A Smooth Solution for a Keldysh Type Equation

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Abstract

We solve the Dirichlet problem for a nonlinear degenerate elliptic equation that arises in modeling weak shock reflection at a wedge. The equation exhibits a nonlinear version of a degeneracy first studied by Keldysh. Using monotone operator techniques, we prove existence of a weak solution in a weighted Sobolev space. For negative boundary data, the solution is smooth at the degenerate boundary. By contrast, we showed in [4] that positive boundary data lead to solutions with unbounded gradients at the degenerate boundary.

Contents

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1 A Nonlinear Keldysh Equation

In this paper we show existence of a weak solution of a boundary value problem for the equation

$$\left((u+x)u_x - \frac{u}{2}\right)_x + u_{yy} = 0\,, \tag{1}$$

in the domain Ω shown in Figure 1. The boundary data that we consider are given by

$$u = g \qquad \text{on} \qquad \partial\Omega\,, \tag{2}$$

where $g = 0$ on Γ_2. The fact that $g = 0$ on Γ_2 implies that equation (1) is degenerate along Γ_2.

In an earlier paper, we solved this problem for data which were positive on the remaining part of the boundary (which is thus nondegenerate), and we found that the solution had a singularity at the degenerate boundary, in the sense that u approached zero at Γ_2 like the square root of the distance from the boundary. In this paper, we assume instead that the boundary data along $\partial\Omega$ is nonpositive, but bounded below by a function of the form $-x/(1+\sqrt{1+Cx})$, where C is a negative constant that depends on the size of Ω, chosen so that $\partial\Omega - \Gamma_2$ is still nondegenerate. We seek a solution u that is elliptic in the interior of Ω, and degenerate along Γ_2. Our main conclusion is that a solution exists, and is smooth up to the boundary in this case.

The problem we study is motivated by an application to multidimensional shock interactions. Equation (1) is a reduction of the unsteady small disturbance equation,

$$\begin{aligned} u_t + u\,u_x + v_y &= 0\,, \\ -v_x + u_y &= 0\,, \end{aligned} \tag{3}$$

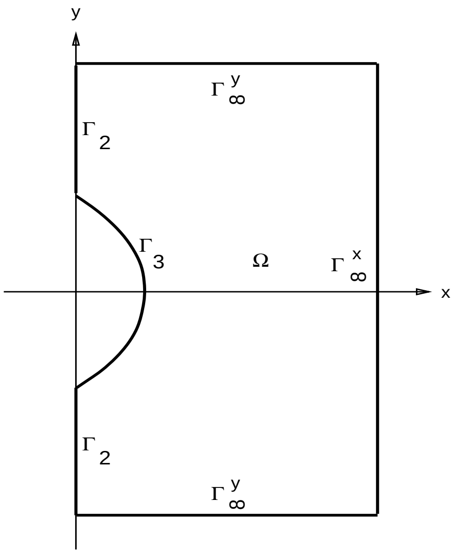

Figure 1: Domain and Boundary Components of the Dirichlet Problem

proposed by several authors to model flow arising from weak shock reflection off a wedge, [2], [15]. Brio and Hunter, [2], obtained equation (3) as an asymptotic limit of the Euler equations. Depending on the Mach number of the incident shock and the wedge angle, different interaction patterns occur. Numerical and experimental results reveal interesting flow structure which is inconsistent with the simplifying approximation of piecewise uniform flow near the interaction of waves. The regions of nonuniform flow are subsonic. The solution in the subsonic region must be completed by solving a free boundary problem for the degenerate elliptic equation and the work reported here is a step in this direction. At the end of the paper we explain why the solution found here seems more appropriate for solving some free boundary problems than does the solution in [4]. However, the free boundary problem for (1) is still open.

We have been led to problem (1) and (2) by looking for solutions of equation (3) in similarity variables $\xi = x/t$, $\eta = y/t$. We study a two-parameter family of Riemann initial data (for further details, see [5]). Now, the reduced equation changes type from hyperbolic to elliptic, according as the flow is supersonic or subsonic. The equation is degenerate along a sonic curve. In [3] we study the flow in the hyperbolic region; for some parameter values this part of the solution can be completely determined without knowledge of the subsonic flow. The flow in the subsonic region is governed by the equation

$$\left((u-\rho)u_\rho + \frac{u}{2}\right)_\rho + u_{\eta\eta} = 0, \tag{4}$$

which is obtained by eliminating v from the self-similar version of equation (3) and then applying a further change of coordinates $\rho = \xi + \eta^2/4$ to straighten the degenerate boundary. Equation (1) is obtained from (4) by letting $x = -\rho$, $y = \eta$. Equation (4) was derived by Harabetian, [8], as an inner expansion for the weak shock reflection problem.

The difficulties in solving equation (1) are caused by its nonlinearity and by the fact that its characteristic form is degenerate. The general study of linear elliptic equations degenerating on the boundary of the domain was initiated by Tricomi, [16], for a class of equations modeled by $xu_{yy} + u_{xx} = 0$. Keldysh pointed out in [9] that there are degenerate elliptic equations for which data along parts of the degenerate boundary must be prescribed, by contrast to Tricomi type equations. Keldysh's example of such an equation, now called the Keldysh equation, is

$$y^\gamma \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} + c(x,y)u = 0, \qquad \gamma > 0\,. \tag{5}$$

A systematic theory of well-posed problems for linear elliptic equations that degenerate along a part of the boundary was established by Fichera, [6]. A summary of Fichera's theory can be found in the monograph of Oleĭnik and Radkevič, [13]. We can use this theory to predict that the nonlinear equation (1) will behave like a Keldysh equation with $\gamma = 1$. Linear techniques presented in [13] and [17] motivate the choice of a weighted Sobolev space that contains weak solutions.

To shed some light on the choice of linearization and partially explain the dichotomy between positive solutions of (1), which are singular at the boundary, and negative solutions, which appear to be regular, we solve a one-dimensional version of problem (1) and (2): find a function $u(x)$ which satisfies (1) on $0 \le x \le L$, with $u(0) = 0$, $u > -x$ in $(0, L)$, and $u(L) = K$. In this case, (1) can be integrated once, to give

$$(u+x)u' - \frac{u}{2} = A,$$

and solving this as a linear equation for $x(u)$ gives

$$\frac{D}{4}u^2 + (AD - 2)(u + A) - x = 0$$

where D is another constant of integration. The boundary condition $u(0) = 0$ gives $A = 0$ or $A = 2/D$; so there are two one-parameter families of solutions which match the left boundary condition and yield a subsonic solution,

$$u(x) = \frac{2}{D}(2 - \sqrt{4 + Dx}) \quad \text{and} \quad u(x) = \frac{2}{\sqrt{D}}\sqrt{x}.$$

Representatives of these solutions are sketched in Figure 2. (We may assume $x \geq 0$, and hence $D > 0$ for the second family.) Now, applying the second boundary condition, $u(L) = K$, we see that if $K > 0$ then only the second family contributes a solution, which is

$$u = K\sqrt{\frac{x}{L}},$$

while if $K < 0$ the solution is given by the first family with

$$D = \frac{4(2K + L)}{K^2}$$

and D is positive or negative according as $K > -L/2$ or $K < -L/2$. The solution with $K > 0$ has a singularity at $x = 0$, while that with $K < 0$ is regular and has slope $-1/2$ at $x = 0$. Furthermore, the solution defined this way, for $K < -L/2$, can be written as

$$u(x) = -\frac{x}{1 + \sqrt{1 + Cx}} \tag{6}$$

with $C = (2K + L)/K^2 < 0$; it is defined for $x \leq -1/C$ and smooth for $x < -1/C$. This is the comparison function we shall use in this paper.

Nonlinear generalizations of Keldysh type equations do not appear to be amenable to any techniques in the literature on degenerate elliptic equations. One reason may be the possibility of singular solutions at the degenerate boundary, as discussed by Kohn and Nirenberg for a class of linear equations including the Keldysh equation, [10]. The standard techniques for nonlinear elliptic equations, for example the p-Laplacian, seek classical solutions by means of a fixed-point procedure, as described by Gilbarg and Trudinger,

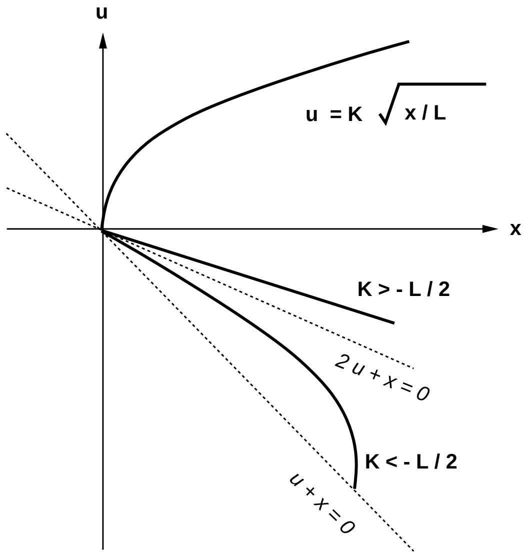

Figure 2: The Three Types of Solution of the Ordinary Differential Equation

[7]. This method requires apriori gradient bounds which cannot be obtained for equation (1). We avoid this difficulty, as we did in [4], by using monotone-operator techniques coupled with sub- and super-solutions to show existence of a weak solution in a weighted Sobolev space.

We showed in [4] that the leading order asymptotics for equation (1) admit solutions with either a square root singularity along the degenerate boundary Γ_2 or smooth solutions of the form $u(x,y) = (-\frac{1}{2} + w(x,y))x$, where $w(0,y) = 0$. For Dirichlet data with a square root singularity at Γ_2, the equation has a weak solution in the weighted Sobolev space $\mathcal{H}_{\frac{1}{2}}$ equipped with the norm

$$\|u\|_{\mathcal{H}_{\frac{1}{2}}} = \int_\Omega \sqrt{x}u_x^2 + u_y^2 .$$

In the current paper, we find a weak solution of a Dirichlet problem where the boundary data is smooth and bounded between $-x/(1+\sqrt{1+Cx})$ and 0. This lower bound is precisely the bottom curve in Figure 2, given by (6) with C chosen so that $x \leq x_0 < -1/C$ in Ω.

The outline of the paper is as follows. In the next section, we first show that $u = 0$ and $u = -x/(1+\sqrt{1+Cx})$ are super- and sub-solutions of the given boundary value problem, (2), for u. We write the problem in terms of the function $h = u - g$, where g is a smooth extension of the boundary data to the interior of Ω, and $h = 0$ on the boundary. We define a weighted Sobolev space $\mathcal{H}_0^1$, the closure of $C_0^1(\overline{\Omega})$ in the norm defined by

$$\|h\|_{\mathcal{H}_0^1} = \int_\Omega x h_x^2 + h_y^2 \,. \tag{7}$$

We obtain a weak formulation for the boundary value problem for h. We set up the problem to use the Browder-Minty theorem for pseudo-monotone operators to obtain the existence of a weak solution in the Hilbert space $\mathcal{H}_0^1$. The Browder-Minty theorem requires that the bivariate form associated with the weak formulation be bounded, continuous, pseudo-monotone, and coercive. By analogy with examples of linear degenerate elliptic equations, we expect a weighted Sobolev space with a norm like (7) to be appropriate for our problem. However, $\mathcal{H}_0^1$ contains functions whose behavior at the degenerate boundary is too singular for the weak form of the nonlinear equation to be defined. We surmount this difficulty by modifying the coefficients of the weak form outside the convex set of functions that lie between the sub- and super-solutions.

In Section 3 we define the operator $T : \mathcal{H}_0^1 \to \mathcal{H}_0^{1*}$ associated with the modified weak bivariate form. Its coefficients are cut off so that they belong to the convex set of functions which lie between the sub- and super-solutions. We define weak solutions for the modified problem in terms of the operator T.

In Section 4 we verify that T satisfies the assumptions of the Browder-Minty theorem. We show that T is continuous by using the calculus of Nemytskii operators. We show that T is pseudo-monotone by verifying that it is of calculus of variations type. The choice of the norm in $\mathcal{H}_0^1$ ensures that T is coercive. By this we establish existence of a weak solution of the modified problem.

In Section 5 we show that this solution is smooth up to the degenerate boundary, and that it satisfies the original problem defined by equations (1) and (2).

In Section 6 we explain what this contributes to the motivating problem of subsonic regions in shock interactions.

2 Weak Formulation

We study the boundary value problem (1) and (2) where g satisfies the conditions

$$\begin{aligned} g = 0 \qquad & \text{on} \qquad \Gamma_2\,, \\ g_{sub} \leq g \leq 0 \qquad & \text{on} \qquad \partial\Omega\,, \end{aligned}$$

where g_{sub} is defined by

$$g_{sub} = -\frac{x}{1+\sqrt{1+Cx}}$$

and

$$0 > C \geq -\frac{1-2\eta}{\max_\Omega\{x\}(1-\eta)^2}\,,$$

for some $\eta > 0$. Equation (1) is elliptic when $u > -x$, degenerate where $u = -x$ and hyperbolic for $u < -x$. Since $g = 0$ on Γ_2, where $x = 0$, the equation degenerates along Γ_2. With the above choice of C, the lower bound for the boundary data, g_{sub}, satisfies $g_{sub} + x \geq \eta x$. Therefore, $\partial\Omega - \Gamma_2$ is nondegenerate. The functions

$$u_{sup} = 0 \qquad \text{and} \qquad u_{sub} = -\frac{x}{1+\sqrt{1+Cx}} \tag{8}$$

are super- and sub-solutions of (1) and (2), since they satisfy the equation in Ω and $u_{sub} \leq g \leq u_{sup}$ on $\partial\Omega$. Notice that u_{sub} corresponds to the solution (6) with an appropriate choice of C. In this paper we show that there exists a strong solution of (1) and (2) which we then show is bounded between u_{sup} and u_{sub}.

Since the equation is degenerate along Γ_2 it is not apriori clear whether a boundary value problem with data imposed along the degenerate boundary will have a solution or not. We follow the approach of Oleĭnik and Radkevič in [13] to construct the Fichera function for associated linear problems to determine what kind of data defines a well-posed problem. Suppose that we have a linear operator

$$L(u) = a^{kj}(x)\,u_{x_k x_j} + b^k(x)u_{x_k} + c(x)\,u,$$

with non-negative characteristic form

$$a^{kj}(x)\xi_k\xi_j \geq 0, \quad \forall\vec{\xi} = (\xi_1, \xi_2)\,, \tag{9}$$

and that Γ_2 is the part of the boundary on which (9) is not positive definite. Let $\vec{n} = (n_1, n_2)$ be the inward normal on Γ_2. Then the function

$$b \equiv (b^k - a^{kj}_{x_j})n_k,$$

is called the *Fichera function.* It is shown in [13] that one must impose boundary data along the part of the degenerate boundary on which the Fichera function is negative. If w is a known solution of (1), we construct the linear equation

$$\left((w+x)u_x - \frac{u}{2}\right)_x + u_{yy} = 0.$$

For the solutions $w = -x/(1+\sqrt{1+Cx})$ and $w \equiv 0$, we find that $b = -1/2$ along Γ_2. This motivates consideration of the boundary conditions (2).

We rewrite the equation to obtain a homogeneous Dirichlet problem. Let $g \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be an extension of the boundary data to the interior of Ω. Then the function $h = u - g$ vanishes on the boundary and satisfies the equation

$$-\left((h+g+x)h_x + h(g_x - \frac{1}{2})\right)_x - h_{yy} = f\,, \tag{10}$$

where

$$f \equiv \left((g+x)g_x - \frac{1}{2}g\right)_x + g_{yy}\,. \tag{11}$$

To obtain a weak formulation of the Dirichlet problem, multiply (10) by a test function $w \in C^1_0(\overline{\Omega})$, and integrate by parts to obtain

$$B_g(h,w) \equiv \int_\Omega \left[(h+g+x)h_x + h(g_x - \frac{1}{2})\right] w_x + h_y w_y = \int_\Omega fw\,, \tag{12}$$

where we have called the bivariate form $B_g(h,w)$. For functions h, $w \in C^1_0(\overline{\Omega})$, introduce the scalar product

$$(h,w) = \int_\Omega x h_x w_x + h_y w_y + hw\,.$$

and complete $C^1_0(\overline{\Omega})$ in the norm defined by the scalar product. As we shall show, this norm is equivalent to (7); we define $\mathcal{H}^1_0$ to be the completion with norm (7). Equivalence of the norms was proved in [4] in the following proposition, which we restate.

Proposition 1 *Let $0 < \alpha < 2$. There exists a constant $K > 0$ such that for every $h \in C_0^1(\overline{\Omega})$ with*

$$\int_\Omega x^\alpha h_x^2 + h_y^2 + h^2 < \infty$$

the following inequality holds:

$$\int_\Omega h^2 \, dx \, dy \leq K \int_\Omega x^\alpha \, h_x^2 \, dx \, dy \, .$$

We use the bivariate form which appears in (12) to define a weak solution.

Definition 1 *For any $f \in L^2(\Omega)$, a function $h \in \mathcal{H}_0^1(\Omega)$ is called a* weak solution *of* (10) *if*

$$B_g(h, w) = \int_\Omega f w \, ,$$

for every $w \in C_0^1(\overline{\Omega})$.

To use the Browder-Minty theorem to show existence of a weak solution we would need to show that $B_g(h, w)$ is continuous, bounded, pseudo-monotone and coercive on $\mathcal{H}_0^1 \times \mathcal{H}_0^1$. However, B_g is not even defined on all of $\mathcal{H}_0^1 \times \mathcal{H}_0^1$, since $\mathcal{H}_0^1$ contains functions that are quite singular. We can obtain an operator with the desired properties by modifying $B_g(h, w)$ outside the set of functions of interest determined by sub- and super-solutions, and thus we show the existence of a weak solution of a modified problem. Using sub- and super-solutions for the modified problem we show that the weak solution so obtained satisfies the original weak form of the problem in Definition 1.

This modification of the problem mimics our approach in [4]. There, we sought a solution which was actually singular. Here, we construct a solution which is regular at the boundary. However, the problem is naturally formulated in a space which contains singular functions.

3 The Modified Problem

In this section we use knowledge of the sub- and super-solutions, given by (8), to modify the weak formulation outside the set of functions that lie between the sub- and super-solutions. Let $\tilde{h}(h) = \tilde{u} - g$ where $\tilde{u}$ is the cut-off function

$$\tilde{u} = \begin{cases} h + g & \text{if } g_{sub} \leq h + g \leq 0 \\ g_{sub} & \text{if } h + g < g_{sub} \\ 0 & \text{if } 0 < h + g \end{cases} \, . \tag{13}$$

Because $g_{sub} \leq g \leq 0$ and because C is chosen so that $g_{sub} + x \geq \eta x$, we know that $g_{sub} \geq -x(1-\eta)$ and therefore,

$$|\tilde{h}| \leq -g_{sub} \leq (1-\eta)x \,. \tag{14}$$

The modified problem for h is

$$\begin{aligned} -\left((\tilde{h}+g+x)h_x + \tilde{h}\left(g_x - \frac{1}{2}\right)\right)_x - h_{yy} &= f \quad \text{in} \quad \Omega \\ h &= 0 \quad \text{on} \quad \partial\Omega \,. \end{aligned} \tag{15}$$

This corresponds to a modified problem for u:

$$\begin{aligned} \left((\tilde{u}+x)u_x - \frac{\tilde{u}}{2}\right)_x + u_{yy} &= 0 \quad \text{in} \quad \Omega \\ u &= g \quad \text{on} \quad \partial\Omega \,. \end{aligned} \tag{16}$$

The sub- and super-solutions, (8), of the original problem are also sub- and super-solutions of (16), and the two problems coincide for all functions between the sub- and the super-solution. Therefore, a solution of problem (16) which is bounded by the sub- and super-solutions solves the original problem.

The bivariate form associated with (15) is

$$\tilde{B}_g(h,w) \equiv \int_\Omega \left[(\tilde{h}+g+x)h_x + \tilde{h}(g_x - \frac{1}{2})\right] w_x + h_y w_y$$

and we define a weak solution for the modified problem in the same way as for the original.

Definition 2 *For any $f \in L^2(\Omega)$, a function $h \in \mathcal{H}_0^1(\Omega)$ is a* weak solution *of* (15) *if*

$$\tilde{B}_g(h,w) = \int_\Omega fw \,,$$

for every $w \in C_0^1(\Omega)$.

Note that we will apply the Browder-Minty theorem to an arbitrary function $f \in L^2(\Omega)$ and then choose the function given by equation (11) to solve (1) and (2).

To show existence of a weak solution we define a nonlinear operator $T : \mathcal{H}_0^1 \to \mathcal{H}_0^{1*}$ such that $\tilde{B}_g(h,w) = \langle T(h), w\rangle$, for every $w \in \mathcal{H}_0^1$. Here the space ${\mathcal{H}_0^1}^*$ is the negative norm weighted Sobolev space dual to $\mathcal{H}_0^1$ under the pairing

$$\langle h, w\rangle = \int_\Omega hw\,.$$

To define T, we note that $w \mapsto \tilde{B}_g(h,w)$ is a linear mapping for each $h \in \mathcal{H}_0^1$. We first show that for each $h \in \mathcal{H}_0^1$ and $w \in C_0^1(\overline{\Omega})$ the mapping $w \mapsto \tilde{B}_g(h,w)$ is bounded in the norm of $\mathcal{H}_0^1$ and therefore can be extended to a bounded linear functional on $\mathcal{H}_0^1$.

Theorem 1 *For each $h \in \mathcal{H}_0^1(\Omega)$, and $w \in C_0^1(\overline{\Omega})$,*

$$|\tilde{B}_g(h,w)| \le \tilde{C}\|w\|_{\mathcal{H}_0^1}\,, \tag{17}$$

where $\tilde{C}$ depends on $\|h\|_{\mathcal{H}_0^1}$.

Proof: We use the Cauchy-Schwartz inequality:

$$\begin{aligned}
|\tilde{B}_g(h,w)| &= \left|\int_\Omega \sqrt{\tilde{u}+x}h_x\sqrt{\tilde{u}+x}w_x + \frac{\tilde{h}}{\sqrt{x}}\left(g_x - \frac{1}{2}\right)\sqrt{x}w_x + h_y w_y\right| \\
&\le \left(\int_\Omega (\tilde{u}+x)h_x^2 + \left(g_x - \frac{1}{2}\right)^2\frac{\tilde{h}^2}{x} + h_y^2\right)^{\frac{1}{2}}\left(\int_\Omega (\tilde{u}+x)w_x^2 + xw_x^2 + w_y^2\right)^{\frac{1}{2}}.
\end{aligned}$$

The bounds (14) for $\tilde{h}$ and the fact that $g \in C^1(\overline{\Omega})$ with $g_{sub} \le g \le 0$ imply

$$|\tilde{B}_g(h,w)| \le \left(D\left(\int_\Omega xh_x^2 + h_y^2\right)^{\frac{1}{2}} + K\right)\left(\int_\Omega xw_x^2 + w_y^2\right)^{\frac{1}{2}}.$$

The estimate (17) follows from this, and in fact $\tilde{C}$ is a linear function. ∎

Corollary 1 *The operator $w \mapsto \tilde{B}_g(h,w)$, defined on $C_0^1(\overline{\Omega})$, is a bounded linear functional for each $h \in \mathcal{H}_0^1(\Omega)$ and extends to a bounded linear functional on $\mathcal{H}_0^1(\Omega)$.*

The proof of Theorem 1 immediately implies the following

Corollary 2 *There exists a nonlinear, bounded mapping $T : \mathcal{H}_0^1 \to {\mathcal{H}_0^1}^*$ such that*

$$\tilde{B}_g(h,w) = \langle T(h), w\rangle\,, \tag{18}$$

for all $h \in \mathcal{H}_0^1(\Omega)$, and $w \in \mathcal{H}_0^1(\Omega)$.

We shall show that T is continuous, pseudo-monotone, and coercive and use the Browder-Minty theorem to obtain the existence of a weak solution for the modified problem $\tilde{B}_g(h,w) = \langle f, w\rangle$. Expressed in terms of T, a weak solution of the modified problem (15), by Definition 2, is a function $h \in \mathcal{H}_0^1$ such that

$$\langle T(h), w\rangle = \langle f, w\rangle\,, \qquad \forall w \in \mathcal{H}_0^1\,.$$

In the next section we investigate the properties of the mapping T and prove the existence of a weak solution of (15).

4 Properties of the Nonlinear Operator T

In this section we show that the operator T satisfies the hypotheses of the following form of the Browder-Minty theorem, as given in the monograph of Renardy and Rogers, [14].

Theorem 2 (The Browder-Minty Theorem; [14], page 367) *Let X be a real, reflexive Banach space and suppose that $T : X \to X^*$ is continuous, coercive and pseudo-monotone. Then for every $f \in X^*$ there exists a solution $u \in X$ of the equation*

$$T(u) = f\,.$$

In this version of the classical Browder-Minty theorem, monotonicity of T is replaced by a slightly relaxed condition of pseudo-monotonicity. The main difference between the two conditions is that pseudo-monotonicity requires that T be monotone only in the highest order derivatives. The lower order terms are taken care of using compactness arguments. We show below that T is of calculus of variations type which implies that T is pseudo-monotone. We show that T is continuous using Nemytskii operators. Coercivity in the norm of $\mathcal{H}_0^1$ is a straightforward calculation.

4.1 Continuity of T

We use a standard approach to show that $T : \mathcal{H}_0^1 \to \mathcal{H}_0^{1*}$ is continuous. By writing T in a particular way we recognize it as having a symbol which is a Nemytskii operator. Then Theorem 3, stated below, establishes continuity of T on $\mathcal{H}_0^1$.

For this purpose we write the operator T in the following way:

$$\langle T(h), w\rangle = \int_\Omega \left[\frac{\tilde{u}+x}{x}\sqrt{x}h_x + (g_x - \frac{1}{2})\frac{\tilde{h}}{\sqrt{x}}\right]\sqrt{x}w_x + \int_\Omega h_y w_y\,.$$

Define the weighted symbol vector

$$\omega = (\omega_1, \omega_2, \omega_3) = (\frac{h}{\sqrt{x}}, \sqrt{x}h_x, h_y)\,.$$

Then the coefficients of the symbol of T can be written as a_1 and a_2 where

$$\begin{aligned} a_1(x, y, \omega) &= \frac{(\tilde{u}+x)\omega_2}{x} + \left(g_x - \frac{1}{2}\right)\widetilde{\omega_1} \\ &= \left(\frac{\widetilde{\omega_1}}{\sqrt{x}} + \frac{g+x}{x}\right)\omega_2 + \left(g_x - \frac{1}{2}\right)\widetilde{\omega_1}\,, \\ a_2(x, y, \omega) &= \omega_3\,, \end{aligned}$$

and $\widetilde{\omega_1}$ is formed from ω_1 by applying the usual cutoffs. The bounds for $\tilde{u}$ imply that

$$\eta \le \frac{(\tilde{u}+x)}{x} \le 1\,,$$

while the bounds for $\tilde{h}$ yield

$$|\widetilde{\omega_1}| \le \sqrt{x}(1-\eta)$$

and, therefore, there is a continuous function $k(x, y)$ such that

$$\left|\left(g_x - \frac{1}{2}\right)\widetilde{\omega_1}\right| \le k(x, y)\,. \tag{19}$$

Both a_1 and a_2 are continuous in Ω and measurable for every $\omega \in \mathbf{R}^3$. Moreover, using (19), we have

$$\begin{aligned} |a_1(x, y, \omega)| &\le C_1|\omega_2| + k(x, y)\,, \\ |a_2(x, y, \omega)| &= |\omega_3|\,. \end{aligned}$$

Hence the conditions of the following theorem are satisfied for a_1 and a_2 with $p = 2$.

Theorem 3 ([12],[14]) *Let*

$$\Omega \times \mathbf{R}^3 \ni (x,y,\omega) \mapsto f(x,y,\omega) \in \mathbf{R}^3$$

satisfy the Caratheodory conditions, and

$$|f(x,y,\omega)| \leq C\,|\omega|^{p-1} + k(x,y)\,,$$

where $p \in (1,\infty)$, $k \in L^q$, $1/p + 1/q = 1$. *Then the Nemytskii operator defined by*

$$a(\omega)(x,y) \equiv f(x,y,\omega(x,y))$$

is a bounded and continuous map from L^p *to* L^q.

The continuity of $T : \mathcal{H}_0^1 \to \mathcal{H}_0^{1^*}$ now follows from the continuity of $\omega(\xi) \to a_i(\xi,\omega(\xi))$ as a map from $L^2(\Omega)$ to $L^2(\Omega)$. Therefore, we have proved

Theorem 4 *The operator* T *is a continuous operator from* $\mathcal{H}_0^1$ *to* $\mathcal{H}_0^{1^*}$.

4.2 Pseudo-Monotonicity of T

We use the notion of pseudo-monotonicity expounded in [14]. For completeness, we reproduce the definitions before giving the proof that our operator fits the hypotheses. (This method was also used to find the singular solution in [4].)

Definition 3 *Let* X *be a real, reflexive Banach space. An operator* $T : X \to X^*$ *is called* pseudo-monotone *if* T *is bounded and if whenever* $u_n \rightharpoonup u \in X$ *and* $\limsup \langle T(u_n), u_n - u\rangle \leq 0$, *it follows that*

$$\liminf_{n\to\infty} \langle T(u_n), u_n - v\rangle \geq \langle T(u), u - v\rangle \quad \forall v \in X.$$

Instead of verifying that T satisfies the properties of Definition 3, we show that T is of calculus of variations type and invoke the following theorem from [14].

Theorem 5 ([14]) *If* T *is of calculus of variations type, then* T *is pseudo-monotone.*

The definition of calculus of variations type is as follows.

Definition 4 *Let X be a reflexive Banach space. An operator $T : X \to X^*$ is said to be of* calculus of variations type *if it is bounded, and it has the representation $T(h) = \hat{T}(h, h)$ where the mapping from $X \times X$ to X^* given by $(h, z) \mapsto \hat{T}(h, z)$ satisfies the following properties:*
1. The mappings $z \mapsto \hat{T}(h, z)$ and $h \mapsto \hat{T}(h, z)$ are bounded and continuous.
2. For each $h \in X$,

$$\langle \hat{T}(h, h) - \hat{T}(h, z), h - z \rangle \geq 0, \quad \forall z \in X . \tag{20}$$

3. If $h_n \rightharpoonup h$ in X, and $\langle \hat{T}(h_n, h_n) - \hat{T}(h_n, h), h_n - h \rangle \to 0$, then for every $z \in X$

$$\hat{T}(h_n, z) \rightharpoonup \hat{T}(h, z) \quad \text{in } X^* .$$

4. If $h_n \rightharpoonup h$ in X, and $\hat{T}(h_n, z) \rightharpoonup \psi$ in X^*, then

$$\langle \hat{T}(h_n, z), h_n \rangle \to \langle \psi, h \rangle .$$

In the remainder of this section, we show that T satisfies the four conditions of Definition 4. We use the representation given by $T(h) = \hat{T}(h, h)$ where $\hat{T}(h, z)$ is defined by

$$\langle \hat{T}(h, z), w \rangle = \int_\Omega \left((\tilde{u} + x) z_x + \tilde{h}(g_x - \frac{1}{2}) \right) w_x + z_y w_y .$$

The second property in Definition 4 is equivalent to requiring that T be monotone in the derivatives of h, while properties 3 and 4 roughly equate to compactness in the lower order terms. This is provided by the following proposition.

Proposition 2 *If $h_n \rightharpoonup h$ in $\mathcal{H}_0^1(\Omega)$, then $\tilde{h}_n/\sqrt{x} \to \tilde{h}/\sqrt{x}$ in $L^2(\Omega)$.*

This proposition follows from the following compactness theorem found in Adams, [1].

Theorem 6 ([1], page 33) *Let $K \subset L^2(\Omega)$ be a bounded subset of $L^2(\Omega)$. Suppose that there exists a sequence $\{\Omega_j\}$ of subdomains of Ω having the following properties:*
1. For each j, $\Omega_j \subset \Omega_{j+1}$,
2. For each j the set of restrictions to Ω_j of the functions in K is precompact in $L^2(\Omega_j)$,
3. For every $\epsilon > 0$ there exists j such that

$$\int_{\Omega - \Omega_j} |u|^2 d\mu < \epsilon, \quad \forall u \in K. \tag{21}$$

Then K is precompact in $L^2(\Omega)$.

The proof of Proposition 2 is now established as follows.

Proof: We may assume that $|\tilde{h}_n|/\sqrt{x} \leq M$, by using the cutoff function. Here, M depends on the size of Ω, assumed to be bounded. This defines a bounded set in L^2. Let Ω_j be the subset of Ω with $x \geq 1/j$. Clearly, $\Omega_j \subset \Omega_{j+1}$.

Denote by $\mathcal{H}_0^1(\Omega_j)$ the space of restrictions of functions in $\mathcal{H}_0^1(\Omega)$ to the set Ω_j. Now, $\mathcal{H}_0^1(\Omega_j)$ is isomorphic to $H^1(\Omega_j)$, (the standard, unweighted Sobolev space), and $H^1(\Omega_j)$ is compactly embedded in $L^2(\Omega_j)$. Hence, the set of restrictions of $\tilde{h}_n/\sqrt{x}$ to Ω_j is precompact in $L^2(\Omega_j)$.

Therefore, it suffices to show that for every $\epsilon > 0$ there exists j such that (21) holds. Let $\epsilon > 0$. Take j large enough that

$$\epsilon \geq 2\left(\frac{1}{j}\right)^2 \max_{\Omega} |y|.$$

Then

$$\int_{\Omega-\Omega_j} \frac{|\tilde{h}_n|^2}{x}\, dxdy \leq \int_{\Omega-\Omega_j} \frac{(1-\eta)^2 x^2}{x}\, dxdy \leq (1-\eta)^2 \max_{\Omega-\Omega_j} |x|\ \mu(\Omega-\Omega_j)$$
$$\leq 2(1-\eta)^2 \max_{\Omega} |y| \left(\frac{1}{j}\right)^2 \leq \epsilon\,.$$

Here we used the bounds (14) for $|\tilde{h}_n| \leq x(1-\eta)$ and the fact that the measure of $\Omega - \Omega_j$, $\mu(\Omega-\Omega_j)$, is less than $2\max_\Omega |y|(1/j)$. Strong convergence of (a subsequence) $\tilde{h}_n$ in $L^2(\Omega)$ now follows from Theorem 6. ■

We are now in a position to prove

Theorem 7 *The operator T is of calculus of variations type.*

Proof: We investigate the properties of $\hat{T}(h,z)$.
1. Boundedness and continuity are proved just as in Theorems 1 and 4.
2. Monotonicity in the second argument is straightforward since

$$\langle \hat{T}(h,h) - \hat{T}(h,z), h-z\rangle = \int_\Omega (\tilde{u}+x)(h-z)_x^2 + (h-z)_y^2 \geq 0\,.$$

3. Let $h_n \rightharpoonup h$ in $\mathcal{H}_0^1$. Then, by Proposition 2, we deduce $\tilde{h}_n \to \tilde{h}$ in $L^2(\Omega)$ and this implies that $\tilde{u}_n \to \tilde{u}$ in $L^2(\Omega)$; we also have $\tilde{h}_n/\sqrt{x} \rightharpoonup \tilde{h}/\sqrt{x}$ in $L^2(\Omega)$.

Therefore, even without the hypothesis that $\langle \hat{T}(h_n,h_n)-\hat{T}(h_n,h),h_n-h\rangle \to 0$, we obtain

$$\langle \hat{T}(h_n,z)-\hat{T}(h,z),w\rangle \to 0, \quad \forall w \in C_0^1(\overline{\Omega}),$$

and therefore, since $C_0^1(\overline{\Omega})$ is dense in $\mathcal{H}_0^1$, $\hat{T}(h_n,z) \rightharpoonup \hat{T}(h,z)$ in ${\mathcal{H}_0^1}^*$.
4. Let $h_n \rightharpoonup h$ in $\mathcal{H}_0^1$, and $\hat{T}(h_n,z) \rightharpoonup \psi$ in ${\mathcal{H}_0^1}^*$. We need to show that

$$\langle \hat{T}(h_n,z),h_n\rangle = \langle \hat{T}(h_n,z),h\rangle + \langle \hat{T}(h_n,z),h_n-h\rangle \to \langle \psi, h\rangle .$$

It suffices to show that $\langle \hat{T}(h_n,z),h_n-h\rangle \to 0$. Now, we have

$$\langle \hat{T}(h_n,z),h_n-h\rangle = \int_\Omega \left[(\tilde{u}_n+x)z_x + \tilde{h}_n(g_x-\frac{1}{2})\right](h_n-h)_x \\ + z_y(h_n-h)_y .$$

The compactness argument provided by Proposition 2, uniform boundedness of $\tilde{u}$ and $\tilde{h}$, and the fact that $h_n \rightharpoonup h$, now yield $\langle \hat{T}(h_n,z),h_n-h\rangle \to 0$. ■

Corollary 3 *The operator T is pseudo-monotone.*

4.3 Coercivity of T

We shall show that there exist constants $\tilde{C}>0$, $D>0$ such that

$$\langle T(h),h\rangle = \tilde{B}_g(h,h) \geq \tilde{C}\|h\|^2_{\mathcal{H}_0^1} - D\|h\|_{\mathcal{H}_0^1} . \tag{22}$$

This will imply that T is coercive; that is,

$$\lim_{h\to\infty} \frac{\langle T(h),h\rangle}{\|h\|_{\mathcal{H}_0^1}} = \infty .$$

Theorem 8 *The operator T is coercive.*

Proof: We estimate $\langle T(h),h\rangle$ as follows:

$$\begin{aligned}\langle T(h),h\rangle &= \int_\Omega (\tilde{u}+x)h_x^2 + h_y^2 + \left(g_x-\frac{1}{2}\right)\tilde{h}h_x \\ &\geq \eta\|h\|^2_{\mathcal{H}^1} - \int_\Omega \left|g_x-\frac{1}{2}\right| \frac{|\tilde{h}|}{\sqrt{x}}\sqrt{x}|h_x| .\end{aligned}$$

The Cauchy-Schwartz inequality and the bound $|\tilde{h}| \leq x(1-\eta)$ yield (22). ■

Using the Browder-Minty Theorem, Theorem 2, we obtain existence of a weak solution of the modified problem. Our conclusion is

Theorem 9 *For every $f \in \mathcal{H}_0^{1^*}(\Omega)$, there exists a function $h \in \mathcal{H}_0^1(\Omega)$ such that,*

$$\langle T(h), w\rangle = \langle f, w\rangle \,, \qquad \forall w \in \mathcal{H}_0^1(\Omega)\,.$$

5 Regularity of the Solution

In this section we show that the solution we have found is a classical solution and that it solves the original problem. We show that for smooth boundary data g with $g_{sub} \leq g \leq 0$, the weak solution u is in $C^2(\Omega)$ and is continuous up to the degenerate boundary. If the boundary data g has slope $g_x = -1/2$ at the degenerate boundary, we show in addition that u is continuously differentiable up to the degenerate boundary.

We show that $u \in C^2(\Omega)$ in two steps. We first work in the set

$$\Omega_\epsilon \equiv \{(x,y) \in \Omega \,|\, \text{dist}\,((x,y), \partial\Omega) \geq \epsilon\}\,.$$

Then, we take the limit $\epsilon \to 0$.

First, we study the operator L defined by the modified problem (16) in the domain Ω_ϵ, and with $\tilde{u}$ defined by

$$\tilde{u}(u) = \begin{cases} u & \text{if } g_{sub} \leq u \leq 0 \\ g_{sub} & \text{if } u < g_{sub} \\ 0 & \text{if } u > 0 \end{cases}\,.$$

We claim that the weak solution $u = h - g$ we constructed in the previous section gives a weak solution of this problem. For this purpose we take f to be defined by equation (11) and use Theorem 9 which establishes existence of a weak solution of the modified problem (16). If necessary, we may further modify the weak solution to be bounded: the modified problem corresponds to a linear equation when u exceeds the given bounds. We can modify the solution also, replacing it by the corresponding sub- or super-solution, which is also a solution, outside the bounds. The ensuing function, which we will also call u, is also a weak solution, and is bounded. The weak solution is in the space $\mathcal{H}^1(\Omega_\epsilon)$ which is isomorphic to $W^{1,2}(\Omega_\epsilon)$ for every $\epsilon > 0$.

To show that u is $C^2(\Omega)$, we use higher-order regularity theory for quasilinear elliptic equations, citing the following theorem proved in Ladyzhenskaya and Ural'tseva, [11].

Theorem 10 ([11], page 284) *Suppose that in regions of the form*

$$\{x \in \overline{\Omega}, \quad |z| \leq M, \quad |p| \leq M_1\}$$

with arbitrary M_1 and $M > 0$, the functions $A^i(x, z, p)$ belong to the class $C^{l-1,\alpha}$, for $l \geq 2$, and that they satisfy the following inequalities:

$$\begin{aligned} \nu(|z|) \sum \xi_i^2 \leq \frac{\partial A^i(x,z,p)}{\partial p_j} \xi_i \xi_j &\leq \mu(|z|) \sum \xi_i^2 \,, \\ \sum_i \left(\left| \frac{\partial A^i}{\partial z} \right| + |A^i| \right) + \sum_{i,j} \left| \frac{\partial A^i(x,z,p)}{\partial x_j} \right| &\leq \mu(|z|)(1 + |p|) \,. \end{aligned} \tag{23}$$

Then, an arbitrary bounded generalized solution $u \in W^{1,2}(\Omega)$ of the equation

$$D_i A^i(x, u, Du) = 0 \,,$$

in an arbitrary region $\Omega' \subset \Omega$, belongs to $C_{l,\sigma}(\Omega')$.

To apply this theorem, we check that the inequalities hold for the coefficients

$$A^1(x,z,p) = (z + x)p_1 - \frac{1}{2} z \quad \text{and} \quad A^2(x,z,p) = p_2$$

of our equation in Ω_ϵ. (In fact, we are applying this theorem to the modified equation, in which A^1 contains a cut-off function for z, which is not as smooth as required in the theorem. The cut-off function can easily be smoothed without changing any of its other essential features. We ignore this in the following estimates, just to avoid introducing further notation.)

The first condition holds for $\nu(|z|) = \min(\epsilon, \eta)$, which corresponds to strict ellipticity of the operator L defined in a domain $\Omega_\epsilon \subset \Omega$. The right hand side inequality follows from the form of the coefficient $z + x$.

The second inequality, (23), is verified as follows. The left side is bounded by

$$\left(2|p_1| + |(z + x)p_1| + \frac{1}{2}|z| + |p_2| + \frac{1}{2} \right) \leq (C_1|z| + C_2)(1 + |p|) \,.$$

We comment that this theorem actually implies that, since the dependence of the A^i on the coefficients is arbitrarily smooth and the equation is homogeneous, the solution is differentiable to any order in the interior of Ω.

Finally, since this theorem holds for every $\epsilon > 0$, we have the following corollary to Theorem 9.

Corollary 4 *A weak solution $u \in \mathcal{H}^1$ of equation* (16) *is $C^{2,\alpha}(\Omega)$.*

We can now replace the modified problem by the original weak form, and complete the proof of existence of a weak solution of (1) and (2). The functions given by (8) are also sub- and super-solutions of the modified problem. Since the solution is continuous, it satisfies $g_{sub} \leq u \leq 0$. This implies that $h = u - g$ satisfies $g_{sub} \leq h \leq -g_{sub}$, and hence it is a weak solution of (10) and u is a weak solution of (1) and (2). But in this case, $\tilde{u} = u$ and equation (16) is the classical version, (1), of the weak form we have solved. Therefore, we have constructed a weak solution u of the original problem which is continuous on Ω_ϵ for every $\epsilon > 0$. This implies that u is continuous in the interior of Ω.

We state these conclusions as a second corollary to Theorem 9.

Corollary 5 *There exists a weak solution h in the sense of Definition 1 to* (10)*, and the corresponding $u = h + g$ is a weak solution of* (1) *and* (2)*. Moreover, u satisfies $g_{sub} \leq u \leq 0$ in $\overline{\Omega}$, and u is a classical solution in Ω.*

We have applied Theorem 10 to the weak solution constructed in this paper. However, it can be applied to any weak solution of (1) by similarly modifying the problem and using sub- and super-solutions. Thus, using the corollary and the arguments described at the beginning of this section we obtain the following theorem.

Theorem 11 *A weak solution $u \in \mathcal{H}^1(\Omega)$ of equation* (1)*, with smooth boundary data g such that $g_{sub} \leq g \leq 0$, is twice continuously differentiable in Ω and continuous up to the degenerate boundary Γ_2.*

In the remainder of this section we show that if the boundary data has slope $-1/2$ at the degenerate boundary, then a weak solution u is continuously differentiable up to the degenerate boundary. In this case we may choose functions

$$u_{sub} = -\frac{x}{1+\sqrt{1+C_1 x}} \quad \text{and} \quad u_{sup} = -\frac{x}{1+\sqrt{1+C_2 x}},$$

for some $C_1 < 0 < C_2$, as sub- and super-solutions in a neighborhood of the degenerate boundary Γ_2. Since the weak solution u is $C^2(\Omega)$, and since u is squeezed between u_{sub} and u_{sup}, we see that u is C^1 up to the degenerate boundary, and that the slope of u at the degenerate boundary equals $-1/2$.

6 Conclusions

We have proved existence of a weak solution in a weighted Sobolev space $\mathcal{H}^1$ of equation (1) with smooth nonpositive Dirichlet boundary data, (2). We have shown that this solution is in $C^2(\Omega)$ and that it is continuous up to the degenerate boundary. Moreover, if the boundary data is chosen so that it has slope $-1/2$ at the degenerate boundary then the solution u is C^1 up to the degenerate boundary.

We have not proved that the solution is unique. In [4] we showed that there is a simple comparison principle for solutions of (1) which have a convexity restriction $u_{xx} \leq 0$. However this hypothesis is probably not reasonable for the solution we have constructed, since, for example, our solution might be the middle curve of Figure 2.

Uniqueness is important when we apply the result of this paper to the motivating Riemann problem for equation (3). We have not completed this problem yet; however, we note here some preliminary results which suggest that the solution we have found here, with $u < 0$ in the interior of Ω and no singularity at the degenerate boundary, is more likely than the singular solution of [4] to lead to a solution in the subsonic region which will match the free boundary condition and the solution in the supersonic region. For this, we need to explain that equation (4) is derived from a system

$$\begin{aligned} (u-\rho)u_\rho - \frac{\eta}{2}u_\eta + v_\eta &= 0\,, \\ \frac{\eta}{2}\,u_\rho + u_\eta - v_\rho &= 0\,, \end{aligned}$$

in which u and v are the components of velocity. We have the following result for solutions of this system.

Proposition 3 *If there is a line segment $\rho = u_m$ along which $u = u_m$, then v is constant along this line if and only if $(u-\rho)u_\rho = 0$ there.*

Proof: If $u \in C^1$ in a neighborhood of the line, then the result follows immediately since in the first equation each of the first two terms separately, and hence the third also is zero on the line. But a solution which is not in C^1 has precisely a square root singularity, and then $\lim(u-\rho)u_\rho \neq 0$ on the line; hence $v_\eta \neq 0$ there and v is not constant. ■

In the motivating application to self-similar flow, the subsonic region is adjacent to a constant state, and thus we have a weak solution only if v

is continuous at the degenerate boundary, and hence constant there. This seems to rule out the singular solution.

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References

[1] R. A. Adams. *Sobolev Spaces.* Academic Press, New York, 1975.

[2] M. Brio and J. K. Hunter. Mach reflection for the two dimensional Burgers equation. *Physica D*, 60:194–207, 1992.

[3] S. Čanić and B. L. Keyfitz. Quasi-one-dimensional Riemann problems and their role in self-similar two-dimensional problems. In preparation.

[4] S. Čanić and B. L. Keyfitz. An elliptic problem arising from the unsteady transonic small disturbance equation. *Journal of Differential Equations*, to appear.

[5] S. Čanić, B. L. Keyfitz, and D. H. Wagner. A bifurcation diagram for oblique shock interactions in the unsteady transonic small disturbance equation. In *Proceedings of the Fifth International Conference on Hyperbolic Problems, Theory, Numerics, and Applications*, to appear.

[6] G. Fichera. Sulle equazioni differenziali lineari ellittico-paraboliche del secondo ordine. *Atti. Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. Ser. I*, 5:1–30, 1956.

[7] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order.* Springer-Verlag, New York, 1977.

[8] E. Harabetian. Diffraction of a weak shock by a wedge. *Communications on Pure and Applied Mathematics*, XL:849–863, 1987.

[9] M. V. Keldysh. On some cases of degenerate elliptic equations on the boundary of a domain. *Doklady Acad. Nauk USSR*, 77:181–183, 1951.

[10] J. J. Kohn and L. Nirenberg. Degenerate elliptic-parabolic equations of second order. *Communications on Pure and Applied Mathematics*, XX:797–872, 1967.

[11] O. A. Ladyzhenskaya and N. N. Ural'tseva. *Linear and Quasilinear Elliptic Equations.* Academic Press, New York and London, 1968.

[12] J. Nečas. *Introduction to the Theory of Nonlinear Elliptic Equations.* John Wiley and Sons, New York, 1986.

[13] O. A. Oleǐnik and E. V. Radkevič. *Second Order Equations with Nonnegative Characteristic Form.* American Mathematical Society, Providence, 1973.

[14] M. Renardy and R. C. Rogers. *An Introduction to Partial Differential Equations.* Springer-Verlag, New York, 1993.

[15] E. G. Tabak and R. R. Rosales. Weak shock focusing and the von Neumann paradox. *Physics of Fluids A*, to appear.

[16] F. Tricomi. Sulle equazioni lineari alle derivate parziali di secondo ordine, di tipo misto. *Rend. Reale Accad. Lincei.*, 14:134–247, 1923.

[17] H. Triebel. *Interpolation Theory, Function Spaces, Differential Operators.* North-Holland, Amsterdam, 1978.