Weak Shock Reflection Modeled by the Unsteady Transonic Small Disturbance Equation

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1 Introduction

In this paper we report on our recent progress in the study of weak shock reflection modeled by the unsteady transonic small disturbance equation (UTSD)

$$\begin{aligned} u_t + uu_x + v_y &= 0 \\ -v_x + u_y &= 0\,. \end{aligned} \tag{1.1}$$

In contrast with our previous work [5] where we focus on shock interaction initial data, in this paper we concentrate on shock reflection initial data. Specifically, we consider here initial data that consist of an incident shock

$$S_1 : \xi = a\eta + \omega_1 \qquad \text{with} \qquad \omega_1 = \frac{1}{2} + a^2\,, \tag{1.2}$$

and "no flow through the wall" boundary condition

$$v(\xi, 0) = 0. \tag{1.3}$$

The incident shock separates the ambient (upstream) state $U_0 = (0,0)$ from the state $U_1 = (1, -a)$ behind the incident shock. The parameter a is the slope of the incident shock wave. This model was suggested in [3, 11] as an asymptotic description of the transition between Mach reflection and regular reflection for weak shocks. A numerical comparison, see [13], between the solutions of the UTSD equation and the solutions of the compressible Euler equations for weak shocks shows the same solutions structures for both equations in the region where the incident and the reflected waves interact. The motivation for the studies in [3, 11, 13] was the resolution of the so called von Neumann paradox. This refers to a weak shock reflection configuration, von Neumann

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reflection, that closely resembles Mach reflection, but for which shock polar analysis shows that simple Mach reflection is impossible [2, 10].

There are now numerical studies of solution structures for (1.1) in the parameter regimes that are believed to correspond to von Neumann reflection [3, 11, 13]. In [11] a study of the solution with $a = 0.5$ was presented. The study in [11] presented a shock polar analysis for the linearized equation and a high-resolution numerical solution showing a supersonic bubble behind the reflected wave. However, global existence of solutions to the boundary value problem for the UTSD has not yet been proved.

This paper presents an overview of the one-parameter family of initial value problems described by (1.1), (1.2) and (1.3) for the entire range of the parameter a. We suggest that this problem allows two qualitatively different classes of solutions. One is the well accepted *regular reflection* solution occurring for $a \geq \sqrt{2}$, and the other is a *complex reflection* occurring for $0 < a < \sqrt{2}$. We pose the corresponding free-boundary problems for the positions of the transonic shocks in each class, and use [7, 8] to suggest a strategy for proof of the global existence of a solution in the case of regular reflection. We conjecture solution structures for the complex reflection solution, basing our arguments on the well-posedness of the corresponding free-boundary problems, on shock polar analysis and on numerical simulations using the software developed in [9].

In addition, we compare our results with the solution found in [11].

2 Background

We study self-similar solutions, *i.e.*, solutions which depend on $\xi = x/t$, $\eta = y/t$. The governing equations (1.1) in self-similar coordinates read

$$\begin{aligned} (u-\xi)u_\xi - \eta u_\eta + v_\eta &= 0 \\ -v_\xi + u_\eta &= 0\,. \end{aligned} \tag{2.1}$$

The Rankine-Hugoniot equations are

$$\frac{d\xi}{d\eta} = -\frac{[v]}{[u]} = \frac{[\frac{1}{2}u^2 - \xi u]}{[v - \eta u]}\,. \tag{2.2}$$

Locally, at a state $U = (u, v)$, the equations are elliptic if $u > \rho = \xi + \eta^2/4$, degenerate if $u = \rho$ and hyperbolic if $u < \rho$. The characteristics in the hyperbolic region are straight lines (when the equation is linearized about a constant state) given by

$$\xi - \xi_0 + \left(\frac{\eta_0}{2} \pm \sqrt{\frac{\eta_0^2}{4} + \xi_0 - u}\right)(\eta - \eta_0) = 0. \tag{2.3}$$

They are tangent to the parabolas $P_u \equiv \{\xi + \eta^2/4 = u\}$. (The parabola $\rho = \xi + \eta^2/4 = u$ is the equivalent of the sonic circle in the full set of Euler equations for gas dynamics.)

It is convenient to work in the coordinate system (ρ, η); the equations become

$$\begin{aligned} (u-\rho)u_\rho - \tfrac{\eta}{2}u_\eta + v_\eta &= 0 \\ \tfrac{\eta}{2}u_\rho - v_\rho + u_\eta &= 0\,. \end{aligned} \tag{2.4}$$

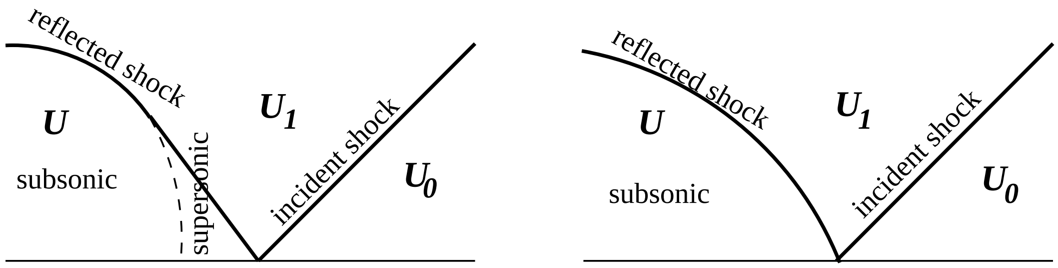

FIGURE 3.1: Supersonic and Transonic Regular Reflection

Eliminating v yields the second-order equation

$$Q(u) \equiv \left((u-\rho)u_\rho + \frac{u}{2}\right)_\rho + u_{\eta\eta} = 0\,. \tag{2.5}$$

In the subsonic region, $u > \rho$, the equations are elliptic, with ellipticity ratio

$$\frac{\Lambda}{\lambda} = \frac{\max\{u-\rho, 1\}}{\min\{u-\rho, 1\}}.$$

Since the region where the UTSD equations are elliptic is unbounded, we introduce a cut-off boundary σ and impose far-field conditions there. To justify the far-field conditions, in [7] we develop an asymptotic expansion in which the reflected shock S_2 approaches the sonic parabola P_1. More precisely, in [7] we prove

PROPOSITION 2.1 *The behavior $\xi(\eta) = -\frac{\eta^2}{4} + 1 + b(\eta)$ for the position of the reflected transonic shock S_2 as $\eta \to \infty$, with $b(\eta) \to 0$ as $\eta \to \infty$, is consistent with the Rankine-Hugoniot conditions.*

3 THE REGULAR REFLECTION SOLUTION ($a \geq \sqrt{2}$)

The regular reflection configuration describes a solution of the initial-value problem (1.1), (1.2) and (1.3) which consists of two shocks, the incident shock S_1 and a reflected shock S_2, meeting at a point on the wall $\Xi_a = \left(\frac{1}{2} + a^2, 0\right) = (\xi_a, 0)$. Shock polar analysis shows that regular reflection can take place only if $a \geq \sqrt{2}$ (we repeat this calculation in [5]). If $a \geq \sqrt{2}$ there are two possible reflected states at the reflection point:

$$U_R = (1 + a^2 - a\sqrt{a^2-2}, 0) \quad \text{and} \quad U_F = (1 + a^2 + a\sqrt{a^2-2}, 0). \tag{3.1}$$

The larger value U_F is always subsonic, and U_R is also subsonic if

$$a < a^* = \sqrt{1 + \frac{\sqrt{5}}{2}} \approx 1.45.$$

The larger value U_F is known as "strong" and the other, U_R, as "weak" shock reflection. Ben-Dor, [1, page 21], calculates weak and strong regular reflections for the compressible Euler equations, and notes that the tangency at the shock polar (where the two states coalesce) is not at the sonic point, so the same phenomenon, of a small range of incident shock angles in which both weak and strong reflections are subsonic, occurs also in the full model. Neither of these solutions can be discarded on theoretical grounds [1, 12]. However, it is noted in [1] that, experimentally, only the weak shock reflection has been observed.

We pose these problems as free-boundary problems for the positions of the transonic shocks.

3.1 Transonic Regular Reflection

In transonic regular reflection the state immediately behind the reflected wave is subsonic. It was remarked in the previous subsection that this is the case for all values of $a \geq \sqrt{2}$ when strong reflection, $U = U_F$ in (3.1), occurs. For $\sqrt{2} \leq a \leq a^*$ both strong and weak reflection are of this type. By causality, if the state immediately behind the shock is subsonic, then it remains so, and the reflected shock remains transonic. If the horizontal flow velocity is also bounded, by $u_{\max}$ say, then the shock cannot cross the sonic parabola corresponding to $u_{\max}$, $\xi + \eta^2/4 = u_{\max}$, and hence is curved. The ellipticity ratio in this case is bounded, and the equations (2.4) or (2.5) are strictly elliptic.

We formulate the free-boundary problem in terms of the first velocity component u and recover v from the system of equations (2.1). To work only with u we eliminate the second velocity component v from the Rankine-Hugoniot conditions (2.2). This can be done in several different ways. To prove the existence of a solution by using a fixed point theorem, it is crucial to eliminate v in such a way to produce an oblique derivative boundary condition for u across the transonic shock S_2. See [7] or [8] for more details. The domain Ω in which the flow is subsonic is bounded by the reflected shock S_2, by the wall $\eta = 0$, and by the cut-off boundary σ. The free-boundary problem for the position of S_2 is as follows: Find a function u and a free boundary $\rho = \rho(\eta)$ so that

$$
\begin{gathered}
Q(u) \equiv \left((u-\rho)u_\rho + \tfrac{u}{2}\right)_\rho + u_{\eta\eta} = 0 \text{ in } \Omega, \\
\frac{d\rho}{d\eta} = -\sqrt{\rho - \frac{u+1}{2}} \quad \text{and} \quad N(u;\rho) \equiv \beta \cdot \nabla u = 0 \text{ on } S_2 \equiv \{\rho = \rho(\eta)\}, \\
u_\eta = 0 \text{ on } \eta = 0, \quad u = f \text{ on } \sigma, \quad u(\Xi_a) = u_R, \quad \rho(0) = \xi_a.
\end{gathered}
$$

Here

$$
\beta = (\beta^1, \beta^2) = \left(\rho' \left[\frac{7u+1}{8} - \rho\right], \frac{5u+3}{8} - \rho\right). \tag{3.2}
$$

This is a free-boundary problem for a quasilinear equation. In [7] we prove the existence of a *global* solution of a modified problem where we introduce a cut-off function $g(\rho - \frac{u+1}{2})$ to keep the term under the square root non-negative. We have been able to prove, so far, that the term $\rho - \frac{u+1}{2}$ is non-negative in a finite subdomain around the reflection point. Therefore, in [7] we prove

Theorem 3.1 *The elliptic free boundary problem above has a solution in a domain of finite size around the reflection point.*

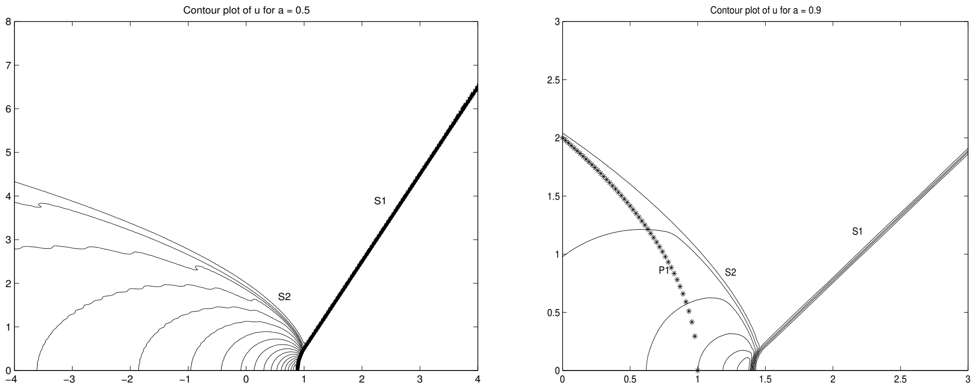

FIGURE 4.1: Complex reflection for $a = 0.5$ (left) and $a = 0.9$ (right)

3.2 SUPERSONIC REGULAR REFLECTION

If $a > a^*$ the state U behind the reflected shock can be either supersonic ($U = U_R$) or subsonic ($U = U_F$). Here we pose a free boundary problem corresponding to weak reflection, in which case $U = U_R$. The solution has the constant value U_R in a region behind the reflected shock, but becomes subsonic at the parabola P_R; the solution is nonconstant and the reflected shock is transonic and curved beyond P_R; see Figure 3.1.

The governing equation (2.5) is degenerate at the sonic curve P_R. We consider a domain Ω bounded by the transonic shock S_2, by the degenerate parabola P_R, by the wall $\eta = 0$ and by the cut-off boundary σ. Equation (2.5) holds in the interior of the domain. The boundary conditions that hold at the free boundary S_2 are again $N(u;\rho) \equiv \beta \cdot \nabla u = 0$, where β is defined in (3.2), and $\frac{d\rho}{d\eta} = -\sqrt{\rho - (u+1)/2}$. The boundary condition at the sonic curve P_R is $u = \rho_R = u_R$. The sonic parabola P_R and the reflected shock S_2 meet at a fixed (known) point. The condition at the wall is $u_\eta(\xi, 0) = 0$ and the far field condition is $u = f$ on σ. This free boundary problem is different from the previous one in that the equation $Q(u) = 0$ is degenerate on P_R. Our preliminary results [6] indicate that techniques extending [7], based on fixed point theorems, elliptic regularization and upper and lower solutions will prove the existence of a solution to this problem.

4 THE COMPLEX REFLECTION SOLUTION ($0 < a < \sqrt{2}$)

When $a < \sqrt{2}$ regular reflection at the wall is not possible. The structure of the global solution depends on the reflected wave, on its interaction with the incident shock, and on the subsonic flow behind the reflected wave. If the reflected wave travels faster than the incident shock, it will interact with the incident shock at a point above the wall, Ξ_T. Our numerical simulations indicate that this is the case for all $a \in (0, \sqrt{2})$. Figure 4.1 shows solution for the values $a = 0.5$ and $a = 0.9$.

To formulate the global problem we pose a quasi-one-dimensional Riemann problem at Ξ_T and solve it locally using the generalized Lax entropy condition [4, 5] to identify admissible shock waves. Then we pose a free boundary problem to complete the solution in the rest of the domain.

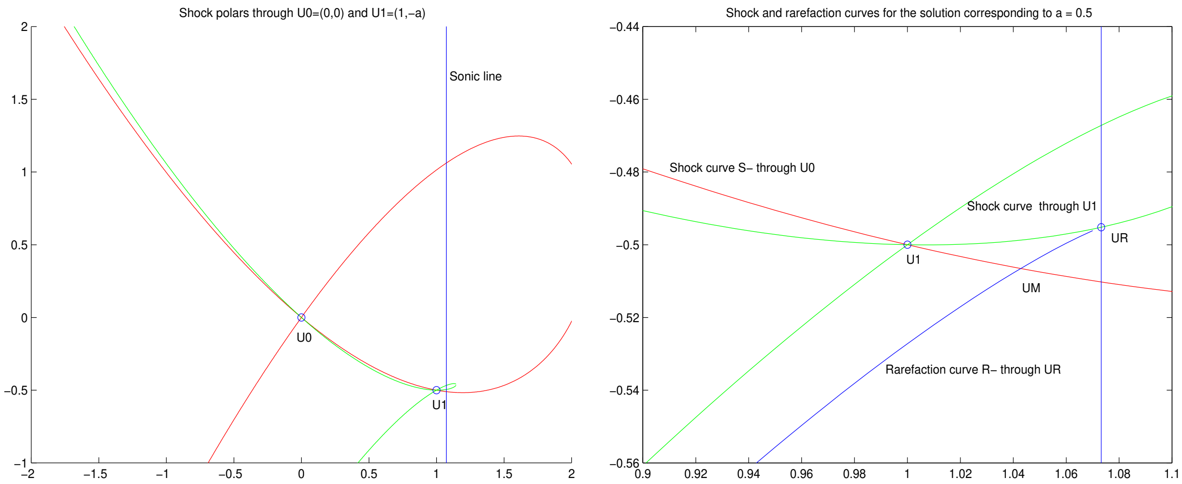

FIGURE 4.2: The figure on the left shows the shock loci through U_0 and U_1 for $a = 0.5$ with $\Xi_T = (1.007, 0.514)$. The figure on the right shows a magnified view of the shock loci in a neighborhood of $U_1 = (1, -a)$. The shock locus through U_0 and the rarefaction locus through U_R intersect at U_M.

The local solution at the interaction point is given by the following theorem.

THEOREM 4.1 *Let $0 < a < \sqrt{2}$ and let $\Xi_T = (\xi_T, \eta_T) \in S_1$ be given, with $\eta_T > 0$ and $\rho_T = \xi_T + \eta_T^2/4 > 1$. Then there exists a unique sonic state U_R that belongs to the shock locus of $U_1 = (1, -a)$ and is such that the quasi-one-dimensional Riemann problem for $U_0 = (0, 0)$ and U_R has an admissible solution consisting of a (minus)-rarefaction wave from U_R to some U_M and a (plus)-shock from U_M to U_0. This solution is unique.*

The terms minus and plus refer to the sign in equation (2.3). A typical set of shock and rarefaction loci is shown in Figure 4.2. Although the interaction point Ξ_T lies outside the parabola P_1 (since U_1 must be supersonic at Ξ_T), we do not have an estimate for the position of Ξ_T.

A sketch of the solution in the physical space (ξ, η) is shown in Figure 4.3. In this

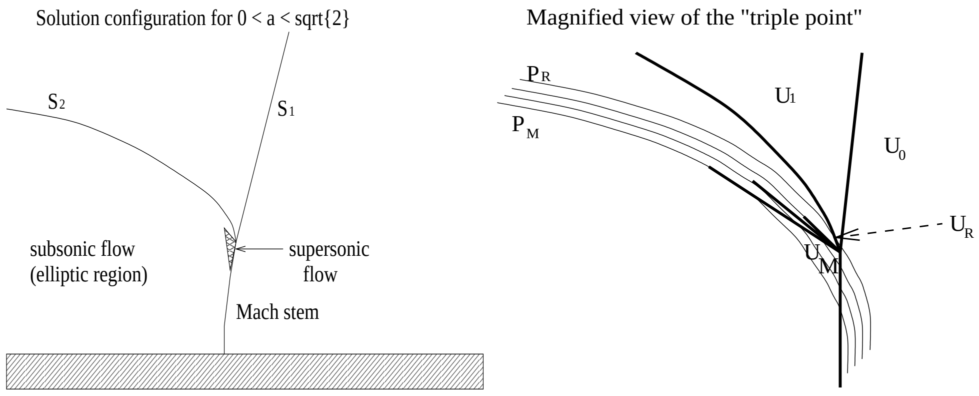

FIGURE 4.3: Sketch of the solution structure for $0 < a < \sqrt{2}$.

figure we also show the sonic parabolas P_u for $u_M < u < u_R$. For a fixed sonic state

U_R there is a unique space-like line L, which is tangent to the parabola P_R at Ξ_T [5]. The solution of the quasi-one-dimensional Riemann problem, with data U_0 and U_R prescribed on L at Ξ_T, consists of a rarefaction wave between U_R and U_M, and a shock between U_0 and U_M. The states in the rarefaction wave become sonic at the parabolas P_{u}. The locus of the sonic points is given by the equation

$$P_{\mathrm{MR}} \quad \equiv \left\{ \quad \xi - \xi_T = \frac{\eta}{2}(\eta - \eta_T) \right\}, \tag{4.1}$$

where (ξ_T, η_T) are the coordinates of the interaction point Ξ_T [5]. Near Ξ_T, the value of u behind the reflected shock S_2 increases away from the interaction point so that $u > \rho$ everywhere behind S_2 except at Ξ_T where $u = \rho$. That is, the shock S_2 is transonic everywhere except at Ξ_T. The domain in which the flow is subsonic is bounded by the Mach stem, MS, by the sonic parabola P_M defined by $\xi + \eta^2/4 = u_M$, by the sonic parabola P_{MR}, and by the reflected shock wave, S_2. See Figure 4.4. Written in

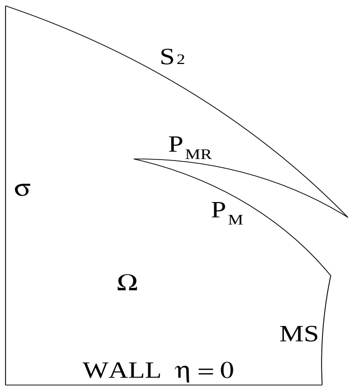

FIGURE 4.4: The domain of subsonic flow in complex reflection

(ρ, η) coordinates, after eliminating the second velocity component v, the free boundary problem for the position of the two transonic shocks S_2 and MS is the following: Find a solution u and the positions of the two free boundaries $\rho = \rho_1(\eta)$, $\rho = \rho_2(\eta)$ so that

$$\begin{gathered} Q(u) \equiv \left((u-\rho)u_\rho + \tfrac{u}{2}\right)_\rho + u_{\eta\eta} = 0 \text{ in } \Omega, \\ \left(\frac{d\rho}{d\eta}\right)^2 = \rho - \frac{u+1}{2} \quad \text{and} \quad N_1(u;\rho) \equiv \beta_1 \cdot \nabla u = 0 \text{ on } S_2 \equiv \{\rho = \rho_1(\eta)\}, \\ \frac{d\rho}{d\eta} = \sqrt{\rho - \frac{u}{2}} \quad \text{and} \quad N_2(u;\rho) \equiv \beta_2 \cdot \nabla u = 0 \text{ on } MS \equiv \{\rho = \rho_2(\eta)\}, \\ u_\eta = 0 \quad \text{on} \quad \eta = 0, \quad u = \rho \text{ on } P_{\mathrm{MR}}, \quad u = \rho_M \text{ on } P_{\mathrm{M}}, \quad \text{and} \quad u = f \text{ on } \sigma. \end{gathered}$$

Here, the coefficients β_1 in the oblique derivative boundary operator $N_1(u;\rho)$ are defined by (3.2). The coefficients β_2 in the the oblique derivative boundary operator $N_2(u;\rho)$ are

$$\beta_2 = \left(\rho' \left[\frac{7u}{8} - \rho\right], \frac{5u}{8} - \rho\right). \tag{4.2}$$

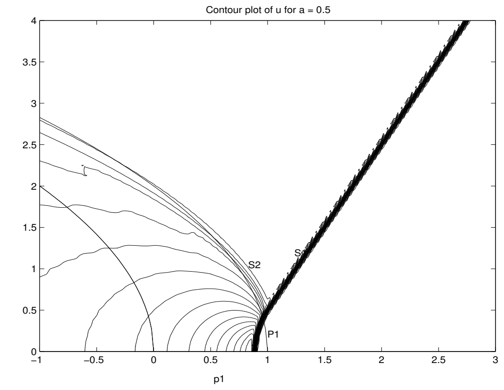

Figure 4.5: Contour plot of u for $a = 0.5$ showing that the reflected wave meets with the incident shock S_1 **outside** of parabola P_1

We recover v from the PDE by integration. The operator Q in this case is degenerate at the portion of the boundary determined by the curves P_{MR} and P_M. Assuming Ξ_T is given, we foresee the following three new difficulties in this problem. First, the problem is posed on a non-convex domain. This is relevant when analyzing smoothness of the solution up to the degenerate boundary. Second, the problem consists of two free boundaries which will require proving existence of a fixed point involving two free-boundary mappings. Third, one boundary condition on the free boundary corresponding to the Mach stem loses uniform obliqueness at the point where it touches the wall; the boundary condition on S_2 also fails to be uniformly oblique, since β_2 is tangent to S_2 at the point where ρ achieves its maximum value. The first problem can be dealt with by an appropriate choice of sub- and super-solutions. The second difficulty requires synchronized estimates for the two free boundaries. The last difficulty seems to be more serious and is a subject of our current research. Finally, we must resolve the difficulty that we expect a solution to this problem to exist for only one value of Ξ_T and we must determine this value.

4.1 Conclusions

We conclude this paper by a comparison between the complex reflection solution conjectured in this paper and the corresponding solution conjectured in [11].

In [11] a solution corresponding to $a = 0.5$ was studied. It was conjectured that the solution consists of an incident shock S_1, a reflected wave S_2, a rarefaction wave and a Mach stem, all meeting at a point Ξ_T above the wall. The difference between the solution conjectured in [11] and the solution conjectured here is that in this paper we argue that the state immediately behind the reflected shock wave has to be sonic, and not supersonic, as suggested in [11]. There are two reasons why we believe this to be the case. First, the solution suggested in [11] apparently violates causality, whereas the solution conjectured in this paper does not. Second, if the state immediately behind the reflected wave, call it U_R again, is supersonic, then the flow U would have to be constant ($U = U_R$) until it becomes subsonic, and the shock between U_1 and U, S_2, would have to be straight. In order for the shock S_2 to become curved it would have to

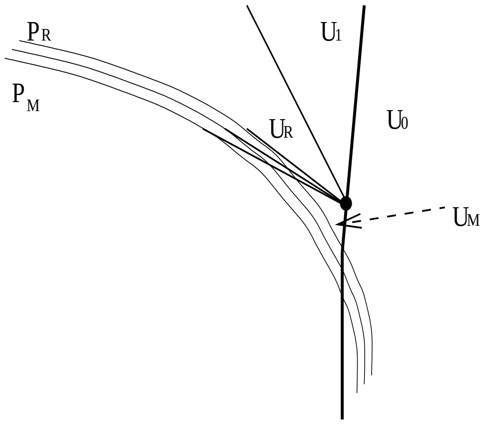

Figure 4.6: Sketch of the solution with supersonic U_R.

become transonic. This would require the state U behind the shock to become subsonic, and this can happen only beyond the parabola P_R. However, P_R lies below the (plus) shock S_2, and so this cannot happen. See Figure 4.6. Therefore, S_2 would have to be a straight line all the way to infinity, and this is not what is seen in numerical simulations, see Figure 4.1, nor is this supported by the asymptotic analysis in [7]. Shown in Figure 4.6 is a sketch of the solution when the state U_R immediately behind the reflected wave is supersonic. The sketch shows the corresponding parabolas P_u, $u_M \leq u \leq u_R$ which all lie below the reflected wave S_2.

To support the conjecture presented in this paper, a global existence result for the free-boundary problem described in Section 4 is needed. This is a subject of our current and future research.

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