Oblique shock interactions and the von Neumann paradox

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Abstract: We present a new theoretical approach to studying weak shock reflection in regimes where shock polar analysis is insufficient. The approach is based on examining the subsonic flow using monotone operator techniques. Our results reveal complicated flow structures in the zone of interaction of waves. Results predicting the region of high velocity gradients in the von Neumann reflection are reported. The precise type and the position of the singularity are described.

Key words: Riemann problem, Shock interactions, von Neumann reflection, Transonic shocks

1. Introduction

When a weak shock wave hits a wedge, different reflection patterns occur depending on the shock strength and the wedge angle. The type of shock interaction that occurs and the transition criteria between different types of reflection have been the subjects of many numerical and experimental studies. The most common theoretical approach is based on shock polar analysis. For certain small Mach numbers and small wedge angles this analysis fails to predict the type of reflection that takes place (the "von Neumann paradox"). The main reason for the inadequacy of this approach is that it assumes that the flow downstream of the reflected wave and near the point of interaction of waves is uniform. However, numerical and experimental studies suggest that in place of a triple–shock configuration dividing the region of uniform flow, as in the case of Mach reflection, for a certain range of parameters, the reflected wave contains a region with high velocity gradients corresponding to a compression wave near the Mach stem (von Neumann reflection).

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For this reason we have taken a different approach in order to understand the singularities in the equations that govern the flow, and to obtain a bifurcation diagram corresponding to different types of reflection. Besides shock polar analysis we have used monotone operator techniques to study the subsonic flow in the neighborhood of wave interactions. Equations that approximate the flow in the interaction zone have been obtained (Brio & Hunter (1992), Tabak & Rosales (1994)) from the compressible gas dynamics equations. The asymptotic equation is the unsteady transonic small disturbance equation, also known as the 2-D Burgers equation:

$$\begin{aligned} u_t + uu_x + v_y &= 0, \\ -v_x + u_y &= 0. \end{aligned} \tag{1}$$

This equation was also found (Morawetz (1994)) to be an approximation to the full potential equation in the neighborhood of wave interactions. Here u and v denote scaled, normalized velocity components.

Instead of modeling the initial–value problem directly, we study a *two-parameter family of Riemann problems* associated with shock reflection. The initial data consist of two oblique shock waves separating constant states. The two shock waves correspond to the incident shock, $S_1 : x = ay$, and the reflected wave, $S_2 : x = -by$, at time $t = 0$ when the incident shock hits the wedge. The two parameters a and b are related to the angle of the wedge and to the strength of the incident shock. In principle, the solution to the shock–reflection problem could be recovered from this Riemann problem by determining the parameter b from the flow angle a and the conditions at infinity. The uniform states separated by S_1 and S_2 satisfy Rankine-Hugoniot conditions across the shocks, so these are *oblique–shock Riemann data.* In scaling the equations, one finds that the initial states can be given by $U_0 = (0, 0)$, $U_1 = (1, -a)$, $U_2 = (1 + a/b, 0)$ (see Brio & Hunter (1992)). Propagation of the singularity at the origin at $t = 0$ (the point where the two shocks meet) corresponds to a corner signal which gives rise to a region of subsonic flow.

We study self-similar solutions of this initial–value problem. The system of equations (1) written in self-similar coordinates changes type from hyperbolic to elliptic, corresponding to the regions where the flow is supersonic or subsonic. For $t > 0$, the second shock wave, S_2, either overtakes the incident shock, S_1, or the incident shock travels faster than the reflected wave, depending on the values of a and b. Now, the corner signal, which gives rise to a region where the flow is subsonic and therefore not necessarily uniform, may interfere with the reflected and/or the incident shock wave. In order to describe the flow for all choices of a and b we

(1) study hyperbolic wave interactions,

(2) study interaction of shock waves with the subsonic region, and

(3) obtain a bifurcation diagram in (a,b)-space describing different types of solutions, based on the type of the hyperbolic interaction and on the free-boundary solution.

In Section 2 we summarize results (Canic & Keyfitz (1995b)) on hyperbolic wave interactions relevant to the study of von Neumann reflection. In Section 3 we present results about the subsonic flow that suggest new, interesting structures in the region of wave interaction. Based on the theory given in Sections 2 and 3 we conjecture the structure of the flow in von Neumann reflection. The bifurcation diagram describing other types of reflection can be found in Canic et al. (1994).

2. Hyperbolic wave interactions

In self-similar coordinates ($\xi = x/t, \eta = y/t$) equation (1) becomes

$$\begin{aligned} (u-\xi)u_\xi - \eta u_\eta + v_\eta &= 0, \\ -v_\xi + u_\eta &= 0. \end{aligned} \tag{2}$$

This system is hyperbolic outside and elliptic inside the parabola $P_u: \quad \xi + \eta^2/4 = u$. This parabola replaces the sonic circle of the Euler equations of compressible gas flow. Finding a self-similar solution of the initial–value problem described above reduces to solving a boundary–value problem in physical space (ξ, η), with data given on a curve sufficiently far from the origin (corresponding to data at infinity at $t = 0$). Because the coefficients of system (2) depend explicitly on ξ and η, this system is not standard for the hyperbolic theory of conservation laws. For example, the characteristics in the hyperbolic region, linearized about a constant state, are straight lines tangent to the sonic parabola.

Hyperbolic wave interactions occur for values of a and b for which either the two shock waves, S_1 and S_2, intersect in the hyperbolic region, or the incident shock meets the wall before interacting with either S_2 or the subsonic region. In these situations, new waves are generated as solutions of a quasi-one-dimensional Riemann problem based at the point of interaction, Ξ_I. In Canic & Keyfitz (1995b) we prove the following:

Theorem 1 *Given Riemann data (U_0, U_2) at Ξ_I with hyperbolic states U_0 and U_2, let $D(U_0)$ be the downstream wave locus of U_0. Then there are three regions, R_1, R_2 and R_3, in the complement of $D(U_0)$ such that if U_2 belongs to any of the three regions, there exists a unique Riemann solution between U_0 and U_2, qualitatively different for each region. If U_2 belongs to the complement of $R_1 \cup R_2 \cup R_3$, there may be none, one, or two admissible solutions.*

More details about the structure of quasi-one-dimensional Riemann solutions can be found in Canic & Keyfitz (1995b) and Canic et al. (1994).

3. Shock interaction with the subsonic region

In order to understand the interaction of a shock with the subsonic region we need to solve a free-boundary problem for a degenerate elliptic equation (2), where the position of a transonic shock is determined by the Rankine-Hugoniot conditions

$$\begin{aligned} \xi'(\eta) &= -\eta/2 \pm \sqrt{\xi + \eta^2/4 - (u+u_0)/2}, \\ \xi'(\eta) &= (v_0 - v)/(u_0 - u), \end{aligned} \tag{3}$$

with $\xi(\eta_*) = \xi_*$ corresponding the point where the shock meets the sonic curve; $\xi = \xi(\eta)$ is the position of the transonic shock, and u and v are the limits of the velocity components along the transonic shock (see Fig. 1a).

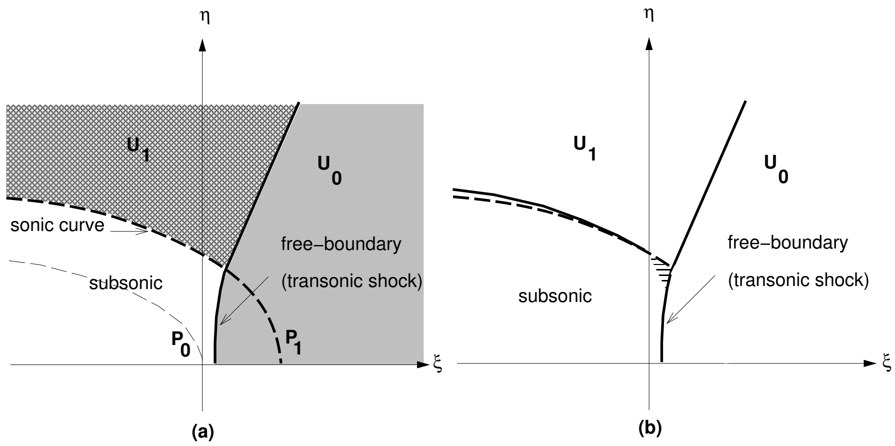

Figure 1. Subsonic boundary-value problems.

In an attempt to capture possible singularities, we first studied a fixed boundary–value problem based on an approximate position of the transonic shock (Canic & Keyfitz (1994), Canic & Keyfitz (1995a)). For this, it is sufficient to solve a second–order, quasilinear, degenerate elliptic PDE obtained from (2):

$$\left((u-\rho)u_\rho + \frac{u}{2}\right)_\rho + u_{\eta\eta} = 0, \quad \text{where} \quad \rho = \xi + \frac{\eta^2}{4}, \tag{4}$$

and then recover v using the fact that $v = 0$ at the wall. Our results reveal interesting solution structures, as follows.

• The *asymptotic behavior* of a solution is described by the following theorem, proved in Canic & Keyfitz (1994).

Theorem 2 *If u is constant along the sonic boundary, then there are exactly two possibilities for the asymptotic behavior of a solution at the sonic boundary. Either $u \in C^1$ up to the boundary, and $u_\rho = 1/2$, or $u \in C^\alpha$, where $\alpha \leq 1/2$, and $u = \sqrt{\rho_M - \rho}\ w$, where $w \neq 0$ at the sonic line, $\rho = \rho_M$.*

• *Global existence* of a subsonic solution for an approximate position of the Mach stem is proved in Canic & Keyfitz (1994) and Canic & Keyfitz (1995a). Using monotone operator techniques adapted for this degenerate elliptic equation we proved that the problem allows two qualitatively different types of solutions, depending on the type of boundary data given along the transonic shock. One solution is *smooth* up to the boundary, another has a *square-root singularity* along the sonic line.

• Another interesting feature of our solution is that the regular subsonic solution (u, v) matches smoothly with the supersonic region, while the singular solution does not (Canic & Keyfitz (1995a)). The singular case would require a weak shock wave emerging from the square–root singularity to match the subsonic with the supersonic flow (see Fig. 1b). The existence of a solution of this free-boundary problem has not yet been proved.

• Finally, the *curvature* of the Mach stem is determined by the type of boundary data along the stem. Namely, the first Rankine-Hugoniot condition implies

$$u' = 2\rho'(1 - 2\rho''), \quad \text{where} \quad \rho(\eta) = \xi(\eta) + \eta^2/4. \tag{5}$$

Here u' denotes the derivative of u along the transonic shock $\xi = \xi(\eta)$. Therefore, if the shock steepens monotonically to vertical at the axis, (i.e., $\xi' > 0$), then the shock is convex, (i.e., $\xi'' > 0$ or $\rho'' > 1/2$), if and only if $u' < 0$. Moreover, the higher the gradient of u, the bigger the curvature of the free boundary. Therefore, in order to have a monotonically increasing, convex Mach stem, as observed in the experiments, the data for the approximate free boundary has to be decreasing. By Theorem 2, the solution u must then have a square–root singularity at the point where the Mach stem meets the sonic curve. This argument proves the following:

Theorem 3 *Assume that the incident shock S_1 does not interact with another shock before meeting the sonic curve, and that the Mach stem $\xi = \xi(\eta)$ is an increasing, convex function. Then the subsonic solution must have a square–root singularity at the point where the Mach stem crosses the sonic curve.*

3.1. A conjecture for von Neumann reflection

We conjecture that the following configuration of S_1 and S_2 leads to von Neumann reflection. The incident shock S_1 between the uniform states U_0 and U_1 interacts with the region where U_1 becomes subsonic, giving rise to a free-boundary problem with a convex Mach stem. The flow at the sonic curve has a square-root singularity, which, in turn, gives rise to a weak shock emerging from the singular region. The shock S_2, separating the states U_1 and U_2, interacts with the region where U_2 becomes subsonic, P_2, leading to a free-boundary problem for another transonic shock. The shock generated by the square-root singularity catches up with the transonic shock continuing from S_2 giving rise to the flow pattern shown in Fig. 2.

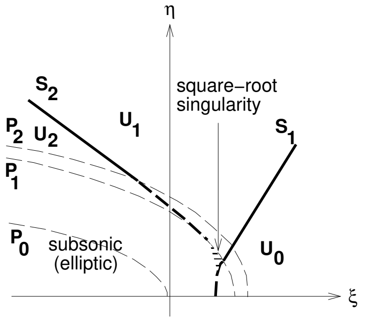

Figure 2. Conjectured structure for von Neumann reflection.

4. Final Remarks

Our results about the square-root behavior of the horizontal velocity component, u, seem to agree with the numerical results described in Tabak & Rosales (1995). Their numerical simulations indicate that the vertical velocity component, v, exhibits a stronger singularity. In the course of our study to prove the existence of a solution of the free-boundary problem we have come across some indications that this might indeed be the case. Nevertheless, this problem is still open.

References

Brio M, Hunter JK (1992) Mach reflection for the two-dimensional Burgers equation. Physica D 60: 194–207.

Canic S, Keyfitz BL (1994) An elliptic problem arising from the unsteady transonic small disturbance equation. J Diff Equations, to appear.

Canic S, Keyfitz BL (1995a) A smooth solution for a keldysh type equation. Comm PDE, submitted.

Canic S, Keyfitz BL (1995b) Quasi-one-dimensional Riemann problems and their role in self-similar two-dimensional problems. In preparation.

Canic S, Keyfitz BL, Wagner DH (1994) A bifurcation diagram for oblique shock interactions in the unsteady transonic small disturbance equation. In: Proc 5th International Conference on Hyperbolic Problems, Theory, Numerics and Applications, to appear.

Morawetz CS (1994) Potential theory for regular and Mach reflection of a shock at a wedge. Comm Pure Appl Math XLVII: 593–624.

Tabak EG, Rosales RR (1994) Weak shock focusing and the von Neumann paradox. Physics of Fluids A, to appear.

Tabak EG, Rosales RR (1995) private communication.