ and so u satisfies $Q(u) = 0$. By a similar modification we ensure that $N(x,y,u,Du) = \psi(u;u_0,U_0')$ defines a uniformly oblique derivative boundary condition. Indeed, since $\nu = (1,-\xi')/\sqrt{1+(\xi')^2}$ is a unit inward normal to Ω at Σ, then if $U \in B_{2A}(U_1)$, we have

$$\beta \cdot \nu = \frac{u + (\xi')^2}{\sqrt{1+(\xi')^2}} \ge B > 0, \tag{2.18}$$

for some $B = B(a_0, A, u_0, u_1)$. Therefore, assuming bounds on U, problem (2.17) is a mixed boundary value problem for a uniformly elliptic operator in a domain with corners, with a uniform oblique derivative boundary condition along Σ, and a Dirichlet condition on σ. In Section 3 we obtain a weak solution to this problem by applying Lieberman's theory to a quasilinear equation.

STEP 3

Update to $\widetilde{\xi}$ the position of the boundary component Σ by using the Rankine-Hugoniot condition (2.6),

$$\widetilde{\xi}'(y) = -\sqrt{-\frac{u(\xi(y), y) + u_0(\xi(y), y)}{2}} \quad \text{with} \quad \widetilde{\xi}(0) = 0, \tag{2.19}$$

where u denotes the subsonic solution obtained in the previous step, and u_0 is the upstream perturbed flow. A mapping J on the set $\mathcal{K}_\eta \subset H_{1+\alpha_\Sigma}$ is defined by $\widetilde{\xi} = J\xi$. We use the following fixed point theorem (a corollary of the Schauder fixed point theorem) to show that this mapping has a fixed point which solves the free boundary problem.

THEOREM 2.3 (COROLLARY 11.2 OF [7]) *Let K be a closed, convex subset of a Banach space $\mathcal{B}$ and let T be a continuous mapping from K into itself such that the image TK is precompact. Then T has a fixed point.*

This theorem is applied in Section 4.

3 THE FIXED BOUNDARY PROBLEM

In this section we show that the nonlinear boundary value problem (2.17) has a solution, for a fixed $\xi \in \mathcal{K}_\eta$, defined in STEP 1 of Section 2.3.

The problem is

$$\begin{array}{rcccccl} Q(u) & \equiv & D_i\left(a^{ij}(u)D_j u\right) & = & (uu_x)_x + u_{yy} & = 0 & \text{in} \quad \Omega, \\ & & N(X,u,Du) & \equiv & \beta(X,u)\cdot Du & = \psi(u) & \text{on} \quad \Sigma, \\ & & & & u & = u_1 & \text{on} \quad \sigma, \end{array} \tag{3.1}$$

where $\beta(X,u) = \beta(u; u_0(\xi(y), y), \xi'(y))$ and $\psi(u) = \psi(u; u_0, U_0')$ are given in (2.9). Our result is as follows (the function spaces are defined in Section 2.2).

THEOREM 3.1 *Let Ω be a Lipschitz domain, with $\Sigma \in H_{1+\alpha_\Sigma}$ for some $0 < \alpha_\Sigma < 1$ and σ smooth. Then there exist positive numbers a_0 and b_0 and values of γ and δ, $0 < \gamma \le \delta < 1$, such that, for any $C^{1+\delta}$ perturbation $U_0(x, y)$ of the constant state U_∞ satisfying*

$$|U_0(x,y) - U_\infty|_0 < a_0 \quad \text{and} \quad |DU_0(x,y)|_0 < b_0 \quad \text{on} \quad \Sigma,$$

there exists a solution u of the problem (3.1). The solution is in $H_{1+\alpha_*}^{(-\gamma)}$ for all α_* with $0 < \alpha_* \le \alpha_\Sigma$.

The theory of oblique derivative boundary value problems for elliptic equations was developed for problems that satisfy three basic hypotheses.

1. The operator Q is uniformly elliptic in the sense that there exist positive numbers λ, $\Lambda > 0$ such that (we use a summation convention throughout)
$$\lambda|\zeta|^2 \leq a^{ij}(z)\zeta_i\zeta_j \leq \Lambda|\zeta|^2, \quad \forall\zeta \in \mathbf{R}^2, z \in \mathbf{R}. \tag{3.2}$$
2. The boundary operator $N(X, u, Du)$ is uniformly oblique, with positive constants B and B_1 such that for all $(X, z, p) \in \Sigma \times \mathbf{R} \times \mathbf{R}^2$
$$\left|\frac{\partial N}{\partial p}\right| \leq B_1, \quad \text{and} \quad \frac{\partial N(X, z, p)}{\partial p} \cdot \nu \geq B > 0, \tag{3.3}$$
where ν is the unit inward normal to Ω at the boundary Σ.
3. The domain Ω satisfies an *exterior cone condition* as defined by Gilbarg and Trudinger, [7, page 203]: at every point $P \in \partial\Omega$ there exists a finite right circular cone $W = W_P$ with vertex P such that $\overline{\Omega} \cap W_P = P$.

The first two conditions do not hold for the operators in (3.1). To remedy this, we modify the problem. It is convenient to divide the proof of Theorem 3.1 into four steps. The first is to introduce a modified nonlinear problem, using a cutoff function, to fulfill hypotheses 1 and 2. The second step is to show that a linearized version of the modified problem has a solution. In the third step we use a fixed point theorem to solve the nonlinear modified problem. And, finally, in the fourth step we show that the solution to the modified problem satisfies the L^∞ bounds imposed by the cutoff function and therefore solves the original nonlinear problem. The four steps of the proof occupy the next four subsections.

3.1 The Modified Problem (Definition)

The uniform shock configuration consists of two constant states, U_∞ and U_1, separated by the shock $x = sy$. We consider perturbations $U_0(x, y)$ of U_∞ which we will show can be chosen small enough that $u(x, y) \in (u_1 - A, u_1 + A)$ for an $A > 0$ satisfying (2.11), and so, as in Proposition 2.2, the downstream flow remains uniformly elliptic if $|u - u_1| < 2A$. Anticipating this bound, we introduce a smooth cutoff function $\varphi(w)$ with $u_1 - 2A \leq \varphi(w) \leq u_1 + 2A$, and we modify the coefficient of the nonlinear operator Q to yield a uniformly elliptic operator. Define the function $\varphi(w)$ in three disjoint intervals to be
$$\varphi(w) \equiv \begin{cases} u_1 - 2A & \text{if} \quad w < u_1 - 2A \\ w & \text{if} \quad u_1 - A \leq w \leq u_1 + A \\ u_1 + 2A & \text{if} \quad u_1 + 2A < w, \end{cases} \tag{3.4}$$

and extend φ to be C^∞ and monotone between the intervals with derivative bounded by 2. With the aid of $\varphi(w)$ we modify the nonlinear differential operator Q and the boundary operator N to

$$\begin{aligned} \widetilde{Q}(u) &\equiv (\varphi(u)u_x)_x + u_{yy}, \\ \widetilde{N}(u) &\equiv \beta^1(\varphi(u))u_x + \beta^2(\varphi(u))u_y. \end{aligned} \tag{3.5}$$

For brevity, in the notation for β we ignore the dependence on the upstream flow U_0 and on the boundary curve $\xi(y)$.

We also modify the argument of ψ in (3.1), so the modified nonlinear problem reads

$$\begin{aligned} \widetilde{Q}(u) &= 0 && \text{in} \quad \Omega, \\ \widetilde{N}(u) &= \psi(\varphi(u)) && \text{on} \quad \Sigma, \\ u &= u_1 && \text{on} \quad \sigma. \end{aligned} \tag{3.6}$$

We first solve a linearized version of the problem.

3.2 The Linearized Problem (Existence)

Let the domain Ω be bounded by a smooth boundary component σ and by a component Σ, defined by the function $x = \xi(y) \in \mathcal{K}_\eta$, as in Figure 2.2. To use existing theory, we need some minimal smoothness of w, the coefficient of the linearized problem. We measure the local Hölder smoothness of Dw by a parameter ϵ, which we will choose later, see equation (3.19), and we control the behavior of w at the corners (global Hölder continuity) with a parameter γ_1, also to be specified later, see (4.8). Assume that

$$w \in H_{1+\epsilon}^{(-\gamma_1)} \tag{3.7}$$

for some γ_1 with $0 < \gamma_1 \le \alpha_\Sigma$. Define linear operators

$$\begin{aligned} Lu &\equiv D_i\left(a^{ij}(X)D_j u\right) &&= (\varphi(w)u_x)_x + u_{yy} && \text{in} \quad \Omega, \\ Mu &\equiv \beta^i(X)D_i u &&= \beta^1(\varphi(w))u_x + \beta^2(\varphi(w))u_y && \text{on} \quad \Sigma. \end{aligned} \tag{3.8}$$

We study the linear problem for the function $u - u_1$ which we denote again by u; this makes the boundary condition on σ homogeneous. The problem is then written

$$\begin{aligned} Lu &= 0 && \text{in} \quad \Omega, \\ Mu &= \psi(X) && \text{on} \quad \Sigma, \\ u &= 0 && \text{on} \quad \sigma, \end{aligned} \tag{3.9}$$

where $\psi(X) = \psi(\varphi(w))$. Because $\varphi(w) \ge u_1 - 2A > 0$ (see (2.15)), the linear operator L is uniformly elliptic in the sense of hypothesis 1; the ellipticity ratio $\mu = \Lambda/\lambda$ is determined by u_∞, u_1, a_0 and A. Because of the lower bound on $\varphi(w)$

and because ξ' satisfies the bounds (2.12), the boundary operator M is uniformly oblique as defined in hypothesis 2, and B depends on the same four parameters as does μ.

The theorem for the linear problem is as follows.

Theorem 3.2 *Let Σ be in $H_{1+\alpha_\Sigma}$ and w be in $H_{1+\epsilon}^{(-\gamma_1)}$ for exponents $0 < \epsilon \leq \alpha_\Sigma < 1$ and $0 < \gamma_1 < 1$. Suppose also that for some constant m,*

$$|D(a^{ij})| \leq m d_V^{\gamma_1 - 1} \tag{3.10}$$

and that $|\psi|_0$ is bounded. Then there exists a unique solution $u \in H_{1+\alpha_\Sigma}^{(-\gamma)}$ of the linear problem (3.9), where γ is determined by the geometry of Ω and by the operators L and M. The solution u satisfies the two estimates

$$|u|_{1+\alpha_\Sigma}^{(-\gamma)} \leq C\left(|\psi|_{\alpha_\Sigma}^{(1-\gamma)} + \sup |d_V^{-\gamma} u|\right), \tag{3.11}$$

$$|u|_{1+\alpha_\Sigma}^{(-\gamma)} \leq C_1\left(|\psi|_{\alpha_\Sigma}^{(1-\gamma)} + |\psi|_0\right), \tag{3.12}$$

where C and C_1 are positive constants which depend on $[\varphi(w)]_{\alpha_\Sigma}$, $[\beta(\varphi(w))]_{\alpha_\Sigma}$, Λ/λ, $|\Sigma|_{1+\alpha_\Sigma}$, $\mathrm{diam}(\Omega)$ and B, and C_1 also depends on m.

Proof: We organize the proof in four parts.

Part 1

We first obtain an a priori L^∞ estimate for a solution u, using the comparison principle with an appropriate barrier function. Since Σ is merely in $H_{1+\alpha_\Sigma}$, we use a regularized distance function to construct the barrier. Proposition 3.3 summarizes the properties of the function ρ, a regularized distance to the boundary component Σ. A proof of existence of a function with these properties can be found in the paper by Lieberman [10].

Proposition 3.3 ([10]) *If the boundary component Σ is in $H_{1+\alpha}$, then the domain Ω has a regularized distance function $\rho(X)$ with the properties*

1. $\rho \in C^{2,\alpha}(\Omega) \cap H_{1+\alpha}(\overline{\Omega})$,
2. $1 \leq \frac{\rho}{d_\Sigma} \leq 2$, *where d_Σ is distance to Σ,*
3. $|D^2\rho| \leq c d_\Sigma^{\alpha-1}$,
4. $|D\rho| \geq \rho_0 > 0$ *in* Ω,
5. $|D\rho| \leq \rho_1$ *in* Ω.

The constants c, ρ_0 and ρ_1 depend on S_1 and S_2, and c depends on $|\Sigma|_{1+\alpha}$ as well.

We use ρ to prove the following lemma.

Lemma 3.4 *Let Ω be the Lipschitz domain shown in Figure 2.2 with $\Sigma \in H_{1+\alpha}$ for some $\alpha < 1$. Suppose $Lu \equiv D_i(a^{ij}(X)D_j u)$ and $Mu \equiv \beta^i(X)D_i u$ are uniformly elliptic in Ω and uniformly oblique on Σ, respectively, satisfying* (3.2) *and* (3.3). *Suppose also that* (3.10) *holds for some constant m and $\gamma_1 > 0$. Then there exists a supersolution $z = g(\rho)$ of problem* (3.9), *satisfying*

$$Lz \le 0 \text{ in } \Omega, \quad Mz \le \psi \text{ on } \Sigma\,, \quad z \ge 0 \text{ on } \sigma\,.$$

The function g can be chosen so that

$$|g|_0 \le C_g |\psi|_0,$$

where $C_g = C_g(\Lambda, \lambda, B, m, \rho_0, \rho_1, c, \mathrm{diam}(\Omega), \gamma_1)$.

Proof: Without loss of generality we can assume $\gamma_1 \le \alpha$. We use the regularized distance function of Proposition 3.3 to construct an upper barrier as follows.

For any smooth function g with $g' < 0$ and $g'' < 0$,

$$\begin{aligned} L(g(\rho)) &= g'' a^{ij} D_i\rho D_j\rho + g' a^{ij} D_{ij}\rho + g' D_i(a^{ij}) D_j \rho \\ &\le g''\lambda\rho_0^2 - g'\overline{C}\rho^{\gamma_1 - 1} \end{aligned} \tag{3.13}$$

in Ω, where $\overline{C} = \overline{C}(m, c, \rho_1, \Lambda, \mathrm{diam}(\Omega))$. Evaluating M on Σ (where $\rho = 0$) gives

$$M(g(\rho)) = g'(0)\beta \cdot D\rho = g'(0) k \beta \cdot \nu \le B g'(0)\,. \tag{3.14}$$

Let $K = \overline{C}/\lambda\rho_0^2$ and $c_0 = 1/B$. We solve $g'' - K\rho^{\gamma_1 - 1} g' = 0$ with the side conditions $g(\rho_{\max}) = 0$ and $g'(0) = -c_0|\psi|_0$. The function

$$g(\rho) = c_0 |\psi|_0 \left(\overline{D} - \int_0^\rho \exp(K\rho^{\gamma_1}/\gamma_1)\, d\rho \right),$$

where $\overline{D}(K, \gamma_1, \mathrm{diam}(\Omega))$ is the maximum value of the integral, is a supersolution, with $C_g = c_0 \overline{D}$. ■

We apply this lemma to problem (3.9) with $\alpha = \alpha_\Sigma$. The hypotheses of uniform ellipticity and uniform obliqueness in Lemma 3.4 are satisfied by the operators L and M of (3.9). The estimate (3.10) on $|D(a^{ij})|$ holds because w is in $H_{1+\epsilon}^{(-\gamma_1)}$ by assumption (3.7), with $\gamma_1 \le \alpha_\Sigma$, and so $|w|_{1;\Omega\cup\Sigma\cup\sigma}^{(-\gamma_1)}$ is finite. Thus, $|Dw| \le \frac{1}{2} m d_V^{\gamma_1 - 1}$ for $m = 2|w|_{1+\epsilon}^{(-\gamma_1)}$ and $D_1\varphi(w) = a'(w) w_x$ satisfies (3.10). The function $-g(\rho)$ is a subsolution. Hence we have a corollary to Lemma 3.4.

Corollary 3.5 *Let u be a solution of the linear problem* (3.9). *Then there exists a constant $C_g = C_g(u_\infty, u_1, a_0, A, \mathrm{diam}(\Omega), m, \gamma_1, |\Sigma|_{1+\alpha_\Sigma})$ such that*

$$|u|_0 \le C_g |\psi|_0.$$

Part 2

Now we prove that any solution of (3.9) satisfies estimate (3.11). First, we show that

$$\begin{aligned} R^{1+\alpha}[Du]_{1+\alpha;B(x_0,R)\cap\Omega} &\leq C\big(|u|_{0;B(x_0,2R)\cap\Omega} \\ &+ R|\psi|_{0;B(x_0,2R)\cap\Omega} + R^{1+\alpha}[\psi]_{\alpha;B(x_0,R)\cap\Omega}\big) \end{aligned} \tag{3.15}$$

for a constant C determined only by $[a^{ij}]_\alpha$, $[\beta]_\alpha$, Λ/λ, B, $|\Sigma|_{1+\alpha}$, and $|\sigma|_{1+\alpha}$ in the three cases

(a) $x_0 \in \sigma$ and $B(x_0, 2R) \cap \Sigma = \emptyset$,

(b) $x_0 \in \Sigma$ and $B(x_0, 2R) \cap \sigma = \emptyset$,

(c) $B(x_0, 2R) \subset \Omega$.

Since our hypotheses are invariant under $C^{1,\alpha}$ changes of independent variable, we may assume in cases (a) and (b) that Σ and σ are straight line segments. In case (a), estimate (3.15) then follows from the proof of Theorem 8.33 of [7]. In case (b), estimate (3.15) follows by combining the proofs of Theorem 8.33 and Theorem 6.26 of [7] (to handle the boundary condition, see also [15, Theorem 4.21]). Finally, in case (c), we use Theorem 8.32 of [7].

We multiply (3.15) by $R^{-\gamma}$ and take $R = \cot(\theta/2)d_V(x_0)/2$, where θ represents a corner angle. Now we use the argument of [7, Lemma 6.20] to obtain (3.11). Thus, (3.11) follows from (3.15) for any γ.

The estimate (3.12) follows from (3.11) once we estimate $\sup|d_V^{-\gamma}u|$. Now, since $u = 0$ on V, we have

$$\sup|d_V^{-\gamma}u| \leq |u|_\gamma. \tag{3.16}$$

Since the hypotheses of Theorem 1 of [14] are satisfied (with $f_1 = 0$, $f_2 = \psi$, and $f_3 = 0$), it follows that

$$|u|_\gamma \leq C(|u|_0 + |\psi|_\alpha^{(1-\gamma)}). \tag{3.17}$$

(Note that we have used that

$$|u|_\gamma = \|u\|_\gamma^{(-\gamma)} \leq C(a, \gamma, \mathrm{diam}(\Omega))\|u\|_a^{(-\gamma)} \tag{3.18}$$

for $a > \gamma$.) In citing [14] we note that the estimate holds for any γ less than a value λ_1, which depends on upper bounds for the corner angles at P_1 and P_2, on S_1 and S_2, on the ellipticity ratio and on the obliqueness ratio. The key point is that γ does not depend on γ_1, the exponent in the coefficients of L. Combining (3.17) and (3.16) with (3.11) and Corollary 3.5 gives estimate (3.12) for any γ in the indicated range.

Part 3

We approximate Ω by a sequence of domains Ω_k using a sequence of smooth curves Σ_k in Ω which approximate Σ; for example,

$$\Omega_k = \{X \in \Omega | \rho > 1/k\}, \qquad \Sigma_k = \{X \in \Omega | \rho = 1/k\},$$

where ρ is the regularized distance to the boundary Σ. The coefficients of the linear problem on each Ω_k converge in the appropriate norms to those of (3.9). The hypotheses of Lemma 3.4 apply uniformly in k, and so we have the uniform estimate $|u_k|_0 \leq C_g|\psi|_0$, with the constant C_g of Corollary 3.5.

In each Ω_k a solution u_k to the corresponding linear problem exists by Theorem 1 of [11]. The solutions satisfy (3.11) and (3.12) uniformly in k.

Part 4

Let $k \to \infty$, so that $\Omega_k \to \Omega$. Since we have (3.11) for each k, the sequence u_k is uniformly bounded in $H^{(-\gamma)}_{1+\alpha}$. By the Arzela-Ascoli theorem there exists a convergent subsequence, which we will again denote by u_k, which converges uniformly to a function u. To see that u solves (3.9) we first note that, since $|u_k|^{(-\gamma)}_{1+\alpha_\Sigma}$ is uniformly bounded, u_k and Du_k are equicontinuous on compact subsets of Ω. Therefore, $u = \lim u_k$ satisfies $Lu = 0$, which is in divergence form, in the weak sense in Ω. To show that the boundary conditions are satisfied by u, choose $x_k \in \Sigma_k$ with $x_k \to x_0 \in \Sigma$. Then by the uniform convergence of equicontinuous sequences in $H^{(-\gamma)}_{1+\alpha_\Sigma}$ we get $\beta_k \cdot D(u_k(x_k)) \to \beta \cdot Du(x_0)$, and since $\psi_k(x_k) \to \psi(x_0)$ we have $\beta \cdot Du = \psi$ on Σ. The condition $u = 0$ on σ is easily seen by a similar argument. Therefore, u is a weak solution of (3.9).

Uniqueness follows from the estimates provided by the barrier functions, Corollary 3.5. This completes the proof of Theorem 3.2. ■

Although the construction gives us only a weak solution, we know, since $w \in H^{(-\gamma_1)}_{1+\epsilon}$, that $u \in C^2(\Omega)$ ([7, Theorem 6.2]).

3.3 The Modified Nonlinear Problem (Existence)

To show that the modified nonlinear problem introduced in Section 3.1 has a solution, we use the following fixed point theorem.

Theorem 3.6 (Theorem 11.3 of [7]) *Let T be a compact mapping of a Banach space $\mathcal{B}$ into itself, and suppose that there exists a constant M such that*

$$||u||_{\mathcal{B}} \leq M$$

for all $u \in \mathcal{B}$ and $\tau \in [0,1]$ satisfying $u = \tau T u$. Then T has a fixed point.

We define $T : H^{(-\gamma_1)}_{1+\epsilon} \to H^{(-\gamma_1)}_{1+\epsilon}$ by letting $u = Tw$ be the unique weak solution of the linear mixed boundary problem (3.9) which we have just constructed. By

Theorem 3.2, $T(H_{1+\epsilon}^{(-\gamma_1)}) \subset H_{1+\alpha_\Sigma}^{(-\gamma)}$, so the operator T is compact if $\gamma_1 < \gamma$ and $\epsilon < \alpha_\Sigma$ [6, 9]. At this point, we take

$$\epsilon = \frac{\alpha_\Sigma}{2}, \tag{3.19}$$

and require $\gamma_1 < \gamma$.

Throughout, α_Σ denotes the smoothness of the boundary Σ, fixed in this section, and also denotes the smoothness of all the coefficients in the boundary condition (the three functions in (2.9)). Thus, we take $\alpha_\Sigma \leq \delta$ in Theorem 3.1. We have a geometric upper bound λ_1 for γ, but we do not further specify α_Σ or γ_1 until we consider the free boundary problem in the next section.

We need to show that there exists an $M > 0$ such that

$$\|u\|_{\mathcal{B}} \leq M$$

for all functions $u \in \mathcal{B} \equiv H_{1+\alpha_\Sigma/2}^{(-\gamma_1)}$, which solve

$$\begin{array}{rcllll} \widetilde{Q}\,u & = & 0 & \text{in} & \Omega, & \\ \widetilde{N}u & = & \tau\psi & \text{on} & \Sigma, & \tau \in [0,1], \\ u & = & u_1 & \text{on} & \sigma, & \end{array} \tag{3.20}$$

for the modified nonlinear operators of equation (3.5). Before proceeding with this estimate, we again replace u by $u - u_1$ and solve the same mixed boundary value problem but with homogeneous Dirichlet data on σ and with $\varphi(u)$ still centered at u_1, so $|\varphi(u) - u_1| \leq 2A$. We again denote $u - u_1$ by u. We can say here that if $\widetilde{Q}\,u = 0$ has a solution in $H_{1+\alpha}^{(-\gamma)}$, then $u \in C^2(\Omega)$, because we can look at it as the solution of a linear problem whose coefficients have sufficient smoothness, as in the comment following the proof of Theorem 3.2.

We have the following estimate.

Lemma 3.7 *Suppose that u is a solution of*

$$\begin{array}{rcrcll} \widetilde{Q}\,u & \equiv & \varphi(u)u_{xx} + u_{yy} + \varphi'(u)u_x^2 & = & 0 & \textit{in} \;\; \Omega, \\ \widetilde{N}u & \equiv & \beta^1(\varphi(u))u_x + \beta^2(\varphi(u))u_y & = & \tau\psi(\varphi(u)) & \textit{on} \;\; \Sigma, \\ & & u & = & 0 & \textit{on} \;\; \sigma. \end{array} \tag{3.21}$$

Then, provided $|DU_0|$ and $|U - U_0|$ are small enough, there exist constants $\alpha_* > 0$ and $M > 0$ such that

$$|u|_{1+\alpha_*}^{(-\gamma)} \leq M, \quad \forall \tau \in [0,1]. \tag{3.22}$$

Proof: The proof has several steps. We first obtain an a priori L^∞ bound for a solution u of (3.21) by constructing upper and lower barriers. Then we obtain a height estimate which captures the behavior of u at the corner points. Next, we quote results based on Harnack inequalities ([7] and [12]) to obtain a global Hölder estimate for u. We use this estimate for the coefficients $\varphi(u)$ and $D\varphi(u)$, and then we obtain an estimate like (3.11) which we apply to problem (3.21) to obtain (3.22). We can explain the overall thrust of the proof by pointing out that an important difference between $\widetilde{Q}$ and the linear operator L is that we must deal with the term $\varphi'(u)u_x^2$. In the first step, we handle the L^∞ bounds by a judicious choice of barrier functions; in the second step, we obtain Hölder estimates by using results on equations with quadratic nonlinearities.

We start by obtaining L^∞ bounds for u. The following operators will be very useful:

$$\begin{aligned} \overline{L}z &\equiv \varphi(u)z_{xx} + z_{yy}, \\ \overline{M}z &\equiv \beta^1(\varphi(u))z_x + \beta^2(\varphi(u))z_y, \\ \overline{Q}z &\equiv \varphi(u)z_{xx} + z_{yy} + \varphi'(u)z_x^2 . \end{aligned}$$

If z solves the linear problem

$$\overline{L}z = 0 \text{ in } \Omega, \quad \overline{M}z = \tau\psi(\varphi(u)) \text{ on } \Sigma, \quad z = 0 \text{ on } \sigma, \tag{3.23}$$

then a solution u of (3.21) is a supersolution of (3.23). We construct a subsolution $g = g(\rho)$ of (3.23), $g(\rho) \le z$, as in Lemma 3.4, now with $g \le 0$, $g' > 0$, $g'' > 0$. Again, ρ is the regularized distance from Σ, introduced in Proposition 3.3. We have

$$\overline{L}(g(\rho)) = g''(\varphi(u)\rho_x^2 + \rho_y^2) + g'(\varphi(u)\rho_{xx} + \rho_{yy}) \ge g''\lambda\rho_0^2 - g'(\Lambda c d_\Sigma^{\alpha_\Sigma - 1}),$$

so we choose g to satisfy $g'' - K\rho^{\alpha_\Sigma - 1}g' = 0$ where $K = \Lambda c/(\lambda\rho_0^2)$ (c is the constant in Proposition 3.3). Choosing the integration constants as in Lemma 3.4, we find

$$|g|_0 \le C_g|\psi|_0 ,$$

where now $C_g = C_g(u_\infty, u_1, a_0, A, \mathrm{diam}(\Omega), \alpha_\Sigma, |\Sigma|_{1+\alpha_\Sigma})$; because the term $a'(u)u_x^2$ is missing from $\overline{L}$, this C_g does not depend on m nor on γ_1. Now,

$$g(\rho) \le z \le u . \tag{3.24}$$

We construct an upper bound for u using the auxiliary system

$$\overline{Q}z = 0 \text{ in } \Omega, \quad \overline{M}z = \tau\psi(\varphi(u)) \text{ on } \Sigma, \quad z = 0 \text{ on } \sigma. \tag{3.25}$$

Since $z = u$ solves (3.21), a supersolution h of (3.25) is an upper bound for u. We construct a supersolution $h = h(\rho)$, again as in Lemma 3.4, with $h \ge 0$, $h' \le 0$, $h'' \le 0$. Indeed, we have the following estimate for $\overline{Q}h$:

$$\begin{aligned} \overline{Q}h &= \left(\varphi(u)\rho_x^2 + \rho_y^2\right) h''(\rho) + \left(\varphi(u)\rho_{xx} + \rho_{yy}\right) h'(\rho) + a'(u)h'(\rho)^2\rho_x^2 \\ &\le h''(\rho)\lambda\rho_0^2 - \Lambda c h'(\rho)\rho^{\alpha_\Sigma - 1} + \rho_1^2 a'(u)h'(\rho)^2. \end{aligned}$$

(Here, ρ_1^2 is the constant in item 5 of Proposition 3.3.) Then the requirement $\overline{Q}h \le 0$ is satisfied by solutions of

$$h'' - K\rho^{\alpha_\Sigma - 1}h' + \ell(h')^2 = 0,$$

where $K = \Lambda c/(\lambda\rho_0^2)$ and $\ell = \sup a'\rho_1^2/(\lambda\rho_0^2)$. The solution is

$$h(\rho) = \frac{1}{\ell}\log\left(-C_0\int_0^\rho \exp\left(\frac{K}{\alpha_\Sigma}r^{\alpha_\Sigma}\right)\,dr + K_1\right),$$

where C_0 and K_1 are positive constants of integration.

The boundary condition $\overline{M}h \le \tau\psi$ is satisfied, as in (3.14), if $h'(0) = -c_0|\psi|_0$, where $c_0 = 1/B$, whence $C_0/K_1 = c_0\ell|\psi|_0$. Finally, $h \ge 0$ on σ is satisfied if $h(\rho) \ge 0$ for all $\rho \le \operatorname{diam}(\Omega)$; we fix $h(d) = 0$, where $d = \operatorname{diam}(\Omega)$, and find

$$-c_0\ell|\psi|_0\int_0^d \exp\left(\frac{K}{\alpha_\Sigma}r^{\alpha_\Sigma}\right)\,dr + 1 = \frac{1}{K_1} > 0\,;$$

this determines K_1, and a solution h exists as long as $|\psi|_0$ is small enough.

We estimate $|h|_0$ as follows. Since h is monotonic, $|h|_0 = h(0)$, and the mean value theorem gives

$$|h|_0 \le C_h|\psi|_0, \tag{3.26}$$

where we can take C_h to be

$$C_h = 2c_0\int_0^d \exp\left(\frac{K}{\alpha_\Sigma}r^{\alpha_\Sigma}\right)\,dr$$

as long as $p \equiv C_h\ell|\psi|_0/2$ is less than $1/2$. Now, since $p = P|\psi|_0$ where P depends on u_∞, u_1, a_0, A, $\operatorname{diam}(\Omega)$, α_Σ, and $|\Sigma|_{1+\alpha_\Sigma}$, we have $p < 1/2$ if $|\psi|_0$ is small enough. Since

$$|\psi|_0 \le K|DU_0|_0, \tag{3.27}$$

where K depends only on u_∞, u_1, a_0 and A, this is controlled by $|DU_0|_0$.

We have found smooth functions g and h such that

$$g \le u \le h \quad \text{in} \quad \overline{\Omega}.$$

The constant C_h depends on the same parameters as does C_g; letting C_{gh} be the maximum of these constants, we have

$$|u|_0 \le C_{gh}|\psi|_0, \tag{3.28}$$

where C_{gh} depends on u_∞, u_1, a_0, A, $\operatorname{diam}(\Omega)$, α_Σ and $|\Sigma|_{1+\alpha}$. Next we estimate $|d_V^{-\gamma}u|$ for a suitable constant γ. Lemma 4.1 of [14] gives positive constants h_0

and λ_1 (same λ_1 as in the discussion before (3.17)) such that if $0 < \gamma < \lambda_1$, then there are a positive constant c_1 and a function w_1 in $C^2(\overline{\Omega(h_0)} - V) \cap C(\overline{\Omega(h_0)})$ such that

$$\overline{L}w_1 \leq 0 \text{ and } c_1 d_V^\gamma \leq w_1 \leq d_V^\gamma \text{ in } \Omega(h_0)\,, \text{ and } \overline{M}w \leq -d_V^{\gamma-1} \text{ on } \Sigma(h_0),$$

where $\Omega(h_0)$ and $\Sigma(h_0)$ are the subsets of Ω and Σ, respectively, on which $d_V \leq h_0$. The constant c_1 is determined only by μ, γ, and (an upper bound for) the opening angles at the points in V. Since λ_1 is determined by μ and the same angles, we may let γ be any value less than λ_1. Finally, it follows from the maximum principle that

$$ud_V^{-\gamma} \geq -(1/c_1)(|u|_0 + |\psi|_0)h_0^{-\gamma} \geq -C|\psi|_0$$

because of our L^∞ estimate for u.

To find an upper corner barrier, observe that substitution into equation (3.21) shows that for large enough K the function $w_{\text{sub}} = (e^{Ku} - 1)/K$ is a subsolution of another linear problem:

$$\overline{L}W = 0 \text{ in } \Omega, \quad \overline{M}W = \tau e^{Ku}\psi(\varphi(u)) \text{ on } \Sigma, \quad W = 0 \text{ on } \sigma,$$

in which the operators are the same as (3.23). A similar argument gives the upper bound for w_{sub} and hence for u, and so we have

$$\sup_\Omega |d_V^{-\gamma}u| \leq C. \tag{3.29}$$

To obtain the bound (3.22) on u, we first obtain local Hölder estimates in the interior of Ω, at the boundary Σ, at the smooth boundary σ and at the corners, and then piece them together to obtain a global Hölder estimate, $u \in H_{\delta*}$ and $\|u\|_{\delta*} \leq C$, where C depends on the same parameters as does C_{gh} (typically, $\delta*$ is very small). These standard estimates, typically obtained by an approach based on Harnack inequalities, can be found in several places in the literature: the interior Hölder estimate with a quadratic nonlinearity in the operator is obtained in Lemma 15.4 in [7]. The local estimate at the boundary Σ follows from the remark immediately after Theorem 2.3 in [12]. The local Hölder estimate at σ is obtained in Corollary 9.29 in [7] with the quadratic nonlinearity handled in Lemma 15.4 in [7]. Finally, Hölder continuity at the corners is a direct consequence of barrier construction presented following (3.28). These local estimates are pieced together in the same manner as presented in the proof of Theorem 8.29 in [7].

Once we have a bound on $u \in H_{\delta*}$, then $\varphi(u)$, $\beta(\varphi(u))$ and $\psi(\varphi(u))$ are in $H_{\delta*}$, so if we apply estimate (3.11) with α_Σ replaced by $\min\{\delta*, \alpha_\Sigma, \gamma\}$ then we get

$$\|u\|^{(-\gamma)}_{1+\min\{\delta*,\alpha_\Sigma,\gamma\}} \leq C$$

with a constant independent of u. This gives (3.22) with $\alpha*$ replaced by the minimum of $\delta*$, α_Σ and γ. Now we have in particular $u \in H_\gamma$ (see (3.18)); therefore, we can repeat the argument which began this paragraph with $\delta*$ replaced by γ, so $\delta*$ disappears from the minimum and we get $\alpha* = \min\{\alpha_\Sigma, \gamma\}$ in (3.22).

This concludes the proof of Lemma 3.7. ■

Provided $\alpha_\Sigma \le 2\gamma$, inequality (3.22) implies the bound in Theorem 3.6 in the norm of $\mathcal{B} = H^{(-\gamma_1)}_{1+\alpha_\Sigma/2}$. Hence, T has a fixed point u, which in fact lies in $H^{(-\gamma)}_{1+\alpha_*}$ for $\alpha_* = \min\{\gamma, \alpha_\Sigma\}$. This proves the following result.

THEOREM 3.8 *Let Ω be a domain with smooth boundary component σ and $\Sigma \in H_{1+\alpha_\Sigma}$. Then the modified nonlinear problem* (3.6) *has a solution $u \in H^{(-\gamma)}_{1+\alpha_*}$ where $\gamma \in (0,1)$ is determined by the corner angles, and $\alpha_* = \min\{\alpha_\Sigma, \gamma\}$.*

As was the case in the linear problem, $u \in C^2(\Omega)$. We note also that away from the corners, Σ is in $H_{2+\alpha_\Sigma}$.

3.4 The Nonlinear Problem (Existence)

In this section we show that the solution of the modified nonlinear problem (3.6) solves the original nonlinear problem (3.1). This is achieved by showing that u satisfies the a priori bounds imposed by the cutoff function $\varphi(u)$, (3.4).

We employ the upper and lower barriers constructed in the proof of Lemma 3.7. Returning to the original notation for u, inequality (3.28) implies that for problem (3.1) with constant Dirichlet data $u = u_1$ on σ,

$$|u - u_1|_0 \le C_{gh}|\psi|_0.$$

Now, using (3.27), we can choose DU_0 small enough that

$$C_{gh}|\psi(\varphi(u), u_0, U_0')|_0 < A$$

on Σ. Therefore, the following proposition holds.

PROPOSITION 3.9 *There exists a constant $b_0 > 0$ such that if $|DU_0| \le b_0$ then the solution u of* (3.6) *satisfies*

$$|u - u_1|_0 < A. \tag{3.30}$$

Therefore, for such U_0 we conclude that $\varphi(u) = u$ everywhere, and the solution $u \in H^{(-\gamma)}_{1+\alpha_*}$ of (3.6) solves the nonlinear problem (3.1).

This completes the proof of Theorem 3.1.

4 The Free Boundary Problem

We now define a nonlinear operator which associates a new boundary to the old boundary by solving the fixed boundary problem of Theorem 3.1 and then using the other Rankine-Hugoniot condition (2.6) to update the boundary Σ.

4.1 Definition of the Operator J

For ξ in the set $\mathcal{K}_\eta$ given in Section 2.3, we let

$$\widetilde{\xi} = J\xi \tag{4.1}$$

be a new approximation to the free boundary defined by

$$\widetilde{\xi}'(y) = -\sqrt{-\frac{u(\xi(y),y) + u_0(\xi(y),y)}{2}}, \quad \widetilde{\xi}(0) = 0, \tag{4.2}$$

where u is the solution of the nonlinear mixed boundary problem (3.1). In (4.2), $u(\xi(y),y)$ denotes the limit of u from the right (from the interior of the elliptic region) at the boundary $x = \xi(y)$ and $u_0(\xi(y),y)$ is the limit of u_0 from the left.

To use the version of the Schauder fixed point theorem stated in Theorem 2.3, we need to show that J is defined on $\mathcal{K}_\eta$ and that J is a precompact mapping of $\mathcal{K}_\eta$ to itself.

Proposition 4.1 *There is an $\eta_0 > 0$ such that the operator J maps $\mathcal{K}_\eta$ into itself for any $\eta \le \eta_0$.*

Proof: This result follow from the L^∞ estimates on u. First, we see that J is defined on $\mathcal{K}_\eta$ as long as η is smaller than a value that depends only on u_0, u_1, a_0 and A, since we can choose η_0 so that ξ' satisfies

$$-\sqrt{s^2 + \left(\frac{a_0}{2} + A\right)} \le \xi' \le -\sqrt{s^2 - \left(\frac{a_0}{2} + A\right)}.$$

Now, assuming C^1-bounds on the initial perturbation $U_0(x,y)$ determined by the constants a_0 (in Proposition 2.2) and b_0 (given by Proposition 3.9), we have a solution u by Theorem 3.1, so that J is defined. Property (K1) is satisfied, by the definition of $\widetilde{\xi}$. We need to verify property (K2): there exists an $\eta_0 > 0$ such that for each $\eta \le \eta_0$ and $\xi \in \mathcal{K}_\eta$ the inequality $|(J\xi)' - s| \le \eta$ holds. We show this in two steps. First, we show that there exist $\eta_0 > 0$ and $\widetilde{a}_0 > 0$ such that if $\eta \le \eta_0$ and $|\psi|_0$ is small enough, then

$$|u - u_1|_0 < C|\psi|_0. \tag{4.3}$$

Then we show that for each fixed $\eta > 0$ there exist constants $\widetilde{a}_1 > 0$ and $\widetilde{a} > 0$ such that $|\psi|_0 < \widetilde{a}_1$ and $|u_0 - u_\infty| < \widetilde{a}$ imply $|(J\xi)' - s| \le \eta$.

The bound (4.3) is similar to the bound obtained in Section 3.4, except that we must replace C_{gh} by a constant independent of $|\Sigma|_{1+\alpha_\Sigma}$ (see equation (3.28)). To effect this, we construct upper and lower barriers for $u - u_1$ in terms of $d \equiv x - sy$ (a constant multiple of the distance to the unperturbed boundary).

We find a lower barrier, $g(d)$, as in the proof of Lemma 3.7, using the same operators. A subsolution $g = g(d)$ of

$$\overline{L}w = 0 \text{ in } \Omega, \quad \overline{M}w = \psi(\varphi(u)) \text{ on } \Sigma, \quad w = 0 \text{ on } \sigma, \tag{4.4}$$

gives a lower bound for the solution u of the nonlinear fixed boundary problem (3.1). Define $g(d) = g_1 d + g_2$ where g_1 and g_2 are constants which will be chosen to make g a subsolution of (4.4). Now, $\overline{L}g(d) = 0$ and

$$\overline{M}g(d) = g_1\big(\beta^1(\varphi(u)) - s\beta^2(\varphi(u))\big).$$

We claim that $\beta^1(\varphi(u)) - s\beta^2(\varphi(u)) > 0$ so we can choose $g_1 > 0$ large enough that $\overline{M}g(d) \geq \psi(\varphi(u))$. To get a subsolution, we then choose g_2 sufficiently negative that $g \leq 0$ on σ.

To estimate $\beta^1 - s\beta^2 = \beta \cdot (1, -s)$, we add and subtract $\beta \cdot (1, -\xi') = \beta \cdot \nu$:

$$\beta^1(\varphi(u)) - s\beta^2(\varphi(u)) \geq \beta \cdot \nu - |\beta \cdot (0, \xi' - s)| \geq B - |\beta^2||\xi' - s|\,.$$

(using (2.18) or (3.3)). Since $\xi \in \mathcal{K}_\eta$, we have $|\xi' - s| \leq \eta$. We estimate $\beta^2(\varphi(u))$ from the definition (2.9), noting that $\varphi(u) \in (u_1 - 2A, u_1 + 2A)$. We see that the denominator of β^2 is bounded away from zero, and there is a constant $B_2(u_\infty, u_1, a_0, A) > 0$ such that $|\beta^2| < B_2$. With this we have

$$\overline{M}g(d) \geq g_1\big(\beta \cdot \nu - B_2\eta\big) \geq g_1(B - B_2\eta).$$

Now take $\eta \leq \eta_0 \leq B/(2B_2)$, so that $\beta \cdot \nu - B_2\eta \geq B/2$, and $g_1 = 2|\psi|_0/B$ to get $\overline{M}g(d) \geq \psi(\varphi(u))$.

Finally, to have $g(d) \leq 0$ on σ we choose $g_2 = -g_1 \mathrm{diam}(\Omega)$. Thus, $g(d)$ is a subsolution and

$$|g|_0 \leq G|\psi|_0\,,$$

where G depends on B (that is, on u_∞, u_1, a_0 and A), and on $\mathrm{diam}(\Omega)$.

An upper bound can also be found as in Lemma 3.7. We seek a supersolution $h(d)$ of

$$\overline{Q}w = 0 \text{ in } \Omega, \quad \overline{M}w = \psi(\varphi(u)) \text{ on } \Sigma, \quad w = 0 \text{ on } \sigma. \tag{4.5}$$

with $h \geq 0$, $h' \leq 0$, $h'' \leq 0$. Since $d = x - sy$, we have

$$\overline{Q}h = (\varphi(u)d_x^2 + d_y^2)h'' + a'(u)\,[h'(d)]^2\,d_x^2 = (\varphi(u) + s^2)h'' + \varphi'(u)(h')^2.$$

Let $E \equiv \sup \varphi'(u)/(\varphi(u) + s^2)$. By a construction similar to that in the proof of Lemma 3.7, $\overline{Q}h \leq 0$, $\overline{M}h \leq \psi$ and $h \geq 0$ are satisfied by

$$h(d) = \frac{1}{E}\log|E\,d - K_1| + K_2\,,$$

with

$$\frac{2E\,\mathrm{diam}(\Omega)}{B}|\psi|_0 < 1, \quad K_1 \geq \frac{B}{2|\psi|_0} \quad \text{and} \quad K_2 = -\frac{1}{E}\log\left(K_1 - \mathrm{diam}(\Omega)\right).$$

We achieve the first requirement by choosing $|\psi|_0$ sufficiently small, by making $|DU_0| \leq \tilde{b}_0$ small enough, and then choose K_1 and K_2. Thus

$$|h|_0 \leq H|\psi|_0,$$

where H, like G, depends on u_∞, u_1, a_0, A, and $\mathrm{diam}(\Omega)$.

Thus, take C in (4.3) to be the maximum of G and H and we obtain

$$|u - u_1|_0 \leq C|\psi|_0. \tag{4.6}$$

We complete the proof of Proposition 4.1 by showing that for any $\eta \leq \eta_0$ we can obtain $|\widetilde{\xi}' - s| \leq \eta$ by choosing the upstream perturbation to be small enough. To be precise, suppose that $|\psi|_0 < |s|\eta/C$ and $|u_0 - u_\infty| < |s|\eta$. Now, since $(J\xi)' = \widetilde{\xi}'$ and s are both negative,

$$|\widetilde{\xi}' - s| = \frac{|(\widetilde{\xi}')^2 - s^2|}{|\widetilde{\xi}' + s|} \leq \frac{|(\widetilde{\xi}')^2 - s^2|}{|s|}.$$

Since $\widetilde{\xi}' = -\sqrt{-(u+u_0)/2}$ and $s = -\sqrt{-(u_1+u_\infty)/2}$ we have

$$|(\widetilde{\xi}')^2 - s^2| = \frac{1}{2}\left|u + u_0 - (u_1 + u_\infty)\right| \leq \frac{1}{2}(|u - u_1| + |u_0 - u_\infty|).$$

Now, by requiring $|\psi|_0 < |s|\eta/C$, where C is given in (4.6), we have bounded $|u - u_1| \leq |s|\eta$; we have also chosen $|u_0 - u_\infty| < |s|\eta$. Then we have

$$|(\widetilde{\xi}')^2 - s^2| \leq \frac{1}{2}(|u - u_1| + |u_0 - u_\infty|) \leq |s|\eta$$

and so we have (K2): $|\widetilde{\xi}' - s| \leq \eta$.

We have shown that for each $\eta \leq \eta_0$, $J\mathcal{K}_\eta \subset \mathcal{K}_\eta$. ■

4.2 Compactness of J and the Main Result

We now prove the last property necessary for applying the Schauder fixed point theorem.

Proposition 4.2 *The set $J\mathcal{K}_\eta$ is precompact in $H_{1+\alpha_\Sigma}$, for all sufficiently small $\eta \leq \eta_1$ and sufficiently small $\alpha_\Sigma \geq 0$.*

Proof: To show that $J(\mathcal{K}_\eta)$ is precompact in $H_{1+\alpha_\Sigma}$ we can apply Lemma 2.1 of [13], which gives a weak Harnack inequality on Σ for nonlinear equations. A routine calculation at a point on Σ, using the functions given explicitly in (2.9), shows that we satisfy hypotheses (2.2) and (2.3) of that lemma.

Now, Lemma 2.1 of [13], with b, c, and γ in that lemma equal to zero, implies that there are constants C and α_0 determined only by the ellipticity ratio μ and $\sup u \sup \varphi'$ such that

$$\mathrm{osc}_{B(r)} u \leq C \left(\frac{r}{\rho}\right)^{\alpha_0} \left[\mathrm{osc}_{B(\rho)} u + \rho \sup |\psi|\right]$$

for any balls $B(r)$ and $B(\rho)$ centered at (x_0, y_0) with $r < \rho < d((x_0, y_0), \sigma)$. As in Lemma 3.7, we can combine this estimate with Hölder estimates for u on σ, near V, and inside Ω (estimates which do not involve η, α_Σ or Σ). It follows that for η sufficiently small (say $\eta \leq \eta_1$, for some $\eta_1 > 0$), there are constants C and α_1 independent of η, α_Σ, and $|\Sigma|_{1+\alpha_\Sigma}$ such that

$$|u|_{\alpha_1} \leq C.$$

(In general α_1 will be much smaller than γ.) From this bound, we immediately obtain

$$|\widetilde{\xi}'|_{\alpha_1} \leq C$$

and hence $J(\mathcal{K}_\eta)$ is contained in a bounded subset of $H_{1+\alpha_1}$. It follows that $J(\mathcal{K}_\eta)$ is a precompact set in $H_{1+\alpha_\Sigma}$ provided $\alpha_\Sigma < \alpha_1$.

To fix ideas, we set α_Σ equal to the minimum of $\alpha_1/2$ and $\gamma/2$:

$$\alpha_\Sigma \equiv \min\left\{\frac{\alpha_1}{2}, \frac{\gamma}{2}\right\}. \tag{4.7}$$

A simple bootstrap argument like the one at the end of Lemma 3.7 shows that we can, in fact, eliminate α_1.

This completes the proof of the proposition. ■

We now specify η in the definition of the set $\mathcal{K}_\eta$ to be the minimum of η_0 and η_1 obtained in the proofs of Propositions 4.1 and 4.2. We have now verified the hypotheses of Theorem 2.3, and we have proved the following result.

Lemma 4.3 *There is a fixed point $\xi \in H_{1+\alpha_1}$ for the operator $J : H_{1+\alpha_\Sigma} \to H_{1+\alpha_\Sigma}$ defined by* (4.1) *and* (4.2).

In setting up our theorems, we defined an auxiliary exponent γ_1 for the linearized problem (see equation (3.7)). A posteriori, we see that we can choose any γ_1 less than γ, and α_Σ less than γ and α_1, and such that $\gamma_1 \leq \alpha_\Sigma$ (see Lemma

3.4). To simplify the discussion, we have chosen both γ_1 and α_Σ to be equal to the minimum of $\gamma/2$ and $\alpha_1/2$:

$$\gamma_1 \equiv \alpha_\Sigma = \min\left\{\frac{\gamma}{2}, \frac{\alpha_1}{2}\right\}. \tag{4.8}$$

From the fixed point ξ, we can use Theorem 3.1 to recover $u \in H^{(-\gamma)}_{1+\alpha_*}$, for $\alpha_* \leq \alpha_\Sigma$, as a solution of the nonlinear mixed boundary value problem (3.1) with a boundary Σ which is a fixed point of the operator J defined in this section. Finally, we can apply the original Rankine-Hugoniot condition (2.4) to determine v at a single point, P_1, say (and hence along Σ), and we can use the original equation (2.1) in Ω to recover $v \in H^{(-\gamma)}_{1+\alpha_*}$.

In Theorem 2.1, the constant a is obtained by taking the minimum of a_0 defined in Proposition 2.2 and $\tilde{a} = |s|\eta$. The constant b is obtained by taking the minimum of b_0 defined in Proposition 3.9, and $\tilde{b}_0$ and $\tilde{b}_1$ defined in the proof of Proposition 4.1.

This completes the proof of Theorem 2.1.

This immediately gives Theorem 1.1.

5 Conclusions

We have proved the existence of a steady transonic shock close to a uniform shock by solving a free boundary problem which couples an elliptic equation with an overdetermined boundary condition along a curve whose position is unknown. The main tool in the proof is the Schauder fixed point theorem; the method is classical, taking advantage of Schauder and Harnack estimates for oblique derivative problems for a quasilinear equation.

The proof could be adapted without much change to any free boundary problem involving a quasilinear elliptic system that can be written as a second order equation $(A_1(u,X)u_x)_x + (A_2(u,X)u_y)_y = 0$ and for which the overdetermined boundary condition on $\Sigma \equiv \{x = \xi(y)\}$ has the form

$$\xi' = f(u, \xi, X)\,, \quad \beta(u, \xi, \xi', X) \cdot Du = g(u, \xi, X)$$

involving an oblique derivative operator. The only properties we used were boundedness and regularity of the coefficients A_1, A_2, β and g, and the possibility of casting the problem in a uniformly elliptic and uniformly oblique form. In subsection 5.2, we summarize the technical reasons that this rather specific form seems amenable to this attack.

Some related problems have this form. For example, a transonic version, using the TSD equation, of Schaeffer's shock at a slightly curved wedge [16] will have this form, now with a simpler upstream condition but a more complicated boundary value problem in the elliptic region (and, again, an artificial cutoff boundary). Stability of a steady transonic shock in a variable area duct is also of this form, and even has a natural cutoff boundary (the sides of the duct) near the shock.

The quasisteady problem, mentioned in the Introduction, for the unsteady TSD equation also has this form. This equation is not uniformly elliptic; however, it may be possible to adapt the strategy in the present paper by proving height estimates at the degenerate elliptic corner and matching them to an elliptic equation in the interior.

5.1 Related Results on Free Boundary Problems

Examples of free boundary problems in which the position of the boundary is coupled to an elliptic equation include the Stefan problem and the dam problem; this area has been active recently. A popular method (Alt and Caffarelli [1], Berestycki, Bonnet and van Duijn [2], and Caffarelli [3] and the references cited there) is to pose the problem in variational form, with the condition on the free boundary imposed in a natural way (for example, using the characteristic function of the set where u is positive), from which the existence of a weak solution follows. Caffarelli has shown that once the free boundary can be shown to have a small amount of smoothness, higher regularity follows. However, because our equation is quasilinear, the solution to the variational problem is difficult to construct when it involves highly nonlinear functions such as the characteristic function of a set with a nonsmooth boundary. Although it is possible that this program could be carried out, it did not appear to be simpler than the method we used in this paper.

5.2 Discussion of the Method of this Paper

The main tools used to obtain our result are the Schauder fixed point theorem and Schauder estimates for oblique derivative problems. The second ingredient is relatively new; nonetheless it is interesting that no solution of a transonic free boundary problem has been presented before.

Our innovation, which enabled us to construct a proof, has been to consider a second order equation for u, the principal component of velocity, rather than for ϕ, the potential. In research on transonic flow, the potential equation is generally preferred, since this equation is naturally second-order. In the potential formulation, one boundary condition on Σ would be simply the continuity of ϕ there, that is, a Dirichlet condition. However, the second condition, for the evolution of the free boundary, would become

$$\xi' = -\sqrt{\frac{\phi_x + \phi_{0,x}}{2}}$$

and it can be seen that no gain in regularity should be anticipated when a solution is calculated, differentiated, and then integrated again to get $J\xi$. Thus, there is no mechanism to produce compactness. We experimented with different ways of splitting the Rankine-Hugoniot condition into a boundary condition and a shock evolution condition, and the one we have chosen seems particularly useful.

Another possible choice might have been to work directly with the system of equations in u and v. However, the theory of boundary value problems for elliptic systems is not as advanced as that for second-order equations. We found it convenient to use the already developed theory for oblique derivative problems, though a theory for first-order systems might also do the trick.

The full potential equation for transonic flow does not appear to lend itself to a tidy transformation into a second-order equation for a velocity component, and we do not at present see how to prove a result for that equation. Some reformulation of the equation may be possible.

Finally, we note that considerable loss of regularity is associated with the decision to use a bounded domain: this introduces corners, and from these follow the critical bounds on regularity. However, we do not see how to solve the underlying elliptic problem on an unbounded domain; nor do we see how to produce a tractable operator J on an infinite interval. It appears to us that the only way to prove a result on the whole shock, or all of downstream space, would be to choose a compact domain and then extend it to infinity.

5.3 Acknowledgements

We thank Cathleen Morawetz and Luis Caffarelli for encouragement and for helpful discussions of transonic flow and free boundary problems. We are also grateful to our universities, Iowa State and the University of Houston, for much appreciated logistical support.

6 References

[1] H. W. Alt and L. A. Caffarelli. Existence and regularity for a minimum problem with free boundary. *Journal für die reine und angewandte Mathematik*, 325:105–144, 1981.

[2] H. Berestycki, A. Bonnet, and C. J. van Duijn. The regularity of the free boundary between two fluids in a porous medium. Preprint.

[3] L. A. Caffarelli. Free boundary problems, a survey. In M. Giaquinta, (ed), *Topics in Calculus of Variations*, pages 31–61. Springer-Verlag, Berlin, 1989.

[4] S. Čanić and B. L. Keyfitz. Quasi-one-dimensional Riemann problems and their role in self-similar two-dimensional problems. *Archive for Rational Mechanics and Analysis*, 144:233–258, 1998.

[5] J. D. Cole and L. P. Cook. *Transonic Aerodynamics*. North-Holland, New York, 1986.

[6] D. Gilbarg and L. Hörmander. Intermediate Schauder estimates. *Archive for Rational Mechanics and Analysis*, 74:297–318, 1980.

[7] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer-Verlag, New York, second edition edition, 1983.

[8] F. John. Formation of singularities in one-dimensional nonlinear wave propagation. *Communications on Pure and Applied Mathematics*, XXVII:377–405, 1974.

[9] G. M. Lieberman. The Perron process applied to oblique derivative problems. *Advances in Mathematics*, 55:161–172, 1985.

[10] G. M. Lieberman. Regularized distance and its applications. *Pacific Journal of Mathematics*, 117:329–352, 1985.

[11] G. M. Lieberman. Mixed boundary value problems for elliptic and parabolic differential equation of second order. *Journal of Mathematical Analysis and Applications*, 113:422–440, 1986.

[12] G. M. Lieberman. Nonlinear oblique boundary value problems for nonlinear elliptic equations. *Transactions of the American Mathematical Society*, 295:509–546, 1986.

[13] G. M. Lieberman. Local estimates for subsolutions and supersolutions of oblique derivative problems for general second order elliptic equations. *Transactions of the American Mathematical Society*, 304:343–353, 1987.

[14] G. M. Lieberman. Optimal Hölder regularity for mixed boundary value problems. *Journal of Mathematical Analysis and Applications*, 143:572–586, 1989.

[15] G. M. Lieberman. *Second Order Parabolic Differential Equations*. World Scientific, Singapore, 1996.

[16] D. G. Schaeffer. Supersonic flow past a nearly straight wedge. *Duke Mathematical Journal*, 43:637–670, 1976.