SHOCK WAVE ADMISSIBILITY FOR QUADRATIC CONSERVATION LAWS

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Abstract. In this work we present a new approach to the study of the stability of admissible shock wave solutions for systems of conservation laws that change type. The systems we treat have quadratic flux functions. We employ the fundamental wave manifold $\mathcal{W}$ as a global framework to characterize shock waves that comply with the viscosity admissibility criterion. Points of $\mathcal{W}$ parametrize dynamical systems associated with shock wave solutions.

The region of $\mathcal{W}$ comprising admissible shock waves is bounded by the loci of structurally unstable dynamical systems. Explicit formulae are presented for the loci associated with saddle-node, Hopf, and Bogdanov-Takens bifurcation, and with straight-line heteroclinic connections. Using Melnikov's integral analysis, we calculate the tangent to the homoclinic part of the admissibility boundary at Bogdanov-Takens points of $\mathcal{W}$. Furthermore, using numerical methods, we explore the heteroclinic loci corresponding to curved connecting orbits and the complete homoclinic locus.

We find the region of admissible waves for a generic, two-dimensional slice of the fundamental wave manifold, and compare it with the set of shock points that comply with the Lax admissibility criterion, thereby elucidating how this criterion differs from viscous profile admissibility.

1. INTRODUCTION

We study systems of two conservation laws in one space variable,

$$U_t + F(U)_x = 0 \ , \quad x \in \mathbf{R} \ , \quad t > 0 \ , \tag{1.1}$$

where the *state variable* $U = U(x,t)$ belongs to the state space $\mathcal{U} = \mathbf{R}^2$, and the *flux function* $F : \mathcal{U} \to \mathbf{R}^2$ is a smooth function of U. Equations of this form model the conservation of such things as mass, momentum, and energy in the one-dimensional time-dependent evolution of a continuum. In order to model

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important engineering applications (*e.g.*, three-phase flow in a porous medium), we do not require system (1.1) to be strictly hyperbolic. Thus we allow for points U such that the Jacobian $F'(U)$ has eigenvalues $\lambda_1(U)$ and $\lambda_2(U)$ that coincide or are not real.

We are particularly interested in quadratic models, *i.e.*, models for which the flux function $F(U)$ depends quadratically on U. Models of this type arise as an approximation of more general flux functions when localizing the problem to a neighborhood of a point in state space. They capture a wide range of nonlinear phenomena, including loss of strict hyperbolicity, that occur in the general case. Quadratic models have been studied by several authors [**24**], [**30**], [**14**], [**16**], [**15**], [**10**], [**19**], [**29**], [**28**], [**2**], [**13**], [**11**], [**27**].

Because of the nonlinearity of the flux function F, solutions of the Cauchy problem for system (1.1) typically form discontinuities, even when the initial data are smooth. An initial-value problem with initial data of the form

$$U(x,t=0) = \begin{cases} U_L & \text{if } x < 0 \ , \\ U_R & \text{if } x > 0 \end{cases} \tag{1.2}$$

is called a Riemann initial-value problem. We concentrate on the simplest kind of discontinuous solution of a Riemann problem, called a *shock wave*, which corresponds to a jump discontinuity separating two constant states U_- and U_+ and traveling at a constant speed s.

Since some weak solutions of system (1.1) are unphysical, we must impose additional criteria, called *admissibility criteria*, to extract physical waves. For this purpose, recall that as a model of fluid dynamics, system (1.1) usually arises by neglecting the dissipative effects of a parabolic term:

$$U_t + F(U)_x = \epsilon \left(D(U)\, U_x\right)_x \ . \tag{1.3}$$

A solution of system (1.1) can be thought of as a singular limit, as $\epsilon \to 0$, of solutions of system (1.3). In particular, shock wave solutions of system (1.1) that are limits of traveling wave solutions of system (1.3) are said to satisfy the *viscosity admissibility criterion*, which we shall adopt to determine admissible waves; see Sec. 2.

The central objective of this work is the study of stability, *i.e.*, continuous dependence, of admissible shock waves with respect to changes of the state U_- on the left side of the shock wave and of the shock speed s. Instead of treating the stability of an individual shock wave, our approach in this paper is to work with the three-parameter family of dynamical systems associated with the set of all shock wave solutions.

The fundamental wave manifold $\mathcal{W}$, introduced in Ref. [**12**], offers a unified framework for such an analysis. Points of $\mathcal{W}$ represent shock waves and parametrize the corresponding dynamical systems. An admissible shock point of $\mathcal{W}$ is defined by the existence of a connecting orbit between the two critical points U_- and U_+ in the corresponding dynamical system.

The class of admissible shock waves forms a region within $\mathcal{W}$, which is bounded by the *admissibility boundary*. A dynamical system corresponding to a point on the admissibility boundary is generically structurally unstable, as described in Sec. 4. In fact, the admissibility boundary is contained in the loci associated with the occurrence of nonhyperbolic critical points, double separatrix connections, and nonhyperbolic limit cycles. In this paper we examine certain codimension-1 bifurcations (saddle-node, transcritical, Hopf, homoclinic, and heteroclinic bifurcations) and codimension-2 bifurcations (Bogdanov-Takens bifurcation and double saddle-saddle connections).

In Sec. 5 we present explicit formulae for certain parts of the admissibility boundary for quadratic models. In particular, portions of the saddle-node, Hopf, and heteroclinic loci (with straight-line separatrices) are shown to be ruled surfaces when a special global coordinate system for $\mathcal{W}$ is used. Furthermore, we calculate the tangent to the homoclinic locus at Bogdanov-Takens points of $\mathcal{W}$. Finally, we describe numerical experiments to locate other points on the admissibility boundary that correspond to global bifurcations. In particular, we find that heteroclinic points with curved separatrices seem to lie on ruled surfaces also.

At the end of Sec. 5, we analyze a generic two-dimensional slice of the three-dimensional bifurcation diagram. We find the region of admissible shock waves and the mechanisms that cause bifurcations as the admissibility boundary is crossed. We also determine the region of shock waves admissible according to the Lax criterion and compare it with the viscosity admissibility region, showing where the two criteria are not equivalent.

2. SHOCK WAVE ADMISSIBILITY

In this section we discuss nonclassical solutions of the conservation law (1.1) involving a single jump discontinuity propagating at constant speed $s \in \mathbf{R}$; the states on the left and right sides of this shock wave are denoted, respectively, by $U_- \in \mathcal{U}$ and $U_+ \in \mathcal{U}$. A weak formulation of the conservation law implies that the quantities $U_-, U_+ \in \mathcal{U}$ and $s \in \mathbf{R}$ must satisfy the Rankine-Hugoniot condition

$$-s\,[U_+ - U_-] + F(U_+) - F(U_-) = 0 \ , \tag{2.1}$$

which embodies the conservation principle across the discontinuity. Because there are solutions of Eq. (2.1) that are not physically meaningful and would lead to nonuniqueness, we must impose additional criteria to extract the physical solution.

For this purpose we use the *viscosity admissibility criterion*, which derives from a parabolic system of equations

$$U_t + F(U)_x = \epsilon\,(D(U)\,U_x)_x \ , \quad x \in \mathbf{R} \ , \quad t > 0 \ ; \tag{2.2}$$

here $\epsilon > 0$, and $D(U)$ is called the *viscosity matrix*. We shall work with viscosity matrices that are independent of U and satisfy the following assumption.

Assumption 2.1. The viscosity matrix D has eigenvalues with positive real parts.

To make a connection with shock wave solutions (U_-, U_+, s) of Eq. (1.1), we seek a *traveling wave solution* that joins the states U_- and U_+, *i.e.*, a smooth solution of Eq. (2.2) of the form $U(x,t) = \widehat{U}((x - st)/\epsilon) =: \widehat{U}(\xi)$ such that $U_- = \lim_{\xi\to-\infty} \widehat{U}(\xi)$

and $U_+ = \lim_{\xi\to\infty} \hat{U}(\xi)$. The viscosity admissibility criterion requires that a discontinuous solution (U_-, U_+, s) of system (1.1) be a limit, as $\epsilon \to 0$, of a traveling wave solution joining U_- and U_+. In order to satisfy Eq. (2.2), $\widehat{U}$ must be an orbit for the dynamical system

$$\hat{U}' = X(\widehat{U}) := D^{-1}\left\{-s\,[\hat{U} - U_-] + F(\hat{U}) - F(U_-)\right\}\,. \tag{2.3}$$

Equation (2.1) implies that U_- and U_+ are critical points of the dynamical system (2.3), *i.e.*, U_- and U_+ satisfy $X(\widehat{U}) = 0$; more generally, all critical points of (2.3) lie on the Hugoniot locus of U_-, the set of states $U_{cr} \in \mathcal{U}$ such that

$$-s\,[U_{cr} - U_-] + F(U_{cr}) - F(U_-) = 0 \tag{2.4}$$

for some $s \in \mathbf{R}$.

In order to define admissible shock waves, we recall the following definitions.

Definition 2.2. A critical point U_{cr} of a dynamical system $U' = X(U)$ is *hyperbolic* if the eigenvalues of the Jacobian matrix $\mathcal{J}(U_{cr}) := X'(U_{cr})$ have nonvanishing real parts.

A hyperbolic critical point U_{cr} of a planar dynamical system is one of the following: a *node* if the eigenvalues of the Jacobian are both real and have equal signs, *i.e.*, $\operatorname{discrm} J(U_{cr}) > 0$ and $\det J(U_{cr}) > 0$; a *saddle point* if the eigenvalues of the Jacobian are both real and of opposite signs, *i.e.*, $\operatorname{discrm} J(U_{cr}) > 0$ and $\det J(U_{cr}) < 0$; or a *spiral* if the eigenvalues of the Jacobian are nonreal, *i.e.*, $\operatorname{discrm} J(U_{cr}) < 0$ and $\det J(U_{cr}) > 0$.

For a hyperbolic critical point U_{cr}, the set

$$W^s(U_{cr}) = \{\widehat{U}(\xi_n) \in \mathcal{U} \mid \widehat{U}(\xi_n) \to U_{cr} \text{ for some sequence } \xi_n \to \infty\} \tag{2.5}$$

is called the *stable manifold* of U_{cr}; similarly, the *unstable manifold* $W^u(U_{cr})$ is defined by considering sequences $\xi_n \to -\infty$. Notice that W^s and W^u are invariant under the flow of X.

Following these definitions, we define admissible shock waves.

Definition 2.3. We say that a discontinuous solution (U_-, U_+, s) of the conservation law (1.1) *admits a viscous profile* if there exists an orbit of the dynamical system (2.3) joining the critical points U_- and U_+.

Necessarily, this orbit lies in the unstable manifold of U_- and the stable manifold of U_+. Therefore when U_- and U_+ are hyperbolic critical points, an *admissible shock wave* is one of the following four types:

(1) a *1-wave*, which corresponds to an orbit joining a repelling node or spiral point U_- to a saddle point U_+;
(2) a *2-wave*, which corresponds to an orbit joining a saddle point U_- to an attracting node or spiral point U_+;
(3) a *transitional wave*, which corresponds to an orbit joining two saddle points U_- and U_+;
(4) an *overcompressive wave*, which corresponds to an orbit joining a repelling node or a spiral point U_- to an attracting node or a spiral point U_+.

Figure 2.1 illustrates the four types of admissible shock waves between the two hyperbolic critical points. To this list we can add the possibility that U_- or U_+ be nonhyperbolic, for example a weak node, weak spiral point, or weak saddle point. As we show in Sec. 5, a shock wave involving a nonhyperbolic critical point belongs to the the boundary of the region of admissible shock waves. The viscosity criterion obviously depends on the viscosity matrix. Throughout this work we take D to be a fixed matrix satisfying the Assumption 2.1.

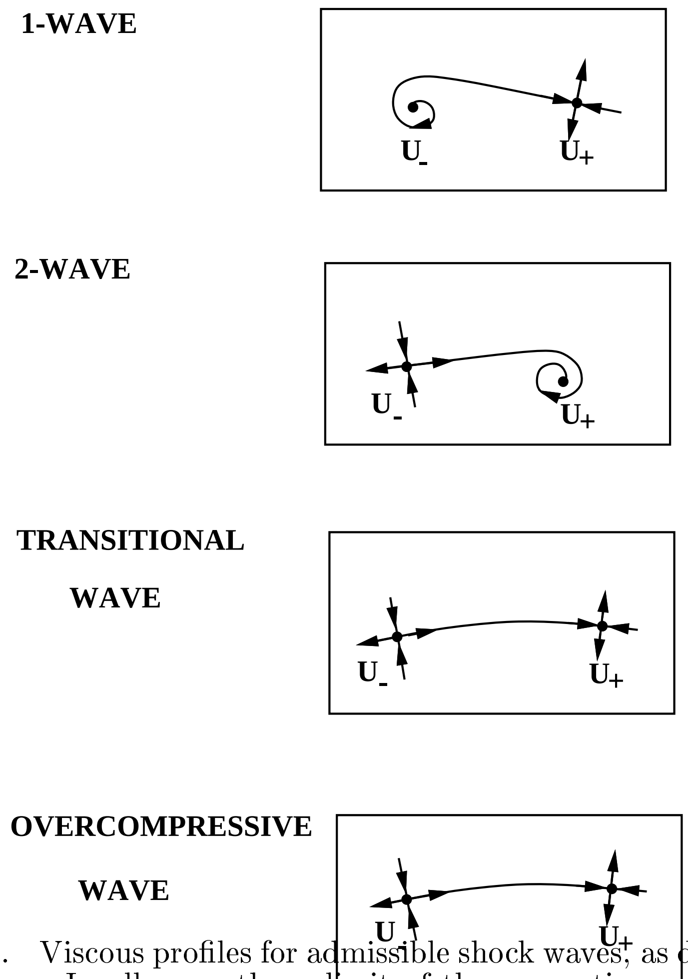

Figure 2.1. Viscous profiles for admissible shock waves, as drawn in state space. In all cases the α-limit of the connecting orbit is U_- and the ω-limit is U_+. The types of critical points connected determine the type of the shock wave.

In the inviscid theory, other shock wave admissibility criteria, such as the Lax characteristic criterion, are used to restrict the class of weak solutions. The Lax criterion requires, for a wave to be admissible, that the characteristics of one and only one family approach the shock wave. This is described by the inequalities

$$s < \lambda_1(U_-) \ , \quad \lambda_1(U_+) < s < \lambda_2(U_+) \tag{2.6}$$

for a Lax 1-shock wave, and

$$\lambda_1(U_-) < s < \lambda_2(U_-) \ , \quad \lambda_2(U_+) < s \tag{2.7}$$

for a Lax 2-shock wave. When applied to gas dynamics, inequalities (2.6) and (2.7) are equivalent to having entropy increases across shock waves [**5**].

The relationship between the classical Lax criterion and the viscosity criterion can be explained as follows. Near a critical point U_{cr}, solutions of the dynamical system approximately satisfy the linearized system

$$\hat{U}' = D^{-1}\left[-s\,I + F'(U_{cr})\right]\left(\hat{U} - U_{cr}\right) \;. \tag{2.8}$$

If $D = I$, then the eigenvalues of system (2.8) are $\lambda^i(U_{cr}) - s$, $i = 1, 2$. Thus the Lax condition is equivalent to requiring that either:

(1) U_- is a repelling node and U_+ is a saddle point for a Lax 1-shock wave; or
(2) U_- is a saddle point and U_+ is an attracting node for a Lax 2-shock wave.

A Lax wave that admits a viscous profile is therefore either a 1-wave or a 2-wave. For D close to a multiple of the identity matrix, a Lax wave has the same configuration of critical points as in (1) or (2). As shown in Ref. [**5**] for weak Lax waves, *i.e.*, with U_+ close to U_-, such a configuration implies the existence of a connecting orbit.

The Lax criterion is not, however, equivalent to the viscous profile criterion. For instance, simple examples of scalar conservation laws have shock waves that satisfy the Lax criterion but not the Oleĭnik criterion, which is equivalent to the viscous profile criterion. More generally, strong Lax waves might not have profiles: although the phase portrait for a Lax wave with a profile is stable, so that nearby Lax waves also have profiles, sufficiently strong perturbations, even within the class of Lax waves, can break the connecting orbit. On the other hand, transitional and overcompressive waves admit viscous profiles but do not comply with the Lax criterion.

3. FUNDAMENTAL WAVE MANIFOLD

3.1 Introduction. The central objective of this paper is to study the stability, *i.e.*, continuous dependence, of admissible shock waves with respect changes of the state U_- on the left of the shock wave and of the shock speed s. Because we adopt the viscosity admissibility criterion to determine admissible waves, the study of stability of admissible waves reduces to bifurcation analysis of the corresponding dynamical system (2.3). In order to determine all admissible shock waves that can appear in a solution of the Riemann problem, we employ a global framework based on the fundamental wave manifold [**12**]. Since shock wave solutions are intrinsically incorporated into the construction of the wave manifold, we are able to treat both classical and nonclassical shock waves in a unified fashion. The central idea in the construction is to parametrize solutions of the Rankine-Hugoniot jump condition.

3.2 Construction of the Fundamental Wave Manifold $\mathcal{W}$. We follow the construction introduced by Isaacson, Marchesin, Palmeira, and Plohr in Ref. [**12**], restricted to the case when $U(x,t) \in \mathcal{U} := \mathbf{R}^2$. The solution set of the Rankine-Hugoniot condition

$$-s\left[U_+ - U_-\right] + F(U_+) - F(U_-) = 0 \tag{3.1}$$

is a subset of the ambient space $\mathcal{P} := \mathcal{U} \times \mathcal{U} \times \mathbf{R}$. Every solution point $(U_-, U_+, s) \in \mathcal{P}$ of the equation (3.1) represents either:

(1) a shock point $(U_+ \neq U_-)$;

(2) a rarefaction point ($U_+ = U_- =: \overline{U}$ and s an eigenvalue of $F'(\overline{U})$); or
(3) a trivial point ($U_+ = U_-$ and s not an eigenvalue of $F'(\overline{U})$).

Although the Rankine-Hugoniot condition represents two equations for five unknowns U_-, U_+, and s, the solution set is not a smooth manifold, since the function $\mathcal{H} : \mathcal{P} \to \mathbf{R}^2$ defined by

$$\mathcal{H}(U_-, U_+, s) := -s\,[U_+ - U_-] + F(U_+) - F(U_-) \tag{3.2}$$

is singular (*i.e.*, its Jacobian is not invertible) at the rarefaction points. This singularity occurs at the intersection of the closures of the sets of trivial points and shock points. Notice that the closure of the trivial points is the diagonal $\Delta = \{(U_-, U_+, s) \in \mathcal{P} \,|\, U_- = U_+\}$. In order to remove the singularity, the technique of blowing up is used [**12**]. We introduce an angle $\Omega \in [0, 2\pi)$ that represents the orientation of the vector $\Delta U := U_+ - U_-$ even in the limit $\Delta U = 0$. In other words, we let $R \in \mathbf{R}$ and $\Omega \in S^1$ serve as polar coordinates for $\Delta U = R\,r(\Omega)$, where $r(\Omega) := (\cos\Omega, \sin\Omega)^{\mathrm{T}}$. We also let $\overline{U} := \frac{1}{2}(U_- + U_+)$ denote the midpoint between U_- and U_+. In this way we obtain the space $\widehat{\mathcal{P}}$:

$$\widehat{\mathcal{P}} := \left\{(\overline{U}, R, \Omega, s) \,|\, (\overline{U}, R, \Omega, s) \in \mathcal{U} \times \mathbf{R} \times S^1 \times \mathbf{R}\right\} \ . \tag{3.3}$$

To each point in $\widehat{\mathcal{P}}$ we associate the left and right states U_- and U_+ through the formulae

$$U_+ = \overline{U} + \tfrac{1}{2} R r(\Omega) \ , \tag{3.4}$$

$$U_- = \overline{U} - \tfrac{1}{2} R r(\Omega) \ . \tag{3.5}$$

Since the antipodal points $(\overline{U}, R, \Omega, s)$ and $(\overline{U}, -R, \Omega + \pi, s)$ in $\widehat{\mathcal{P}}$ map to the same U_- and U_+, we define $\mathcal{P}^*$ to be the space obtained from $\widehat{\mathcal{P}}$ after identification of such antipodal points. In fact, we can restrict Ω to the segment $[0, \pi]$, identifying pairs $(\overline{U}, R, 0, s)$ and $(\overline{U}, -R, \pi, s)$, so that $\mathcal{P}^* = \mathcal{U} \times M_2 \times \mathbf{R}$, where M_2 is a Möbius strip.

Definition 3.1. The 3-dimensional manifold $\mathcal{P}^*$ is called the *blow-up* of $\mathcal{P} = \mathcal{U} \times \mathcal{U} \times \mathbf{R}$ over its diagonal Δ.

At this point we are ready to construct the zero-set of the Rankine-Hugoniot relation (3.1) as a subset of the space $\mathcal{P}^*$. First note that there exists a matrix-valued function A such that

$$F(U_+) - F(U_-) = R\,A(\overline{U}, R, \Omega)\, r(\Omega) \ ; \tag{3.6}$$

for example,

$$A(\overline{U}, R, \Omega) := \int_{-\frac{1}{2}}^{\frac{1}{2}} F'(\overline{U} + \alpha R\, r(\Omega))\, d\alpha \tag{3.7}$$

satisfies Eq. (3.6). Note that we can regard A as being defined on both $\widehat{\mathcal{P}}$ and $\mathcal{P}^*$.

With the aid of A and the function $\mathcal{H}$ defined by the equation (3.2), we lift the Rankine-Hugoniot condition to $\mathcal{P}^*$ by looking at the solution set of the equation

$$R\,\left[-s + A(\overline{U}, R, \Omega)\right]\, r(\Omega) = 0 \ . \tag{3.8}$$

This equation explicitly displays the trivial solutions $R = 0$ that we want to eliminate.

Equation (3.8) suggests defining the function $\mathcal{F} : \widehat{\mathcal{P}} \to \mathbf{R}^2$ by

$$\mathcal{F}(\overline{U}, R, \Omega, s) := \left[-s + A(\overline{U}, R, \Omega)\right] r(\Omega) \ . \tag{3.9}$$

The zero-set of $\mathcal{F}$ contains both nontrivial shock solutions and their $R \to 0$ limits, which correspond to rarefaction points, but does not contain trivial points. Notice that $\mathcal{F}$ is well-defined only on $\widehat{\mathcal{P}}$, and not on $\mathcal{P}^*$, since $\mathcal{F}(\overline{U}, R, \Omega, s) = -\mathcal{F}(\overline{U}, -R, \Omega + \pi, s)$. Therefore we define the *blow-up* of the zero-set of $\mathcal{H}$ to be the zero-set of $\mathcal{F}$ modulo the identification of antipodal points.

Definition 3.2. The *fundamental wave set* $\mathcal{W} \subseteq \mathcal{P}^*$ is the image of the zero set $\widehat{\mathcal{W}}$ of $\mathcal{F}$ under the identification map $\widehat{\mathcal{P}} \to \mathcal{P}^*$.

Under the assumption that zero is a regular value of $\mathcal{F}$, we have the following theorem [**12**].

Theorem 3.3. *The fundamental wave set $\mathcal{W}$ is a 3-dimensional submanifold of the blow-up $\mathcal{P}^*$, and $\widehat{\mathcal{W}} \subset \widehat{\mathcal{P}}$ is a two-fold covering manifold for $\mathcal{W}$.*

Since the points of $\mathcal{W}$ are solutions of the Rankine-Hugoniot condition, a *shock curve* is a slice of $\mathcal{W}$ with fixed U_- or fixed U_+, *i.e.*, the intersection of $\mathcal{W}$ with a surface $U_- =$ const. or $U_+ =$ const. A singularity of such a slice, at which the intersection is not transverse, corresponds to a bifurcation of the shock curve, *viz.*, *secondary bifurcation*. (Primary bifurcation, which causes the branching of shock curves at $U_+ = U_-$, has been removed by the blowing-up process.)

Rarefaction points lie on the $R = 0$ slice of $\mathcal{W}$. These are points $(\overline{U}, 0, \Omega, s) \in \mathcal{W}$ that satisfy

$$\left[-s + F'(\overline{U})\right] r(\Omega) = 0 \tag{3.10}$$

and form the two-dimensional *characteristic manifold* $\mathcal{C} \subset \mathcal{W}$. A *rarefaction curve* is an integral curve of the line field $-\sin\Omega \, d\overline{u} + \cos\Omega \, d\overline{v} = 0$ within $\mathcal{C}$.

The set of points $(\overline{U}, 0, \Omega, s) \in \mathcal{C}$ for which s is a multiple eigenvalue of $F'(\overline{U})$ is called the *coincidence locus* $\mathcal{E}$. The following lemma, proved in Ref. [**12**], characterizes the coincidence locus.

Lemma 3.4. *The real eigenvalue s of $F'(\overline{U})$ has algebraic multiplicity greater than 1 if and only if the 2×2 matrix $(\mathcal{F}_\Omega, \mathcal{F}_s)$ is singular at $(\overline{U}, 0, \Omega, s)$.*

As a consequence of this lemma it follows that the coincidence locus is a one-dimensional submanifold of $\mathcal{C}$ if zero is a regular value for the function $\det(\mathcal{F}_\Omega, \mathcal{F}_s)$.

When studying shock waves in $\mathcal{W}$, one must determine which part of a shock curve corresponds to physical waves. Under the Lax criterion, the admissible parts of the shock curves are bounded by points such that the shock speed coincides with the characteristic speed at one of the states U_- or U_+. In analogy with gas dynamics, such a shock wave is called a *sonic shock wave*. The locus of these points in $\mathcal{W}$ is called the *left sonic locus* $\mathcal{S}_L$. The *right sonic locus* $\mathcal{S}_R$ is defined similarly. Thus the boundaries of the Lax regions within $\mathcal{W}$ consist of the 2-dimensional *sonic loci* $\mathcal{S}_L$ and $\mathcal{S}_R$.

3.3 Fundamental Wave Manifold for Quadratic Flux Functions. Using the properties of the space $\mathcal{P}^*$, we show that the fundamental wave manifold $\mathcal{W} \subset \mathcal{P}^*$ for a quadratic model has the global topological structure of $M_2 \times \mathbf{R}$ [**21**].

Proposition 3.5. *For a quadratic model,* $A(\overline{U}, R, \Omega) = F'(\overline{U})$.

Proof. This follows directly from the definition (3.7) of A and the affine linearity of the Jacobian matrix $F'(U)$. $\square$

This proposition implies that the function $\mathcal{F}$ of Eq. (3.9), the zero-set of which defines $\mathcal{W}$, can be written as

$$\mathcal{F}(\overline{U}, R, \Omega, s) = \left[-s + F'(\overline{U})\right] r(\Omega) \ . \tag{3.11}$$

We shall make use of the following notation introduced in Ref. [**13**]:

$$U =: \begin{pmatrix} u \\ v \end{pmatrix}, \qquad F =: \begin{pmatrix} f \\ g \end{pmatrix} \ , \tag{3.12}$$

where

$$\begin{aligned} f(u,v) &:= \tfrac{1}{2}\left(a_1 u^2 + 2b_1 uv + c_1 v^2\right) + d_1 u + e_1 v \ , \\ g(u,v) &:= \tfrac{1}{2}\left(a_2 u^2 + 2b_2 uv + c_2 v^2\right) + d_2 u + e_2 v \ . \end{aligned} \tag{3.13}$$

Associated to a given quadratic model are the functions

$$\begin{aligned} \alpha(\Omega) &:= \tfrac{1}{2}\{(a_2 + b_1)\cos 2\Omega + (b_2 - a_1)\sin 2\Omega + a_2 - b_1\} \ , \\ \beta(\Omega) &:= \tfrac{1}{2}\{(b_2 + c_1)\cos 2\Omega + (c_2 - b_1)\sin 2\Omega + b_2 - c_1\} \ , \\ \gamma(\Omega) &:= \tfrac{1}{2}\{(d_2 + e_1)\cos 2\Omega + (e_2 - d_1)\sin 2\Omega + d_2 - e_1\} \ , \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} \widetilde{\alpha}(\Omega) &:= \tfrac{1}{2}\{(a_1 - b_2)\cos 2\Omega + (b_1 + a_2)\sin 2\Omega + a_1 + b_2\} \ , \\ \widetilde{\beta}(\Omega) &:= \tfrac{1}{2}\{(b_1 - c_2)\cos 2\Omega + (c_1 + b_2)\sin 2\Omega + b_1 + c_2\} \ , \\ \widetilde{\gamma}(\Omega) &:= \tfrac{1}{2}\{(d_1 - e_2)\cos 2\Omega + (e_1 + d_2)\sin 2\Omega + d_1 + e_2\} \ . \end{aligned} \tag{3.15}$$

These functions are such that

$$\begin{aligned} (-\sin\Omega, \cos\Omega)\, F'(\overline{U}) \begin{pmatrix} \cos\Omega \\ \sin\Omega \end{pmatrix} &= \alpha(\Omega)\overline{u} + \beta(\Omega)\overline{v} + \gamma(\Omega) \ , \\ (\cos\Omega, \sin\Omega)\, F'(\overline{U}) \begin{pmatrix} \cos\Omega \\ \sin\Omega \end{pmatrix} &= \widetilde{\alpha}(\Omega)\overline{u} + \widetilde{\beta}(\Omega)\overline{v} + \widetilde{\gamma}(\Omega) \ . \end{aligned} \tag{3.16}$$

After multiplying Eq. (3.11) on the left by $(\cos\Omega, \sin\Omega)$ and $(-\sin\Omega, \cos\Omega)$, we obtain the two respective equations

$$s = \widetilde{\alpha}(\Omega)\overline{u} + \widetilde{\beta}(\Omega)\overline{v} + \widetilde{\gamma}(\Omega) \ , \tag{3.17}$$

and

$$\alpha(\Omega)\overline{u} + \beta(\Omega)\overline{v} + \gamma(\Omega) = 0 \ . \tag{3.18}$$

Therefore $\mathcal{W}$ can be regarded as the solution set of Eq. (3.18) within $(\overline{u}, \overline{v}, R, \Omega)$-space.

In the ensuing analysis it will prove very useful to employ a distinguished set of coordiates for $\mathcal{W}$. These coordinates, (R, κ, Ω), are defined by the following construction. Let $U_c(\Omega) = (u_c(\Omega), v_c(\Omega))^{\mathrm{T}}$ be a particular solution of Eq. (3.18) such that

$$\begin{aligned} \alpha(\Omega)u_c(\Omega) + \beta(\Omega)v_c(\Omega) + \gamma(\Omega) = 0 \ , \\ \alpha'(\Omega)u_c(\Omega) + \beta'(\Omega)v_c(\Omega) + \gamma'(\Omega) = 0 \ . \end{aligned} \tag{3.19}$$

Such a solution exists for all Ω if the determinant

$$\mathcal{D}(\Omega) = \alpha(\Omega)\beta'(\Omega) - \beta(\Omega)\alpha'(\Omega) \tag{3.20}$$

never vanishes. Geometrically this means that the elliptic region is bounded as a subset of $\mathcal{U}$. Indeed, requiring that Eqs. (3.19) hold is equivalent to saying $\det(\mathcal{F}_\Omega, \mathcal{F}_s) = 0$ at the point (u_c, v_c). By Lemma 3.4, it follows that U_c parametrizes the projection of the coincidence locus $\mathcal{E}$ onto state space, *i.e.*, the boundary of the elliptic region. For simplicity we shall work with the following assumption.

Assumption 3.6. The determinant $\mathcal{D}(\Omega)$ is nonzero for all $\Omega \in [0, \pi)$.

Under Assumption 3.6, the particular solution U_c can be written as

$$\begin{aligned} u_c &= -\left[\gamma\beta' - \beta\gamma'\right]/\mathcal{D} \ , \\ v_c &= -\left[\alpha\gamma' - \gamma\alpha'\right]/\mathcal{D} \ , \end{aligned} \tag{3.21}$$

and the speed s of a point U_c is equal to $\lambda_c := \widetilde{\alpha}u_c + \widetilde{\beta}v_c + \widetilde{\gamma}$. A short calculation shows that $\mathcal{D}$ and the numerators of u_c and v_c are linear in $\sin 2\Omega$ and $\cos 2\Omega$, so that the elliptic boundary is a conic section, and the size of the elliptic region is measured by the constant

$$\eta := -4\mathcal{D}\left[\alpha_0 u_c + \beta_0 v_c + \gamma_0\right] \ , \tag{3.22}$$

where $\alpha_0 := \frac{1}{2}(a_2 - b_1)$, $\beta_0 := \frac{1}{2}(b_2 - c_1)$, and $\gamma_0 := \frac{1}{2}(d_2 - e_1)$ are the Ω-independent parts of α, β, and γ. The constancy of η is assured by the following lemma.

Lemma 3.7. *The function $\mathcal{D}\left[\alpha_0 u_c + \beta_0 v_c + \gamma_0\right]$ is independent of Ω.*

Proof. We write the functions α, β, and γ in the form $\alpha = \alpha_0 + \alpha_1 \cos 2\Omega + \alpha_2 \sin 2\Omega$, *etc.* A straightforward calculation shows that

$$\begin{aligned} \mathcal{D}\left[\alpha_0 u_c + \beta_0 v_c + \gamma_0\right] &= \alpha_0(\beta\gamma' - \gamma\beta') + \beta_0(\gamma\alpha' - \alpha\gamma') + \gamma_0(\alpha\beta' - \beta\alpha') \\ &= \det \begin{vmatrix} \alpha & \alpha' & \alpha_0 \\ \beta & \beta' & \beta_0 \\ \gamma & \gamma' & \gamma_0 \end{vmatrix} \ . \end{aligned} \tag{3.23}$$

Noting that

$$\begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 \\ \beta_0 & \beta_1 & \beta_2 \\ \gamma_0 & \gamma_1 & \gamma_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ \cos 2\Omega & -2\sin 2\Omega & 0 \\ \sin 2\Omega & 2\cos 2\Omega & 0 \end{pmatrix} = \begin{pmatrix} \alpha & \alpha' & \alpha_0 \\ \beta & \beta' & \beta_0 \\ \gamma & \gamma' & \gamma_0 \end{pmatrix} \ , \tag{3.24}$$

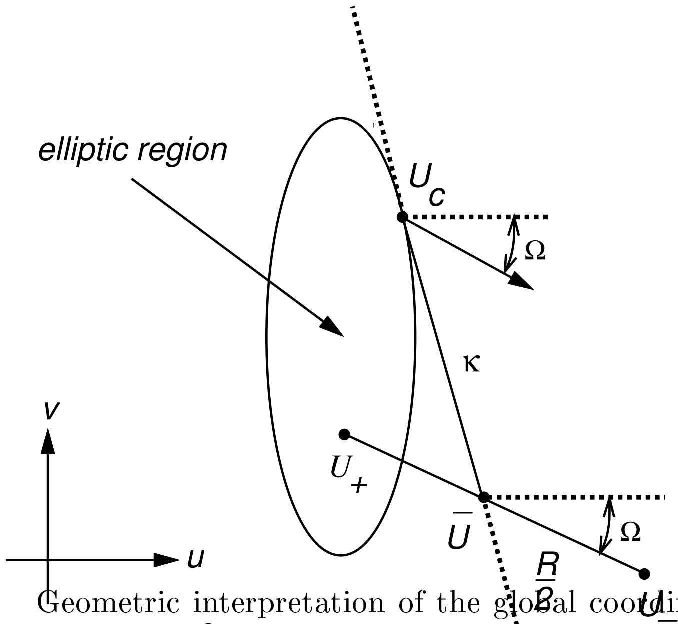

Figure 3.1. Geometric interpretation of the global coordinates (R, κ, Ω). The coordinate Ω, which is the inclination angle of the line segment joining U_- and U_+, defines a point $U_c(\Omega)$ on the boundary of the elliptic region; Ω is also the angle of the eigenvector at $U_c(\Omega)$. The coordinate κ parameterizes the line tangent to the elliptic boundary at $U_c(\Omega)$, and R measures the strength of the shock wave.

we obtain

$$\det \begin{vmatrix} \alpha & \alpha' & \alpha_0 \\ \beta & \beta' & \beta_0 \\ \gamma & \gamma' & \gamma_0 \end{vmatrix} = 2 \det \begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 \\ \beta_0 & \beta_1 & \beta_2 \\ \gamma_0 & \gamma_1 & \gamma_2 \end{vmatrix} . \tag{3.25}$$

This proves the lemma. $\square$

In terms of the parametrization of the elliptic boundary, the general solution of the equation (3.18) can be written as

$$\begin{aligned} \overline{u} &= u_c(\Omega) - \beta(\Omega)\kappa \ , \\ \overline{v} &= v_c(\Omega) + \alpha(\Omega)\kappa \ , \end{aligned} \tag{3.26}$$

with $\kappa \in \mathbf{R}$. If we define the function

$$\widetilde{\mathcal{D}}(\Omega) := \alpha(\Omega)\widetilde{\beta}(\Omega) - \beta(\Omega)\widetilde{\alpha}(\Omega) \ , \tag{3.27}$$

then the corresponding speed s is given by

$$s = \lambda_c(\Omega) + \kappa\widetilde{\mathcal{D}}(\Omega) \ . \tag{3.28}$$

In this way we conclude that the coordinates (R, κ, Ω) serve as global coordinates for the wave manifold $\mathcal{W}$ associated with a quadratic model satisfying Assumption 3.6. The coordinate R determines the strength of a wave, κ is the "distance" from the elliptic region in state space, and Ω is the angle of the shock wave (or of the eigenvector direction, in case of a rarefaction wave). For a geometric interpretation of these coordinates, see Fig. 3.1.

Associated to each point of $\mathcal{W}$ is a left state U_- and a speed s, and therefore a dynamical system (2.3). For quadratic models, the dynamical system associated with $(R, \kappa, \Omega) \in \mathcal{W}$ generically has four critical points, two of which are chosen to serve as left and right states in a shock wave. Therefore, generically there are $\binom{4}{2}$ points in $\mathcal{W}$, for $R > 0$, that correspond to the same dynamical system, with different choices of critical points for U_- and U_+. Interchanging the roles of U_- and U_+ corresponds to taking $R < 0$, giving $\binom{4}{2}$ more critical points. Therefore the mapping $\mathcal{W} \to \mathcal{X}_2(\mathbf{R}^2)$, where $\mathcal{X}_2(\mathbf{R}^2)$ is the space of all planar dynamical systems with quadratic flux functions, is a $2\binom{4}{2}$–fold covering of $\mathcal{W}$ in small neighborhoods of generic points in $\mathcal{W}$. Folds of this map correspond to dynamical systems in which collision of critical points occurs.

The foregoing defines the framework for the analysis of admissible shock waves.

4. ADMISSIBILITY BOUNDARY

In this section we employ the fundamental wave manifold to determine the set of admissible shock waves. As discussed in the previous section, each point in $\mathcal{W}$ can be regarded as a shock wave between the two critical points U_- and U_+. A point in $\mathcal{W}$ can be admissible or not, in the sense of the following definition.

Definition 4.1. A point of $\mathcal{W}$ is *admissible* if there exists a connecting orbit between the critical points U_- and U_+ in the corresponding dynamical system (2.3). Admissible points form a subset $\mathcal{A} \subseteq \mathcal{W}$ called the *admissibility region*; the (topological) boundary $\partial\mathcal{A}$ of this region is the *admissibility boundary*.

Portions of the admissibility boundary are expected to be smooth (even analytic) hypersurfaces in $\mathcal{W}$, and crossing these portions transversally leads to a bifurcation of the vector field (2.3) that influences the connection between the critical points U_- and U_+. In order to determine the admissibility boundary, we shall study the set of points in $\mathcal{W}$ for which the corresponding vector field is not *structurally stable*. We address the problem of structural stability in the next subsection.

4.1 Stability of Polynomial Vector Fields in the Plane. Let X denote a *planar polynomial vector field of degree* $\leq n$, *i.e.*, a vector field of the form

$$X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}\ , \tag{4.1}$$

where $P = \Sigma p_{ij} x^i y^j$ and $Q = \Sigma q_{ij} x^i y^j$, $0 \leq i + j \leq n$. The space $\mathcal{X}_n$ of all planar polynomial vector fields of degree n is endowed with the structure of the affine space $\mathbf{R}^N$, where $N = (n+1)(n+2)$, and each vector field $X \in \mathcal{X}_n$ is identified with the N-tuple $(p_{00}, p_{10}, ..., p_{0n}; q_{00}, q_{10}, ..., q_{0n}) \in \mathbf{R}^N$. Using the usual metric on $\mathbf{R}^N$, $\mathcal{X}_n$ becomes a complete metric space.

In order to understand the bifurcations of a planar polynomial vector field, one uses the Poincaré compactification to capture the bifurcations at infinity. Following Refs. [**23**] and [**7**], we define the Poincaré compactification $\wp(X)$ of the vector field

$X \in \mathcal{X}_n$ to be the unique analytic extension of the vector field on the sphere $S^2 = \{ y \in \mathbf{R}^3 \mid \|y\| = 1 \}$ obtained by central projection of $\mathbf{R}^2 = \{ y \in \mathbf{R}^3 \mid y_3 = 1 \}$ into S^2. This vector field is constructed as follows. With the maps $\psi^+, \psi^- : \mathbf{R}^2 \to S^2$ defined by

$$\begin{aligned} \psi^+(x) &:= \frac{(x_1, x_2, 1)}{(x_1^2 + x_2^2 + 1)^{\frac{1}{2}}} \ , \\ \psi^-(x) &:= -\frac{(x_1, x_2, 1)}{(x_1^2 + x_2^2 + 1)^{\frac{1}{2}}} \ , \end{aligned} \tag{4.2}$$

a vector field $(\psi^+)'(X)$ on the upper hemisphere is induced by the field on $\mathbf{R}^2$, and similarly $(\psi^-)'(X)$ is obtained on the lower hemisphere. In order to avoid unbounded growth of these induced vector fields as y approaches the equator $S^1 \subset S^2$, one multiplies these vector fields by the factor y_3^{n-1}, thereby obtaining the Poincaré field $\wp(X)$. As proved in Ref. [**7**], this Poincaré field $\wp(X)$ is the unique analytic extension of the tangent field obtained by central projection onto $S^2 \backslash S^1$ such that S^1 is an invariant set. Note that the Poincaré field $\wp(X)$ is not symmetric when n is even.

The Poincaré compactification enables us to define the stability of a planar vector field $X \in \mathcal{X}_n$.

Definition 4.2.

(1) A vector field $X \in \mathcal{X}_n$ is said to be *structurally stable* if there is a neighborhood $V \subseteq \mathcal{X}_n$ and a map $h : V \to \mathrm{Hom}(S^2, S^1)$ (homeomorphisms of S^2 that preserve S^1) such that $h(X) = Id$ and $h(Y)$ maps orbits of $\wp(X)$ onto orbits of $\wp(Y)$ for every $Y \in V$.

(2) A vector field $X \in \mathcal{X}_n$ is said to be *regularly structurally stable* (see Ref. [**31**]) if, in addition, the homeomorphism h, restricted to every periodic orbit of X, is continuously differentiable along every smooth curve in V passing through X.

In order to characterize the set of regularly structurally stable polynomial vector fields, we shall make use of the following definitions.

Definition 4.3. A nonhyperbolic critical point U_{cr} for which the Jacobian matrix $\mathcal{J}(U_{cr})$ has exactly one eigenvalue equal to zero is called a *saddle-node*.

An invariant manifold that is tangent to the center eigenspace, *i.e.*, the eigenspace associated with the zero eigenvalue, is called a *center manifold*. Higher-order terms in the projection of a smooth vector field X onto a center manifold determine the type of a saddle-node point. A *saddle-node of the first type* is a saddle-node for which the Taylor expansion of the projection of X onto a center manifold begins with the second order terms. (This is usually what is meant by the term "saddle-node.") When the Taylor expansion of the projection starts with the third order terms, the saddle-node is called a *saddle-node point of the second type*, or a *weak saddle point*.

Recall that a saddle point has four hyperbolic sectors, bounded by four *separatrices*. A saddle-node of the first type has one parabolic sector bounded by the two *strong separatrices*, *i.e.*, separatrices tangent to the nonzero eigenspace, and two hyperbolic sectors, each bounded by a strong separatrix and separated from each other by the *weak separatrix*, *i.e.*, a separatrix tangent to the null eigenspace.

Definition 4.4. An orbit that is a separatrix for two different critical points, or twice a separatrix for the same critical point, is called a *double separatrix*. If the two critical points involved are saddle points, a double separatrix is called a *saddle*, or *heteroclinic, connection*. If a double separatrix connects a saddle point to itself, it is called a *homoclinic loop*.

Definition 4.5. A nonhyperbolic critical point U_{cr} for which the Jacobian matrix $\mathcal{J}(U_{cr})$ has a pair of purely imaginary eigenvalues is called a *Hopf point*.

In discussing the stability of vector fields, the hyperbolicity of periodic orbits plays an important role. The hyperbolicity of a periodic orbit γ is determined by the derivative of the Poincaré map. In order to define the Poincaré map, let L be a line segment transverse to γ at a point p on the orbit, and let $L_0 \subseteq L$ be a sufficiently small segment containing p. The *first return* or *Poincaré map* $P : L_0 \to L$ is defined to be the map that associates to each point $q \in L_0$ the first point $P(q)$ where the orbit of q returns and intersects L. Note that since γ is periodic, p is a fixed point for P.

Since P is a local diffeomorphism of the same class as the vector field [**22**], we can use the derivative of P to describe the behavior of the vector field in the neighborhood of γ. A classical formula of Poincaré gives that for a planar vector field X, $\log P'(p) = \int_\gamma \operatorname{div} X$ (see Ref. [**1**]).

Definition 4.6. We say that γ is a *hyperbolic* periodic orbit if p is a hyperbolic fixed point of the Poincaré map $P : L_0 \to L$, *i.e.*, if $P'(p) \neq 1$.

The following theorem of J. Sotomayor and R. Paterlini [**31**] characterizes the set of regularly structurally stable planar polynomial vector fields.

Theorem 4.7. *A vector field $X \in \mathcal{X}_n$ is regularly structurally stable if and only if it belongs to the set $\mathcal{RS}_n := \mathcal{RS}_n(1) \cap \mathcal{RS}_n(2) \cap \mathcal{RS}_n(3)$, where $\mathcal{RS}_n(i)$, $i = 1, 2, 3$, is defined by property i below:*

(1) *All the critical points of $\wp(X)$ are hyperbolic.*
(2) *All periodic orbits of $\wp(X)$ are hyperbolic.*
(3) *All double separatrices are contained in S^1.*

Furthermore, the regular bifurcation set, $\mathcal{RB}_n^1 = \mathcal{X}_n - \mathcal{RS}_n$, is contained in a countable union of analytic submanifolds of codimension ≥ 1.

4.2 Constituent Components of the Admissibility Boundary. To determine the admissibility boundary $\partial\mathcal{A}$ we study the regular bifurcation set $\mathcal{RB}_n^1$ in $\mathcal{W}$ and determine the subset that consists of bifurcations that influence the connection between the left and right states in a shock wave. In this paper, we focus on two mechanisms responsible for the loss of regular structural stability, namely occurrence of a nonhyperbolic critical point or the occurrence of a double separatrix.

In order to use Theorem 4.7, we require some mild assumptions on the second order terms of the flux function F and the viscosity matrix D that exclude pathological behavior at infinity. The set of vector fields that remain is *generic*, in the sense that it is a subset of $\mathcal{X}_n$ that is open and dense for the metric of $\mathcal{X}_n$. More precisely, let $\mathcal{NP}$ denote the set of all vector fields $X \in \mathcal{X}_n$ having at least one nonhyperbolic critical point at infinity, and $\mathcal{NO}$ the set of all vector fields such that S^1 is a nonhyperbolic periodic orbit. As proved in Ref. [**7**], the set $\mathcal{G} := \mathcal{X}_n - (\mathcal{NP} \cup \mathcal{NO})$ is generic in $\mathcal{X}_n$. Therefore we shall work with the following assumption.

Assumption 4.8. The vector field $X(U) = D^{-1}\{-s\,[U - U_-] + F(U) - F(U_-)\}$ belongs to the set $\mathcal{G}$.

The conditions on the coefficients of F and D that guarantee this assumption will be determined in Sec. 5; we shall use the following polar coordinate transformation of the Poincaré vector field $\wp(X)$. To each $(\Theta, \zeta) \in \mathbf{R} \times (-1, 1)$ we associate the point

$$(x, y, z) = (1 + \zeta^2)^{-\frac{1}{2}}(\cos\Theta, \sin\Theta, \zeta) \in S^2 \backslash \{(0, 0, \pm 1)\}\ . \tag{4.3}$$

Expressed in (Θ, ζ)-coordinates, $\wp(X)$ is

$$(1 + \zeta^2)^{\frac{1-n}{2}} \left[\left(\sum_{i=0}^{n} \zeta^i T_{n-i}(\Theta) \right) \frac{\partial}{\partial\Theta} - \zeta \left(\sum_{i=0}^{n} \zeta^i R_{n-i}(\Theta) \right) \frac{\partial}{\partial\zeta} \right], \tag{4.4}$$

where

$$\begin{aligned} T_k(\Theta) &= -P_k(\cos\Theta, \sin\Theta)\sin\Theta + Q_k(\cos\Theta, \sin\Theta)\cos\Theta\ , \\ R_k(\Theta) &= P_k(\cos\Theta, \sin\Theta)\cos\Theta + Q_k(\cos\Theta, \sin\Theta)\sin\Theta\ . \end{aligned} \tag{4.5}$$

Here, P_k and Q_k are the homogeneous polynomials of degree k such that $P = \sum P_k$, and $Q = \sum Q_k$.

When studying critical points of the induced vector field at infinity, three cases can occur (see Ref. [**7**]):

(1) every point on the equator is a critical point, *i.e.*, $T_n(\Theta) \equiv 0$;
(2) real zeros of $T_n(\Theta)$ are critical points in S^1; and
(3) the equator is a closed orbit, *i.e.*, $T_n(\Theta)$ has no real zeros.

In the first case, the vector field X is nonhyperbolic everywhere at infinity. In the second case, since the Jacobian matrix evaluated at the critical points on the equator is given by

$$\begin{pmatrix} T_n'(\Theta) & T_{n-1}(\Theta) \\ 0 & -R_n(\Theta) \end{pmatrix}, \tag{4.6}$$

the corresponding critical point at infinity is hyperbolic if and only if

$$T_n'(\Theta)\, R_n(\Theta) \neq 0\,. \tag{4.7}$$

In the case when $T_n(\Theta)$ has no real zeros, *i.e.*, the equator is a closed orbit, it is necessary that n be odd, since T_n is a polynomial of degree $n+1$ in $\tan\Theta$; following Refs. [**1**] and [**31**], the condition for a nonhyperbolic periodic orbit at infinity is given by requiring that

$$\log P'(p) = \int_0^{2\pi} R_n(\Theta) T_n{}^{-1}(\Theta)\, d\Theta \neq 0\ . \tag{4.8}$$

Notice that for quadratic vector fields F, this case never occurs.

We concentrate now on describing the manifolds that comprise the admissibility boundary. Nonhyperbolic critical points determine the following loci in $\mathcal{W}$.

Definition 4.9. The locus of points in $\mathcal{W}$ corresponding to dynamical systems having a nonhyperbolic critical point of a saddle-node type is called the *saddle-node locus* $\mathcal{SN}$, whereas points having a nonhyperbolic critical point of Hopf type comprise the *Hopf locus* $\mathcal{HF}$.

With respect to the occurrence of a double separatrix, we distinguish two classes of regularly structurally unstable dynamical systems loci.

Definition 4.10. The points of $\mathcal{W}$ such that the corresponding dynamical system has two saddle points connected by a double separatrix constitute the *heteroclinic locus* $\mathcal{HT}$, whereas the points of $\mathcal{W}$ for which the corresponding dynamical system has a double separatrix forming a homoclinic loop comprise the *homoclinic locus* $\mathcal{HO}$.

These loci can be subdivided further according to the role played by U_- and U_+. For instance, a point of the saddle-node locus $\mathcal{SN}$ lies on the *characteristic manifold* $\mathcal{C}$ if $U_{cr} := U_+ = U_-$ is the nonhyperbolic critical point. If, however, $U_{cr} := U_- \neq U_+$, this saddle-node point lies on the *left sonic locus* $\mathcal{S}_L$, and similarly for the *right sonic locus* $\mathcal{S}_R$. Otherwise, neither U_- nor U_+ is the nonhyperbolic critical point. (Another way to understand the distinction is by considering a one-parameter family of dynamical systems crossing the saddle-node locus. If this curve crosses $\mathcal{C}$, $\mathcal{S}_L$, or $\mathcal{S}_R$, then typically the family undergoes a transcritical bifurcation; at other points of the saddle-node locus, the family typically undergoes saddle-node bifurcation.) The Hopf and homoclinic loci can be subdivided in a similar fashion. Furthermore, a point on the heteroclinic locus can be classified according to whether U_- or U_+ is involved in the heteroclinic orbit.

This classification can be understood more systematically as follows. Two points of $\mathcal{W}$ can be regarded as *equivalent* provided that their corresponding dynamical systems are the same. Thus two equivalent points of $\mathcal{W}$ differ only as to which critical points are chosen as U_- and U_+. In terms of the map $\mathcal{W} \to \mathcal{X}_n$ that associates a dynamical system to each point of $\mathcal{W}$, the equivalence class $[\,p\,]$ containing a point of $p \in \mathcal{W}$ is the preimage of the image of p. We call this equivalence class the *extension* of p.

In the next section we restrict to the case of quadratic models, and give explicit formulae for some of the loci defined here.

5. ADMISSIBLE WAVES IN QUADRATIC MODELS

Our goal is to determine the region of admissible shock waves in $\mathcal{W}$ for generic quadratic flux functions, using the results and ideas developed in the previous sections. We study regularly structurally unstable dynamical systems

$$\dot{U} = D^{-1}\{-s\,[U - U_-] + F(U) - F(U_-)\}\ , \quad U \in \mathbf{R}^2\ , \tag{5.1}$$

for a fixed viscosity matrix D, F quadratic in U, and parameters $(s, U_-) \in \mathbf{R}^3$. The focus is on bifurcations corresponding to the occurrence of either (a) a nonhyperbolic critical point or (b) a double separatrix.

5.1 The Admissibility Boundary. To simplify the analysis in $\mathcal{W}$, we use the notation introduced in Sec. 3; we also make use of the following functions related to the flux function F:

$$\begin{aligned}
\mu(\Omega) &:=(-\sin\Omega,\cos\Omega)\, F''(0)\begin{pmatrix}\cos\Omega\\ \sin\Omega\end{pmatrix}^2 = \alpha(\Omega)\cos\Omega+\beta(\Omega)\sin\Omega\ ,\\
\widetilde{\mu}(\Omega) &:=(\cos\Omega,\sin\Omega)\, F''(0)\begin{pmatrix}\cos\Omega\\ \sin\Omega\end{pmatrix}^2 = \widetilde{\alpha}(\Omega)\cos\Omega+\widetilde{\beta}(\Omega)\sin\Omega\ ,
\end{aligned}\tag{5.2}$$

$$\begin{aligned}
\nu(\Omega) &:=(-\sin\Omega,\cos\Omega)\, F''(0)\begin{pmatrix}\cos\Omega\\ \sin\Omega\end{pmatrix}\otimes\begin{pmatrix}-\sin\Omega\\ \cos\Omega\end{pmatrix}\\
&=-\,\alpha(\Omega)\sin\Omega+\beta(\Omega)\cos\Omega\ ,\\
\widetilde{\nu}(\Omega) &:=(\cos\Omega,\sin\Omega)\, F''(0)\begin{pmatrix}\cos\Omega\\ \sin\Omega\end{pmatrix}\otimes\begin{pmatrix}-\sin\Omega\\ \cos\Omega\end{pmatrix}\\
&=-\,\widetilde{\alpha}(\Omega)\sin\Omega+\beta(\Omega)\cos\Omega\ ,
\end{aligned}\tag{5.3}$$

$$\begin{aligned}
\rho(\Omega) &:=(-\sin\Omega,\cos\Omega)\, F''(0)\begin{pmatrix}-\sin\Omega\\ \cos\Omega\end{pmatrix}^2\\
&=-\,(\alpha'(\Omega)+\widetilde{\alpha}(\Omega))\sin\Omega+(\beta'(\Omega)+\widetilde{\beta}(\Omega))\cos\Omega\ ,\\
\widetilde{\rho}(\Omega) &:=(\cos\Omega,\sin\Omega)\, F''(0)\begin{pmatrix}-\sin\Omega\\ \cos\Omega\end{pmatrix}^2\\
&=-\,(\widetilde{\alpha}'(\Omega)-\alpha(\Omega))\sin\Omega+(\widetilde{\beta}'(\Omega)-\beta(\Omega))\cos\Omega\ .
\end{aligned}\tag{5.4}$$

Also, for every $\Omega\in\mathbf{R}$ mod 2π, let $\mathrm{O}(\Omega)$ be the rotation matrix

$$\mathrm{O}(\Omega) := \begin{pmatrix}\cos\Omega & -\sin\Omega\\ \sin\Omega & \cos\Omega\end{pmatrix}\ .\tag{5.5}$$

Finally, define the matrix $\Delta(\Omega)$ to be

$$\Delta(\Omega) := \mathrm{O}(\Omega)^{-1}D^{-1}\mathrm{O}(\Omega)\ ;\tag{5.6}$$

its entries are denoted Δ_{ij}, $i,j=1,2$.

Lemma 5.1. *Let (R,κ,Ω) be a point of $\mathcal{W}$ with corresponding midpoint $\overline{U}$. Then the following equations hold:*

$$(\cos\Omega,\sin\Omega)F'(\overline{U})\begin{pmatrix}-\sin\Omega\\ \cos\Omega\end{pmatrix} = \tfrac{1}{2}\frac{\eta}{\mathcal{D}}-\tfrac{1}{2}\mathcal{D}'\kappa\ ,\tag{5.7}$$

$$(-\sin\Omega,\cos\Omega)F'(\overline{U})\begin{pmatrix}-\sin\Omega\\ \cos\Omega\end{pmatrix} = s+\mathcal{D}\kappa\ ,\tag{5.8}$$

where $\mathcal{D}$, η, and s are given by Eqs. (3.20), (3.22), and (3.17) respectively.

Proof. To prove the first equality, we differentiate the second of Eqs. (3.16) with respect to Ω to find that

$$(\cos\Omega,\sin\Omega)F'(\overline{U})\begin{pmatrix}-\sin\Omega\\ \cos\Omega\end{pmatrix} = (\widetilde{\alpha}'-\alpha)\overline{u}+(\widetilde{\beta}'-\beta)\overline{v}+\widetilde{\gamma}'-\gamma\ .\tag{5.9}$$

Now $\widetilde{\alpha}' = 2(\alpha - \alpha_0)$, and similarly for $\widetilde{\beta}$ and $\widetilde{\gamma}$; therefore, by Eq. (3.18), the right-hand side of Eq. (5.9) is $-2(\alpha_0\overline{u} + \beta_0\overline{v} + \gamma_0)$. Substituting Eqs. (3.26) yields

$$-2(\alpha_0\overline{u} + \beta_0\overline{v} + \gamma_0) = -2(\alpha_0 u_c + \beta_0 v_c + \gamma_0) - 2(\alpha\beta_0 - \beta\alpha_0)\kappa \ . \tag{5.10}$$

Notice that $\mathcal{D}' = \alpha\beta'' - \beta\alpha''$; also, $\alpha'' = -4(\alpha - \alpha_0)$, and similarly for β and γ. Consequently the second term on the right-hand side of Eq. (5.10) is $-\frac{1}{2}\mathcal{D}'\kappa$, while definition (3.22) implies that the first term is $\frac{1}{2}\eta/\mathcal{D}$. Thus Eq. (5.7) is verified.

In a similar fashion, differentiating the first of Eqs. (3.16) with respect to Ω gives

$$(-\sin\Omega, \cos\Omega)F'(\overline{U})\begin{pmatrix} -\sin\Omega \\ \cos\Omega \end{pmatrix} = (\alpha' + \widetilde{\alpha})\overline{u} + (\beta' + \widetilde{\beta})\overline{v} + \gamma' + \widetilde{\gamma} \ . \tag{5.11}$$

From Eqs. (3.17), (3.19), and (3.26) it follows that

$$(\alpha' + \widetilde{\alpha})\overline{u} + (\beta' + \widetilde{\beta})\overline{v} + \gamma' + \widetilde{\gamma} = s - (\alpha'\beta - \alpha\beta')\kappa = s + \mathcal{D}\kappa \ . \tag{5.12}$$

Equation (5.8) results. □

Theorem 5.2. *Let (R, κ, Ω) be a point of $\mathcal{W}$ with corresponding speed s and left and right states U_- and U_+. Then the dynamical system (5.1) is equivalent to*

$$\begin{aligned}\dot{X} = \Delta\left[\pm\tfrac{1}{2}R\begin{pmatrix} \widetilde{\mu} & \widetilde{\nu} \\ \mu & \nu \end{pmatrix} + \kappa\begin{pmatrix} 0 & -\frac{1}{2}\mathcal{D}' \\ 0 & \mathcal{D} \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{2}\frac{\eta}{\mathcal{D}} \\ 0 & 0 \end{pmatrix}\right] X \\ + \tfrac{1}{2}\Delta\begin{pmatrix} \widetilde{\mu}x^2 + 2\widetilde{\nu}xy + \widetilde{\rho}y^2 \\ \mu x^2 + 2\nu xy + \rho y^2 \end{pmatrix} \ ,\end{aligned} \tag{5.13}$$

where $X = 0$ corresponds to $U = U_\pm$.

Proof. We apply the change of variables $X := \mathrm{O}(\Omega)^{-1}[U - U_\pm]$ to obtain an equivalent dynamical system. Because F is quadratic in U, X satisfies

$$\dot{X} = \Delta \mathrm{O}^{-1}\left\{[-s + F'(U_\pm)]\,\mathrm{O}\,X + \tfrac{1}{2}F''(0)\,[\mathrm{O}\,X]^2\right\} \ . \tag{5.14}$$

Using the relationship $U_\pm = \overline{U} \pm \frac{1}{2}Rr(\Omega)$, Eq. (5.14) becomes

$$\begin{aligned}\dot{X} = \Delta \mathrm{O}^{-1}\left[-s + F'(\overline{U})\right]\mathrm{O}\,X \pm \tfrac{1}{2}R\,\Delta\,\mathrm{O}^{-1}F''(0)\cdot r(\Omega)\otimes[\mathrm{O}\,X] \\ + \tfrac{1}{2}\Delta\,\mathrm{O}^{-1}F''(0)\,[\mathrm{O}\,X]^2 \ .\end{aligned} \tag{5.15}$$

Invoking the definitions (5.2)–(5.4), we obtain that

$$\begin{aligned}\dot{X} = \Delta \mathrm{O}^{-1}\left[-s + F'(\overline{U})\right]\mathrm{O}\,X \\ \pm \tfrac{1}{2}R\,\Delta\begin{pmatrix} \widetilde{\mu} & \widetilde{\nu} \\ \mu & \nu \end{pmatrix} X + \tfrac{1}{2}\Delta\begin{pmatrix} \widetilde{\mu}x^2 + 2\widetilde{\nu}xy + \widetilde{\rho}y^2 \\ \mu x^2 + 2\nu xy + \rho y^2 \end{pmatrix} \ .\end{aligned} \tag{5.16}$$

According to Lemma 5.1 and Eq. (3.11),

$$\mathrm{O}^{-1}\left[-s + F'(\overline{U})\right]\mathrm{O}\,X = \begin{pmatrix} 0 & \frac{1}{2}\frac{\eta}{\mathcal{D}} - \frac{1}{2}\mathcal{D}'\kappa \\ 0 & \mathcal{D}\kappa \end{pmatrix} X \ . \tag{5.17}$$

This equation, when combined with Eq. (5.16), yields system (5.13). □

Notice that the vector field (5.13) can be written in the form

$$\begin{aligned}\dot{X} = &\begin{pmatrix} \mathcal{J}_{11} & \mathcal{J}_{12} \\ \mathcal{J}_{21} & \mathcal{J}_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &+ \tfrac{1}{2} \begin{pmatrix} (\Delta_{11}\widetilde{\mu} + \Delta_{12}\mu)x^2 + 2(\Delta_{11}\widetilde{\nu} + \Delta_{12}\nu)xy + (\Delta_{11}\widetilde{\rho} + \Delta_{12}\rho)y^2 \\ (\Delta_{21}\widetilde{\mu} + \Delta_{22}\mu)x^2 + 2(\Delta_{21}\widetilde{\nu} + \Delta_{22}\nu)xy + (\Delta_{21}\widetilde{\rho} + \Delta_{22}\rho)y^2 \end{pmatrix} ,\end{aligned} \tag{5.18}$$

where

$$\begin{aligned} \mathcal{J}_{11} &= \pm\tfrac{1}{2}R(\Delta_{11}\widetilde{\mu} + \Delta_{12}\mu) , \\ \mathcal{J}_{12} &= \pm\tfrac{1}{2}R(\Delta_{11}\widetilde{\nu} + \Delta_{12}\nu) + \kappa(-\tfrac{1}{2}\Delta_{11}\mathcal{D}' + \Delta_{12}\mathcal{D}) + \tfrac{1}{2}\Delta_{11}\frac{\eta}{\mathcal{D}} , \\ \mathcal{J}_{21} &= \pm\tfrac{1}{2}R(\Delta_{21}\widetilde{\mu} + \Delta_{22}\mu) , \\ \mathcal{J}_{22} &= \pm\tfrac{1}{2}R(\Delta_{21}\widetilde{\nu} + \Delta_{22}\nu) + \kappa(-\tfrac{1}{2}\Delta_{21}\mathcal{D}' + \Delta_{22}\mathcal{D}) + \tfrac{1}{2}\Delta_{21}\frac{\eta}{\mathcal{D}} . \end{aligned} \tag{5.19}$$

This notation will be used below. Also notice that for a fixed shock angle Ω, the two parameter family of dynamical systems (5.13) in $\mathcal{W}$, with parameters κ and R, has a particularly simple form, since the coefficients of the vector field (5.13) depend linearly on R and κ.

5.2 Generic Flux Functions. Before we proceed with determining the boundary of the admissibility region in $\mathcal{W}$, we must determine conditions on the second order terms of the flux function F and the viscosity matrix D that assure that the corresponding vector field (5.1) belongs to the class $\mathcal{G} := \mathcal{X}_n - (\mathcal{NP} \cup \mathcal{NO})$. Since this field is quadratic, S^1 is never a periodic orbit for the Poincaré field, so that the set $\mathcal{NO}$ is empty. Therefore, following the results from the previous section, the vector field (5.1) is generic if and only if there are no nonhyperbolic critical points at infinity.

Critical points at infinity correspond to the zeros of the function $T_2(\Theta)$, defined by the first of Eqs. (4.5). Polar coordinate transformation of the vector field (5.1) gives that

$$\begin{pmatrix} R_2(\Theta) \\ T_2(\Theta) \end{pmatrix} = \tfrac{1}{2}\mathrm{O}(\Theta)^{-1}D^{-1}F''(0)\begin{pmatrix} \cos\Theta \\ \sin\Theta \end{pmatrix}^2 . \tag{5.20}$$

In terms of the functions μ and $\widetilde{\mu}$ defined by Eqs. (5.2),

$$\begin{aligned} R_2(\Theta) &= \tfrac{1}{2}\left[\Delta_{11}(\Theta)\widetilde{\mu}(\Theta) + \Delta_{12}(\Theta)\mu(\Theta)\right] , \\ T_2(\Theta) &= \tfrac{1}{2}\left[\Delta_{21}(\Theta)\widetilde{\mu}(\Theta) + \Delta_{22}(\Theta)\mu(\Theta)\right] . \end{aligned} \tag{5.21}$$

Definition 5.3. An angle Θ_* such that $T_2(\Theta_*) = 0$ is called a *viscosity angle*. A viscosity angle Θ_* is called *exceptional* if $R_2(\Theta_*) = 0$.

This definition, together with the condition (4.7) for the hyperbolicity of critical points at infinity, implies the following proposition.

Proposition 5.4. *The vector field (5.1) is generic if and only if all viscosity angles are distinct and nonexceptional.*

Notice that genericity does not depend on the parameters U_- and s, but rather on the second order terms of the flux function F and on the viscosity matrix D.

Remark. When $D = I$ and the quadratic model is written in the normal form of Schaeffer-Shearer [**24**], genericity has a simple interpretation: having an exceptional viscosity angle corresponds to lying on the boundary between the Cases I and II, whereas a viscosity angle is a double root of the equation $T_2 = 0$ when the model lies on the boundary between the Cases III and IV. It has been proved in Ref. [**3**] that, at the boundary between the Cases I and II, the corresponding vector field undergoes a transcritical bifurcation at infinity, and that, at the boundary between the Cases III and IV, the vector field undergoes a saddle-node bifurcation at infinity.

5.3 Saddle-Node Locus. The saddle-node locus $\mathcal{SN}$ comprises points of $\mathcal{W}$ for which one of the critical points U_{cr} of the corresponding dynamical system has an eigenvalue equal to zero, *i.e.*, the determinant of the Jacobian vanishes:

$$\det\left\{D^{-1}\left[-s + F'(U_{cr})\right]\right\} = 0 \ . \tag{5.22}$$

Notice that this equation is independent of D. The next result gives an explicit formula for the left-hand side of Eq. (5.22) in the case when U_{cr} is U_- or U_+.

Theorem 5.5. *With U_-, U_+, and s written in terms of R, κ, and Ω,*

$$\det\left[-s + F'(U_\pm)\right] = \pm\tfrac{1}{2}R\left\{\mp\tfrac{1}{2}\widetilde{\mathcal{D}}R + \left[\widetilde{\mu}\mathcal{D} + \tfrac{1}{2}\mu\mathcal{D}'\right]\kappa - \tfrac{1}{2}\mu\frac{\eta}{\mathcal{D}}\right\} \ . \tag{5.23}$$

We use the following lemma to prove this theorem.

Lemma 5.6. *The function $\widetilde{\mathcal{D}}(\Omega)$ defined by Eq. (3.27) satisfies*

$$\widetilde{\mathcal{D}}(\Omega) = \mu\widetilde{\nu} - \nu\widetilde{\mu} \ . \tag{5.24}$$

Proof. According to definitions (5.2)–(5.4),

$$\begin{aligned} \mu\widetilde{\nu} - \nu\widetilde{\mu} &= (\alpha\cos\Omega + \beta\sin\Omega)(-\widetilde{\alpha}\sin\Omega + \widetilde{\beta}\cos\Omega) \\ &\qquad - (-\alpha\sin\Omega + \beta\cos\Omega)(\widetilde{\alpha}\cos\Omega + \widetilde{\beta}\sin\Omega) \\ &= \alpha\widetilde{\beta} - \beta\widetilde{\alpha} \ , \end{aligned} \tag{5.25}$$

which is $\widetilde{\mathcal{D}}$. $\square$

We now prove Theorem 5.5.

Proof. Notice that the Jacobian of the vector field (5.1) at the critical point $U_\pm$ equals

$$D^{-1}\left[-s + F'(U_\pm)\right] \ . \tag{5.26}$$

Moreover, by Theorem 5.2, the Jacobian of the equivalent vector field (5.13), evaluated at the critical point $X = \mathrm{O}(\Omega)^{-1}(U - U_\pm) = 0$, equals

$$\Delta \begin{pmatrix} \pm\frac{R}{2}\widetilde{\mu} & \pm\frac{R}{2}\widetilde{\nu} - \frac{1}{2}\mathcal{D}'\kappa + \frac{1}{2}\frac{\eta}{\mathcal{D}} \\ \pm\frac{R}{2}\mu & \pm\frac{R}{2}\nu + \mathcal{D}\kappa \end{pmatrix} . \tag{5.27}$$

The matrices (5.26) and (5.27) are related by a similarity transformation based on $\mathrm{O}(\Omega)$; therefore

$$\det\left[-s + F'(U_\pm)\right] = \det \begin{pmatrix} \pm\frac{R}{2}\widetilde{\mu} & \pm\frac{R}{2}\widetilde{\nu} - \frac{1}{2}\kappa\mathcal{D}' + \frac{1}{2}\frac{\eta}{\mathcal{D}} \\ \pm\frac{R}{2}\mu & \pm\frac{R}{2}\nu + \kappa\mathcal{D} \end{pmatrix} . \tag{5.28}$$

Formula (5.23) follows from this equation and Lemma 5.6. □

By solving Eq. (5.22) when U_{cr} is one of the critical points U_- or U_+, one obtains the loci $\mathcal{C}$, $\mathcal{S}_L$, and S_R.

Corollary 5.7. *The characteristic manifold $\mathcal{C}$ and the sonic loci $\mathcal{S}_L$ and $\mathcal{S}_R$ are ruled surfaces in $\mathcal{W}$. More precisely,*
(a) $R = 0$ defines the characteristic manifold $\mathcal{C}$,
(b) $-A(\Omega)R + B(\Omega)\kappa + C(\Omega) = 0$ defines the left sonic locus S_L, and
(c) $A(\Omega)R + B(\Omega)\kappa + C(\Omega) = 0$ defines the right sonic locus S_R.
Here the coefficients A, B, and C are given by

$$A = -\tfrac{1}{2}\widetilde{\mathcal{D}} , \tag{5.29}$$

$$B = \widetilde{\mu}\mathcal{D} + \tfrac{1}{2}\mu\mathcal{D}' , \tag{5.30}$$

$$C = -\tfrac{1}{2}\mu\frac{\eta}{\mathcal{D}} . \tag{5.31}$$

5.4 Hopf Locus. For a nonhyperbolic critical point U_{cr} such that the two eigenvalues are purely imaginary, the discriminant of the Jacobian is negative and the trace of the Jacobian vanishes:

$$\operatorname{discrm}\left\{D^{-1}\left[-s + F'(U_{cr})\right]\right\} < 0 , \tag{5.32}$$

$$\operatorname{tr}\left\{D^{-1}\left[-s + F'(U_{cr})\right]\right\} = 0 . \tag{5.33}$$

The Hopf locus $\mathcal{HF}$ comprises points in $\mathcal{W}$ with corresponding dynamical systems that contain a nonhyperbolic critical point of this type. The points of the Hopf locus for which U_{cr} equals U_- or U_+ form the left and right Hopf loci $\mathcal{HF}_L$ and $\mathcal{HF}_R$.

Theorem 5.8. *With U_-, U_+, and s written in terms of R, κ, and Ω,*

$$\begin{aligned} tr\left\{D^{-1}\left[-s + F'(U_\pm)\right]\right\} = &\pm \tfrac{1}{2} R\left(\Delta_{11}\widetilde{\mu} + \Delta_{12}\mu + \Delta_{21}\widetilde{\nu} + \Delta_{22}\nu\right) \\ &+ \kappa\left(-\tfrac{1}{2}\Delta_{21}\mathcal{D}' + \Delta_{22}\mathcal{D}\right) + \tfrac{1}{2}\Delta_{21}\frac{\eta}{\mathcal{D}} . \end{aligned} \tag{5.34}$$

Proof. Again because matrices (5.26) and (5.27) are related by a similarity transformation, the left-hand side of Eq. (5.34) is the trace of matrix (5.27). □

Equations (5.32) and (5.33), with U_{cr} taken to be U_- and U_+, define the left and right Hopf loci, respectively.

Corollary 5.9. *The left and right Hopf loci $\mathcal{HF}_L$ and $\mathcal{HF}_R$ are subsets of ruled surfaces in $\mathcal{W}$. More precisely,*
(a) $-\widetilde{A}(\Omega)R+\widetilde{B}(\Omega)\kappa+\widetilde{C}(\Omega)=0$ on left Hopf locus $\mathcal{HF}_L$, and
(b) $\widetilde{A}(\Omega)R+\widetilde{B}(\Omega)\kappa+\widetilde{C}(\Omega)=0$ on right Hopf locus $\mathcal{HF}_R$.

Here the coefficients $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{C}$ are defined by

$$\widetilde{A}(\Omega)=\tfrac{1}{2}\left(\Delta_{11}\widetilde{\mu}+\Delta_{12}\mu+\Delta_{21}\widetilde{\nu}+\Delta_{22}\nu\right)\ , \tag{5.35}$$

$$\widetilde{B}(\Omega)=-\tfrac{1}{2}\Delta_{21}\mathcal{D}'+\Delta_{22}\mathcal{D}\ , \tag{5.36}$$

$$\widetilde{C}(\Omega)=\tfrac{1}{2}\Delta_{21}\frac{\eta}{\mathcal{D}}\ . \tag{5.37}$$

The boundary of the Hopf locus generically consists of points such that the discriminant, as well as the trace, of the Jacobian is zero (*cf.* inequality (5.32)); thus the boundary is the intersection of the ruled surfaces above with $\mathcal{C}$, $\mathcal{S}_L$, and $\mathcal{S}_R$. This intersection is the Bogdanov-Takens locus, which is discussed below. For points on the ruled surfaces of Corollary 5.4 that are neither Hopf nor Bogdanov-Takens points, $U_\pm$ is a saddle point with eigenvalues σ_1 and σ_2 such that $\sigma_1=-\sigma_2$.

5.5 Bogdanov-Takens and Homoclinic Loci. In contrast to the local phenomena that determine the saddle-node and Hopf loci, the existence of a homoclinic orbit is a nonlocal phenomenon. Although we do not know an explicit formula for the homoclinic locus, the tangent space of this locus can be calculated at the points where it intersects $\mathcal{C}$. For a point in this intersection, the corresponding dynamical system has a Bogdanov-Takens critical point at $\overline{U}$.

Definition 5.10. The *Bogdanov-Takens locus $\mathcal{BT}$* consist of points of $\mathcal{W}$ for which one of the critical points U_{cr} of the corresponding dynamical system satisfies

$$\begin{aligned}\det\left\{D^{-1}\left[-s+F'(U_{cr})\right]\right\}&=0\ ,\\ \operatorname{tr}\left\{D^{-1}\left[-s+F'(U_{cr})\right]\right\}&=0\ .\end{aligned} \tag{5.38}$$

Thus the Jacobian at U_{cr} is nilpotent. Formulae for points on the Bogdanov-Takens locus such that U_{cr} equals U_- or U_+ follow immediately from Eqs. (5.23) and (5.34).

Majda and Pego [**20**] have given, for strictly hyperbolic systems of conservation laws, a criterion for a viscosity matrix D to be strictly stable, in the sense that for every state $\overline{U}$, the Cauchy problem for the linearized equation $V_t+F'(\overline{U})V_x=\epsilon D(\overline{U})V_{xx}$ is uniformly well-posed in L^2 as $\epsilon\to 0$. For systems of conservation laws with elliptic regions, however, even $D=I$ fails to satisfy the Majda-Pego criterion. In this context, a more useful question is the following: given a viscosity matrix, for which states $\overline{U}$ is the linearized equation uniformly well-posed? This question is related to the Bogdanov-Takens locus in a manner that we now make precise.

Definition 5.11. A state $\overline{U}$ is a *Majda-Pego point for the viscosity matrix D* provided that $F'(\overline{U})$ has distinct real eigenvalues and that the Cauchy problem for $V_t+F'(\overline{U})V_x=\epsilon D(\overline{U})V_{xx}$ is uniformly well-posed in L^2 as $\epsilon\to 0$.

Proposition 5.12. *A state $\overline{U}$ is a Majda-Pego point for the viscosity matrix D if and only if* $\operatorname{tr}\left[-s+F'(\overline{U})\right]$ *and* $\operatorname{tr}\left\{D^{-1}\left[-s+F'(\overline{U})\right]\right\}$ *have the same (nonzero) sign for each eigenvalue s of $F'(\overline{U})$. In particular, the boundary of the set of Majda-Pego points in contained in the union of the projections onto state space of the coincidence locus $\mathcal{E}$ and the Bogdanov-Takens locus $\mathcal{BT}\cap\mathcal{C}$.*

Proof. Let R be a matrix of eigenvectors of $F'(\overline{U})$, and let λ_1 and λ_2 be the diagonal elements of $R^{-1}F'(\overline{U})R$. Then, as shown in Ref. [**20**], $\overline{U}$ is a Majda-Pego point if and only if the diagonal elements d_{11} and d_{22} of $R^{-1}DR$ are positive. A short calculation shows that

$$\operatorname{tr}\left\{D^{-1}\left[-\lambda_k+F'(\overline{U})\right]\right\} = d_{kk}\cdot(\det D)^{-1}\operatorname{tr}\left[-\lambda_k+F'(\overline{U})\right]$$

for $k=1,2$, from which the first statement follows. On the boundary of the set of Majda-Pego points, either $\operatorname{tr}\left[-s+F'(\overline{U})\right]=0$ or $\operatorname{tr}\left\{D^{-1}\left[-s+F'(\overline{U})\right]\right\}=0$ for some eigenvalue s of $F'(\overline{U})$. In the first case, $\overline{U}\in\mathcal{E}$, and in the second case, $\overline{U}\in\mathcal{BT}$. □

Recall that the universal unfolding of a vector field with a (nondegenerate) Bogdanov-Takens critical point contains dynamical systems with homoclinic orbits; see Ref. [**8**]. A normal form for this unfolding is the system

$$\begin{aligned}\dot{x} &= y\ ,\\ \dot{y} &= \mu_1+\mu_2 y+ax^2+bxy\ ,\end{aligned}\tag{5.39}$$

where $a,b\neq 0$. Notice, however, that all dynamical systems corresponding to points of $\mathcal{W}$ have the state U_- as a critical point. For dynamical systems having the origin as a critical point, a universal unfolding of a Bogdanov-Takens point has been given by Hirschberg and Knobloch [**9**] in the following normal form:

$$\begin{aligned}\dot{x} &= y+\mathcal{O}(3)\ ,\\ \dot{y} &= \nu x+\mu y+Kx^2+Cxy+\mathcal{O}(3)\ .\end{aligned}\tag{5.40}$$

As parameterized by (R,κ,Ω), the dynamical systems in a neighborhood of a point on the Bogdanov-Takens locus form a three-parameter unfolding. The two-parameter unfolding obtained by restricting to fixed $\Omega=\Omega_0$ is, in fact, universal (in the Hirschberg-Knobloch sense), provided that Ω_0 is not a viscosity angle. More precisely:

Theorem 5.13. *Consider a quadratic model for which $\eta\neq 0$. Suppose that Ω_0 is such that $\widetilde{B}(\Omega_0)\neq 0$, and let $\kappa_0:=-\widetilde{C}(\Omega_0)/\widetilde{B}(\Omega_0)$, so that $(0,\kappa_0,\Omega_0)\in\mathcal{BT}\cap\mathcal{C}$. Furthermore, assume that this Bogdanov-Takens point is nondegenerate, in the sense that*

$$\widetilde{\mu}(\Omega_0)\Delta_{21}(\Omega_0)+\mu(\Omega_0)\Delta_{22}(\Omega_0)\neq 0\ ,\tag{5.41}$$

i.e., Ω_0 is not a viscosity angle.

Then, in a neighborhood of $(0, \kappa_0, \Omega_0)$*, the homoclinic loci* $\mathcal{HO}_L$ *and* $\mathcal{HO}_R$ *form two-dimensional manifolds whose boundaries are the Bogdanov-Takens points. Furthermore, the ruled surfaces*

$$\frac{5}{7}\widetilde{A}(\Omega)R + \widetilde{B}(\Omega)\kappa + \widetilde{C}(\Omega) = 0 \ , \tag{5.42}$$

$$-\frac{5}{7}\widetilde{A}(\Omega)R + \widetilde{B}(\Omega)\kappa + \widetilde{C}(\Omega) = 0 \tag{5.43}$$

are tangent to $\mathcal{HO}_L$ *and* $\mathcal{HO}_R$*, respectively, along* $\mathcal{BT} \cap \mathcal{C}$.

Proof. The proof, which we only sketch, consists of three steps:

(1) First we write system (5.13) as a perturbation in R and κ around the Bogdanov-Takens point $(0, \kappa_0, \Omega_0) \in \mathcal{BT} \cap \mathcal{C}$. Throughout this analysis we keep the angle Ω_0 fixed.
(2) Next we introduce a small parameter ϵ and use it to rescale the dynamical system. This transformation "blows up" the bifurcating critical point of Bogdanov-Takens type into two less degenerate critical points; furthermore, the dynamical system for $\epsilon = 0$ is Hamiltonian and has a homoclinic loop, as in Fig. 5.1. With this device we can track the homoclinic bifurcation as R and κ vary in a neighborhood of $R = 0$ and $\kappa = \kappa_0$.
(3) The rescaled system can be written as a small perturbation of a Hamiltonian system, *viz.*, $\dot{X} = f(X) + \epsilon g(X) + O(\epsilon^2)$. An analysis based on Melnikov's integral for the homoclinic connection (see, *e.g.*, Ref. [**8**]) yields the line tangent to the intersection of the homoclinic locus with the plane $\Omega = \Omega_0$.

Figure 5.2 shows the bifurcation diagram corresponding to this unfolding. For more details see Ref. [**3**]. □

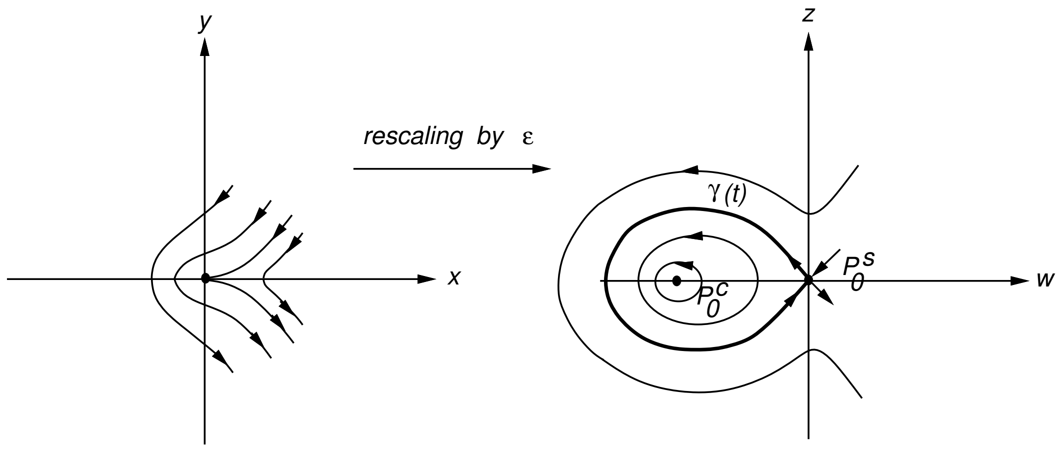

Figure 5.1. The blow-up of a Bogdanov-Takens critical point using a rescaling transformation. The blown-up system is a perturbation of a Hamiltonian system. The critical points of the Hamiltonian system are the saddle point $P_0^s = (0, 0)$ and the center $P_0^c = (\mp\widehat{R}, 0)$.

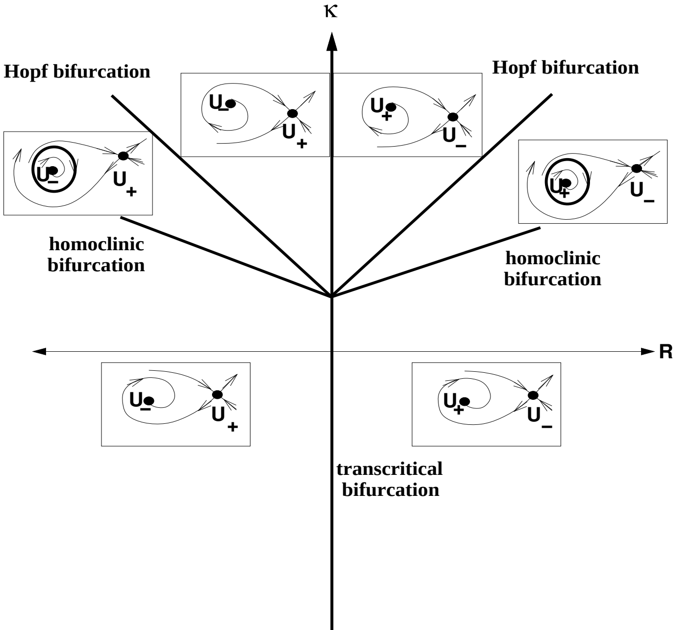

Figure 5.2. The unfolding of a Bogdanov-Takens point in $\mathcal{W}$, as depicted in the (R,κ)-plane for fixed (generic) Ω. The bifurcation diagram shows the loci of Hopf and transcritical bifurcations as well as the tangent manifolds to the homoclinic bifurcation loci. The phase portraits for dynamical systems corresponding to various points in the the (R,κ)-plane are illustrated in the boxes.

Our result concerning homoclinic loci corresponds closely to results of Keyfitz [**17**,**18**], who considers general systems of conservation laws near a point on the projection of the coincidence locus $\mathcal{E}$. Since a general system can be approximated by a quadratic model in the vicinity of a particular state, Theorem 5.13 allows a slight generalization of Keyfitz' results to the case when the (unperturbed) viscosity matrix D is not a multiple of the identity matrix. When $D = I$, condition (5.41) fails precisely at the points of $\mathcal{E}$ where genuine nonlinearity fails, or equivalently where the secondary bifurcation locus intersects $\mathcal{E}$. We emphasize, however, that when D is not a multiple of the identity matrix, $\mathcal{BT} \cap \mathcal{C}$ is distinct from $\mathcal{E}$. In particular, the formation of homoclinic orbits is not tied to the cusping of the (projected) shock curves that occurs at the coincidence locus.

Numerical experiments indicate that the global structure of the homoclinic locus is as indicated in Fig. 5.5, which is discussed below.

5.6 Heteroclinic Locus. The heteroclinic locus $\mathcal{HT}$ comprises points of $\mathcal{W}$ for which the corresponding dynamical system has two saddle points joined by a double separatrix. For convenience, we denote by $\mathcal{HT}_0$ the points of $\mathcal{HT}$ for which the double separatrix joins U_- and U_+. By contrast, if U_-, but not U_+, is one of the critical points joined by the double separatrix, the point belongs to $\mathcal{HT}_L$, and a similar definition applies to $\mathcal{HT}_R$.

For quadratic models, a large class of dynamical systems with heteroclinic orbits joining U_- and U_+ has been determined explicitly [**13**], namely those for which the separatrix is a straight line segment. The existence of straight line connections between the states U_- and U_+ has been proved equivalent (see Ref. [**13**]) to the existence of a constant $\xi \neq 0$ such that

$$\xi D \Delta U = \tfrac{1}{2} F''(0)(\Delta U)^2 \ , \tag{5.44}$$

where $\Delta U = U_+ - U_-$. Assuming that $\Delta U = R(\cos\Omega, \sin\Omega)^{\mathrm{T}}$, it follows that (5.44) holds if and only if

$$(-\sin\Omega, \cos\Omega) D^{-1} F''(0) \begin{pmatrix} \cos\Omega \\ \sin\Omega \end{pmatrix}^2 = 0 \ , \tag{5.45}$$

which is equivalent to the condition

$$\Delta_{21}(\Omega)\widetilde{\mu}(\Omega) + \Delta_{22}(\Omega)\mu(\Omega) = 0 \ , \tag{5.46}$$

where the functions μ and $\widetilde{\mu}$ are defined by Eqs. (5.2). Notice that this is the requirement that Ω be a viscosity angle.

In order to determine the constant ξ in Eq. (5.44), suppose that Ω_* is a viscosity angle. Then Eq. (5.44) implies that

$$\begin{aligned} \xi &= (\cos\Omega_*, \sin\Omega_*) D^{-1} F''(0) \begin{pmatrix} \cos\Omega_* \\ \sin\Omega_* \end{pmatrix}^2 \\ &= \Delta_{11}(\Omega_*)\widetilde{\mu}(\Omega_*) + \Delta_{12}(\Omega_*)\mu(\Omega_*) \ . \end{aligned} \tag{5.47}$$

Therefore, recalling definition 5.3, the condition $\xi \neq 0$ is equivalent to the requirement that Ω_* be a nonexceptional viscosity angle.

Let $\mathcal{HT}_0^{\text{straight}}$ denote the set of all points in $\mathcal{W}$ for which U_- and U_+ are connected by a straight line profile. The following theorem summarizes the results of Ref. [**13**] regarding the straight line transitional shock waves, rephrased in terms of the fundamental wave manifold.

Theorem 5.14. *The straight-line heteroclinic locus $\mathcal{HT}_0^{\mathit{straight}}$ is contained in the union of the planes $\Omega = \Omega_*$ such that Ω_* is a nonexceptional viscosity angle. If $\widetilde{\mathcal{D}}(\Omega_*) \neq 0$, then the intersection of $\mathcal{HT}_0^{\mathit{straight}}$ with the plane $\Omega = \Omega_*$ is an open subset of this plane and is bounded by the two lines that are the intersections of the plane $\Omega = \Omega_*$ with the sonic loci $\mathcal{S}_L$ and $\mathcal{S}_R$.*

For points on the boundary of $\mathcal{HT}_0$ (except those on $\mathcal{C}$), the dynamical system has an orbit joining a saddle point to a saddle-node. The other points on the viscosity planes of Theorem 5.14 correspond to orbits connecting a saddle point to the distinguished strong manifold of a node. The condition $\widetilde{\mathcal{D}}(\Omega_*) \neq 0$ fails when the intersection of $\mathcal{S}_L$, $\mathcal{S}_R$, and the plane $\Omega = \Omega_*$ is a single line, which corresponds to the locus of doubly sonic shock waves. Also, recall that at viscosity angles Ω_*, a Bogdanov-Takens point $\mathcal{BT} \cap \mathcal{C}$ is degenerate.

Certain extensions $\mathcal{HT}_L^{\text{straight}}$ and $\mathcal{HT}_R^{\text{straight}}$ of the locus $\mathcal{HT}_0^{\text{straight}}$ can be characterized explicitly. Here the locus $\mathcal{HT}_L^{\text{straight}}$ consists of points (R, κ, Ω) that have the same speed s and left end state U_- at some point $(R_*, \kappa_*, \Omega_*) \in \mathcal{HT}_0^{\text{straight}}$; the locus $\mathcal{HT}_R^{\text{straight}}$ is defined analogously.

Theorem 5.15. *The straight-line heteroclinic loci* $\mathcal{HT}_L^{straight}$ *and* $\mathcal{HT}_R^{straight}$ *are contained in ruled surfaces within* $\mathcal{W}$*, namely those defined by the equations*

$$\widehat{A}(\Omega;\Omega_*)R+\widehat{B}(\Omega;\Omega_*)\kappa+\widehat{C}(\Omega;\Omega_*)=0\ , \tag{5.48}$$

$$-\widehat{A}(\Omega;\Omega_*)R+\widehat{B}(\Omega;\Omega_*)\kappa+\widehat{C}(\Omega;\Omega_*)=0 \tag{5.49}$$

for each viscosity angle Ω_* *with* $\mu(\Omega_*)\neq 0$*. Here* $\widehat{A}$*,* $\widehat{B}$*, and* $\widehat{C}$ *are the continuous functions such that*

$$\begin{aligned}\widehat{A}(\Omega;\Omega_*)&=\tfrac{1}{2}\widetilde{\mathcal{D}}(\Omega_*)\\ \sin(\Omega-\Omega_*)\,\widehat{B}(\Omega;\Omega_*)&=\mu(\Omega_*)\widetilde{\mathcal{D}}(\Omega)-[\alpha(\Omega)\cos\Omega_*+\beta(\Omega)\sin\Omega_*]\,\widetilde{\mathcal{D}}(\Omega_*)\\ \sin(\Omega-\Omega_*)\,\widehat{C}(\Omega;\Omega_*)&=\mu(\Omega_*)\,[\lambda_c(\Omega)-\lambda_c(\Omega_*)]\\ &\quad-\widetilde{\mathcal{D}}(\Omega_*)\{-\sin\Omega_*\,[u_c(\Omega)-u_c(\Omega_*)]+\cos\Omega_*\,[v_c(\Omega)-v_c(\Omega_*)]\}\ .\end{aligned} \tag{5.50}$$

Proof. Let Ω_*^1 be a viscosity angle, and κ^1, R^1 arbitrary point lying in the transitional wedge of the plane $\Omega=\Omega_*^1$ defined by Theorem 5.14. Then the states U_-^1 and U_+^1 that correspond to the point $(R^1,\kappa^1,\Omega_*^1)$ are connected by a straight-line heteroclinic orbit. We are interested in the set of points that have the same dynamical system as the point $(R^1,\kappa^1,\Omega_*^1)$, but for which U_+ is not the critical point that is involved in the heteroclinic connection. More precisely, let s^1 be the shock speed corresponding to a dynamical system determined by $(R^1,\kappa^1,\Omega_*^1)$, and U_-^1 be the corresponding left state. We seek points (R,κ,Ω) such that

$$\begin{aligned}s^1&=s\ ,\\ U_-^1&=U_-\ .\end{aligned} \tag{5.51}$$

When expressed in (R,κ,Ω)-coordinates, conditions (5.51) become

$$\begin{aligned}\lambda_c(\Omega_*^1)+\kappa^1\widetilde{\mathcal{D}}(\Omega_*^1)&=\lambda_c(\Omega)+\kappa\widetilde{\mathcal{D}}(\Omega)\ ,\\ u_c(\Omega_*^1)-\beta(\Omega_*^1)\kappa^1-\tfrac{1}{2}R^1\cos\Omega_*^1&=u_c(\Omega)-\beta(\Omega)\kappa-\tfrac{1}{2}R\cos\Omega\ ,\\ v_c(\Omega_*^1)+\alpha(\Omega_*^1)\kappa^1-\tfrac{1}{2}R^1\sin\Omega_*^1&=v_c(\Omega)+\alpha(\Omega)\kappa-\tfrac{1}{2}R\sin\Omega\ ,\end{aligned} \tag{5.52}$$

as follows from Eqs. (3.26), (3.28), and (3.5). We solve for R^1 and κ^1 using the last two conditions: under the assumption $\mu(\Omega_*^1)\neq 0$ one obtains that

$$\begin{pmatrix}-\frac{1}{2}R^1\\ \kappa^1\end{pmatrix}=-\frac{1}{\mu(\Omega_*^1)}\begin{pmatrix}\alpha^1 & \beta^1\\ -\sin\Omega_*^1 & \cos\Omega_*^1\end{pmatrix}\begin{pmatrix}u_c^1-(u_c-\beta\kappa-\frac{1}{2}R\cos\Omega)\\ v_c^1-(v_c+\alpha\kappa-\frac{1}{2}R\sin\Omega)\end{pmatrix}\ . \tag{5.53}$$

With these R^1 and κ^1, the first equation in (5.52) holds if and only if

$$\sin(\Omega-\Omega_*^1)\left[\widehat{A}(\Omega;\Omega_*^1)R+\widehat{B}(\Omega;\Omega_*^1)\kappa+\widehat{C}(\Omega;\Omega_*^1)\right]=0\ , \tag{5.54}$$

where $\widehat{A}(\Omega;\Omega_*^1)$, $\widehat{B}(\Omega;\Omega_*^1)$, and $\widehat{C}(\Omega;\Omega_*^1)$ are defined by Eqs. (5.50). Notice that for $\Omega=\Omega_*^1$ the surface defined by Eq. (5.54) is singular. One removes the singularity by dividing this equation by $\sin(\Omega-\Omega_*^1)$, obtaining Eq. (5.48).

A similar calculation for $\mathcal{HT}_R^{\text{straight}}$ leads to Eq. (5.49). $\square$

For certain classes of quadratic dynamical systems, the only possible heteroclinic connections are straight-line connections; for instance, as proved by Chicone [**4**], quadratic gradient systems form such a class. In general, however, heteroclinic orbits need not lie along straight lines.

Example. We present a model studied in Ref. [**2**], for which the flux function F is

$$F = \begin{pmatrix} -\frac{1}{2}u^2 + \frac{1}{2}v^2 + v \\ uv - u \end{pmatrix} \tag{5.55}$$

and the viscosity matrix is $D = I$. The elliptic region is a circle of radius 1 centered at the origin. In this case, dynamical system (2.3) has the form

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(-u^2 + v^2) - su + v - \frac{1}{2}(-u_-^2 + v_-^2) - su_- - v_- \\ uv - u - sv - u_-v_- + u_- + sv_- \end{pmatrix} \tag{5.56}$$

with parameters $U_- = (u_-, v_-)^{\mathrm{T}}$ and s. The viscosity angles associated with the model are $\Omega_*^1 = 0$, $\Omega_*^2 = \pi/3$, and $\Omega_*^3 = 2\pi/3$. Since $D = I$, the locus $\mathcal{HT}_0^{\text{straight}}$ is the union of three straight lines contained in the viscosity planes $\Omega = \Omega_*^i$, $i = 1, 2, 3$. (By contrast, in the generic case, $\mathcal{HT}_0^{\text{straight}}$ is a union of wedges; see Theorem 5.14.) Following the terminology introduced in Ref. [**13**], the U_--projections of these lines onto state space $\mathcal{U}$ are called the *bifurcation lines*. For instance, the bifurcation line corresponding to Ω_*^1 is $\{(u, v) | v = 1\}$.

After the translation $X = U - U_-$, the system (5.56) becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(-x^2 + y^2) - (u_- + s)x + (v_- + 1)y \\ xy + (v_- - 1)x + (u_- - s)y \end{pmatrix}. \tag{5.57}$$

Let U_- lie on the bifurcation line corresponding to Ω_*^1 between the intersection points of this line with the other two bifurcation lines. We choose the shock speed s such that the corresponding dynamical system is Hamiltonian. As noted in Ref. [**2**], system (5.57) is Hamiltonian if and only if $s = 0$. For this choice of parameters, system (5.57) takes the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(-x^2 + y^2) - u_-x + 2y \\ xy + u_-y \end{pmatrix}, \tag{5.58}$$

which derives from the Hamiltonian function

$$H(x, y) = y\left(-\tfrac{1}{2}x^2 + \tfrac{1}{6}y^2 - u_-x + y\right). \tag{5.59}$$

The heteroclinic orbits are given by

$$\begin{aligned} \Gamma_1 &= \{(x, 0) \,|\, x \in (0, -2u_-)\}, \\ \Gamma_2 &= \{(x, y) \,|\, -\tfrac{1}{2}x^2 + \tfrac{1}{6}y^2 - u_-x + y = 0, x \in (-2u_-, 0)\}. \end{aligned} \tag{5.60}$$

We choose U_+ to be the critical point inside the region bounded by the two heteroclinic connections. This determines a point $(R_H, 0, \Omega_H)$ in $\mathcal{W}$ that is associated with the dynamical system (5.58) and this particular choice of critical points.

In the following, we find points in $\mathcal{W}$ that lie in the plane $\Omega = \Omega_H$ and satisfy the property that the curved heteroclinic connection is preserved. For this purpose, we write the dynamical system corresponding to a point (R, κ, Ω_H) as a perturbation, in parameters R and κ, of the Hamiltonian system (5.58):

$$
\begin{aligned}
\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} &= \begin{pmatrix} \frac{1}{2}(-x^2+y^2) - u_- x + 2y \\ xy + u_- y \end{pmatrix} \\
&\quad + \kappa \begin{pmatrix} \beta(\Omega_H) - 1 & \alpha(\Omega_H) \\ \alpha(\Omega_H) & -[\beta(\Omega_H)+1] \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\
&\quad + \frac{R - R_H}{2} \begin{pmatrix} \cos\Omega_H & -\sin\Omega_H \\ -\sin\Omega_H & -\cos\Omega_H \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} .
\end{aligned}
\tag{5.61}
$$

Since we work only with small perturbations, we introduce a new parameter $\epsilon > 0$ that determines the order of perturbation, and write $\widehat{\kappa} := \epsilon^{-1}\kappa$ and $\widehat{R} := \epsilon^{-1}(R - R_H)$. System (5.61) now reads

$$
\dot{X} = f(X) + \epsilon g(X) \ , \tag{5.62}
$$

where $f(X)$ is the Hamiltonian part and

$$
\begin{aligned}
g(X) &= \begin{pmatrix} g_1(x,y) \\ g_2(x,y) \end{pmatrix} \\
&= \begin{pmatrix} \widehat{\kappa}\,[\beta(\Omega_H) - 1] + \frac{1}{2}\widehat{R}\cos\Omega_H & \widehat{\kappa}\alpha(\Omega_H) - \frac{1}{2}\widehat{R}\sin\Omega_H \\ \widehat{\kappa}\alpha(\Omega_H) - \frac{1}{2}\widehat{R}\sin\Omega_H & -\widehat{\kappa}\,[\beta(\Omega_H)+1] - \frac{1}{2}\widehat{R}\cos\Omega_H \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} .
\end{aligned}
\tag{5.63}
$$

We use Melnikov's function to determine the tangent to the manifold along which curved heteroclinic connection is preserved. Melnikov's function for this case is constructed in a manner similar to the one used to treat homoclinic connections [**8**]. One obtains

$$
\begin{aligned}
M(\widehat{R}, \widehat{\kappa}) &= \int_{\Gamma_2} [g_2(x,y)\,dx - g_1(x,y)\,dy] \\
&= \int_{\operatorname{int}\Gamma_1 \cup \Gamma_2} \operatorname{div} g(x,y)\,dx\,dy - \int_{\Gamma_1} g_2(x,0)\,dx.
\end{aligned}
\tag{5.64}
$$

After some straightforward computations, we obtain the line

$$
m_1 R + m_2 \kappa + m_3 = 0 \ , \tag{5.65}
$$

where

$$
\begin{aligned}
m_1 &:= u_-^2 \sin\Omega_H \ , \\
m_2 &:= 6u_- - 2\sqrt{3}(3 - u_-^2)\tanh^{-1}\left[\frac{u_-}{\sqrt{3}} - 2u_-^2\alpha(\Omega_H)\right], \\
m_3 &:= u_-^2 R_H \sin\Omega_H \ .
\end{aligned}
\tag{5.66}
$$

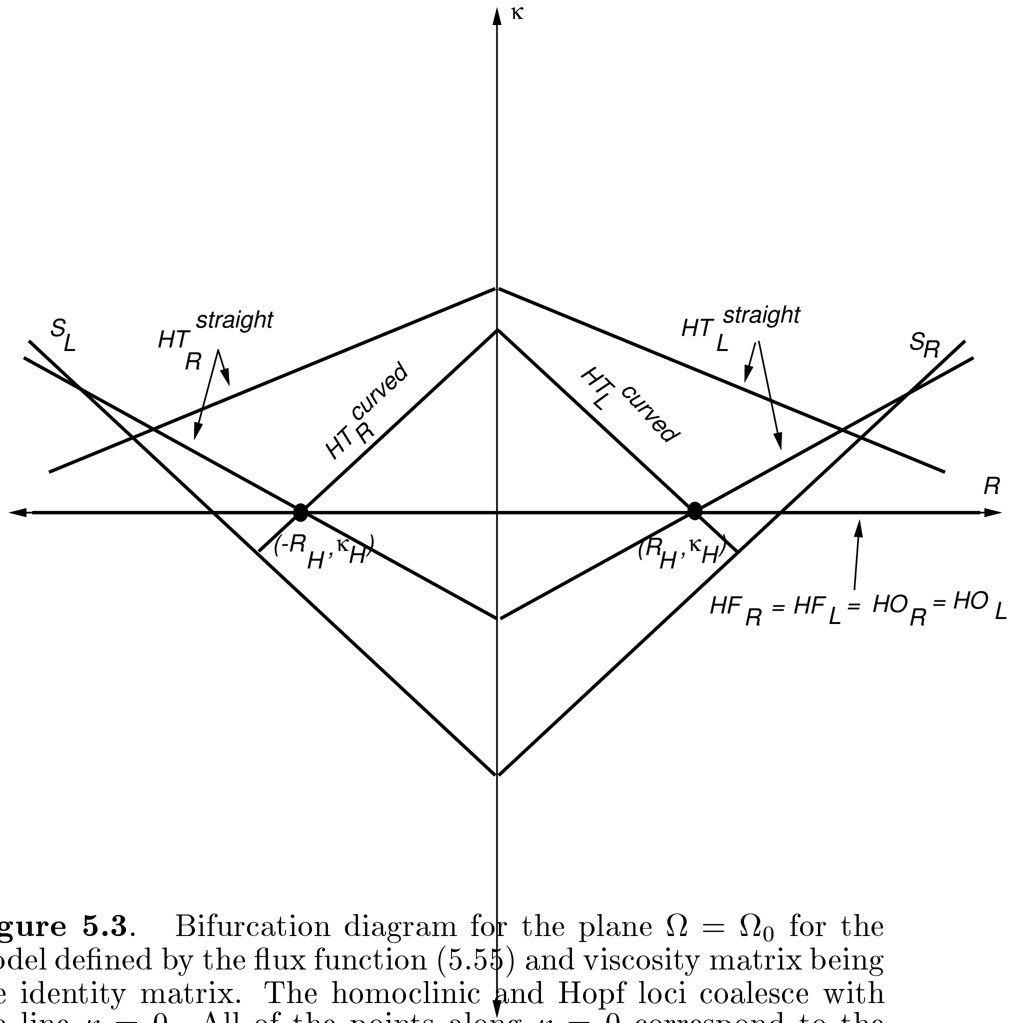

Figure 5.3. Bifurcation diagram for the plane $\Omega = \Omega_0$ for the model defined by the flux function (5.55) and viscosity matrix being the identity matrix. The homoclinic and Hopf loci coalesce with the line $\kappa = 0$. All of the points along $\kappa = 0$ correspond to the Hamiltonian systems. The point (R_H, κ_H) is the unique Hamiltonian system in the plane $\Omega = \Omega_0$ for which the corresponding dynamical system admits a straight-line and a "curved" heteroclinic connection emerging from the state U_-. The unfolding of this point gives the line tangent to the locus $\mathcal{HT}_L^{\text{curved}}$ at the point (R_H, κ_H).

This line is tangent at the point $(R_H, \kappa_H, \Omega_H)$ to the intersection of the plane $\Omega = \Omega_H$ with the heteroclinic bifurcation manifold corresponding to "curved" connecting orbits; see Fig. 5.3.

Such "curved" heteroclinic connections affect the solution of the Riemann problem. As shown in Ref. [**2**], "curved" transitional shock waves, as well as the straight line ones, can lead to nonuniqueness of solution for Riemann problems. As we shall see, "curved" heteroclinic connections can also occur in dynamical systems on the boundary of the region of admissible shock waves.

In order to understand the influence of the existence of "curved" heteroclinic connections to the solution of a Riemann problem, we have performed numerical experiments on some of these models using the Riemann Problem Package developed by E. Isaacson, D. Marchesin, and B. Plohr. We have adapted this program to solve the following problem: given an angle Ω_0, find the points (R, κ) such that the dynamical system corresponding to (R, κ, Ω_0) has a separatrix joining two saddle points. A typical set of results from such an experiment is diagrammed in Fig. 5.4, in which the numerically determined points on the heteroclinic locus are shown as circles. The relationships of this locus with other bifurcation loci is also indicated. (Note that only certain subsets of the lines that are labeled $\mathcal{HF}$ and $\mathcal{HT}$ correspond to Hopf and heteroclinic bifurcations. For instance, the continuations of these lines past $R = 0$ do not belong to these loci.)

The numerical evidence indicates that the intersection of a plane $\Omega = \Omega_0$ with the curved heteroclinic loci consists of straight lines in the (R, κ)-plane, and that these lines vary with Ω_0. This numerical evidence is presented, for one particular choice of Ω_0, in Fig. 5.4. Furthermore, in the example above for which the tangent to the curved heteroclinic locus was calculated, this tangent line coincides with the numerically observed heteroclinic locus. We are led, therefore, to the following conjecture.

Conjecture 5.16. *The curved heteroclinic loci $\mathcal{HT}_L^{\text{curved}}$ and $\mathcal{HT}_R^{\text{curved}}$ are contained in ruled surfaces within $\mathcal{W}$.*

The boundary of $\mathcal{HT}^{\text{curved}}$ consists of two types of points. One is the simultaneous equilibrium and curved heteroclinic bifurcation. This corresponds to the intersection of the saddle-node surface in $\mathcal{W}$ with the surface $\mathcal{HT}^{\text{curved}}$. The unfolding of such points has been studied by Schecter [**25**,**26**].

Another type of a bifurcation that can serve as a boundary of $\mathcal{HT}^{\text{curved}}$ is the occurrence of two straight-line heteroclinic connections. In this case, the locus $\mathcal{HT}^{\text{curved}}$ emanates from the intersection of the $\mathcal{HT}_L^{\text{straight}}$ and the extension of $\mathcal{HT}^{\text{straight}}$. The extension of $\mathcal{HT}^{\text{straight}}$ consists of points of $\mathcal{W}$ for which the other two critical points, those not serving as U_- and U_+ in the corresponding dynamical system, are connected by a straight line heteroclinic orbit.

5.7 Shock Wave Admissibility Regions. Using the analysis and numerical results from the previous sections, we can construct the region $\mathcal{A} \subseteq \mathcal{W}$ of admissible shock waves. We focus on a typical slice of the wave manifold $\mathcal{W}$ for fixed shock angle $\Omega = \Omega_0$. Figure 5.5 illustrates the bifurcation diagram corresponding to the quadratic model for which the flux function F has the following form

$$F(u, v) = \begin{pmatrix} -3u^2 + v^2 + v \\ uv - u \end{pmatrix} , \tag{5.67}$$

and the inverse of the viscosity matrix is given by

$$D^{-1} = \begin{pmatrix} 1 & 0.7 \\ 0.7 & 1 \end{pmatrix} . \tag{5.68}$$

The example given by Eqs. (5.67) and (5.68) belongs to the class of quadratic models for which the corresponding dynamical system (5.1) generically has three saddle points and one antisaddle point (*i.e.*, node or spiral point). We took the slice of $\mathcal{W}$ that corresponds to $\Omega = 0.75$ and determined the region $\mathcal{A}$, which is shaded

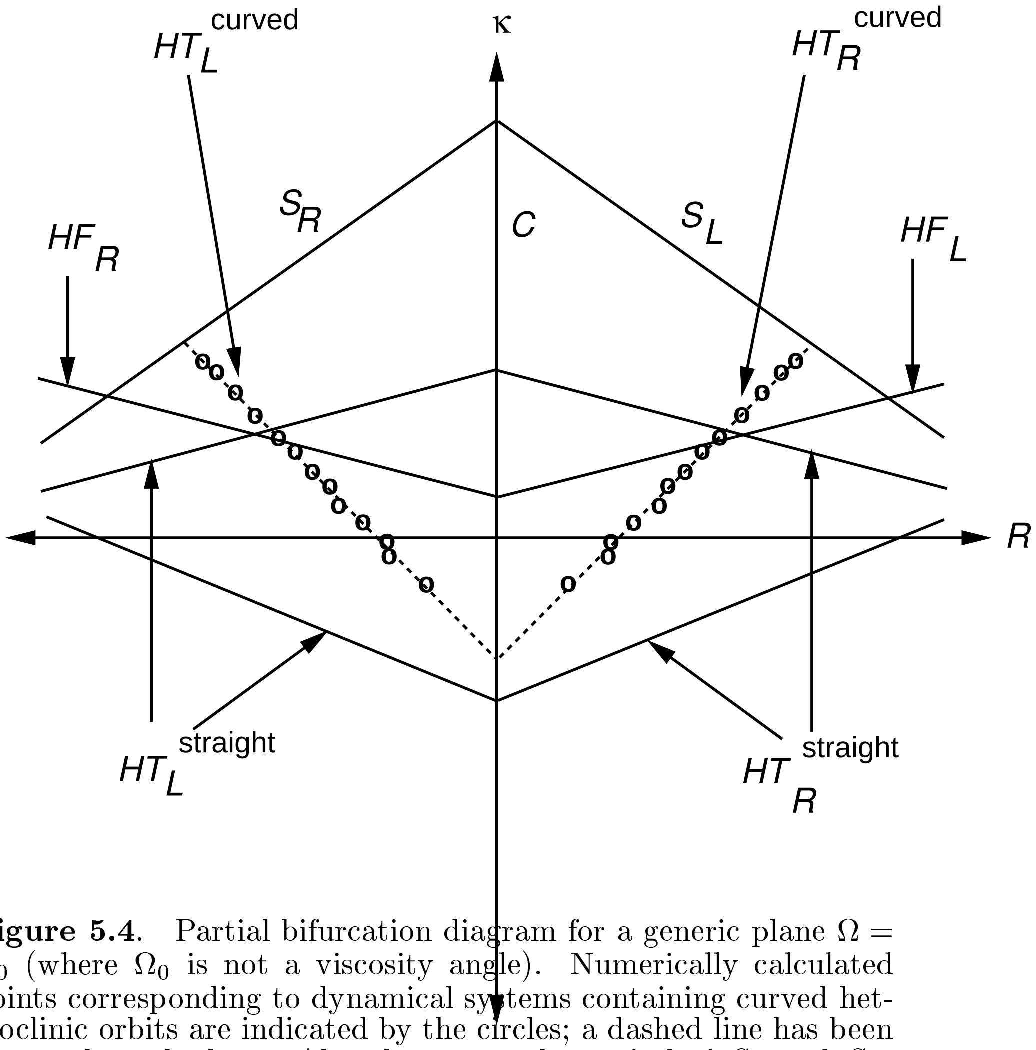

Figure 5.4. Partial bifurcation diagram for a generic plane $\Omega = \Omega_0$ (where Ω_0 is not a viscosity angle). Numerically calculated points corresponding to dynamical systems containing curved heteroclinic orbits are indicated by the circles; a dashed line has been drawn through them. Also shown are the sonic loci S_L and S_R, the Hopf loci $\mathcal{HF}_L$ and $\mathcal{HF}_R$, and the straight-line heteroclinic loci $\mathcal{HT}_L^{\text{straight}}$ and $\mathcal{HT}_R^{\text{straight}}$ corresponding to the three possible viscosity angles. Only certain portions of the lines labeled $\mathcal{HF}$ and $\mathcal{HT}$ lie in the Hopf and heteroclinic loci.

light gray in Fig. 5.6. In the same figure we have also distinguished the set of points for which the shock wave between U_- and U_+ satisfies the Lax criterion. The boundary of Lax-admissible waves consists of portions of the bifurcation manifolds across which at least one of the critical points becomes nonhyperbolic, in the sense of Def. 2.2. On the other hand, the admissibility boundary $\partial\mathcal{A}$ for the viscous shock waves involves portions of all bifurcation loci studied in previous sections.

As an example of the mechanism that is responsible for a shock wave to become inadmissible, we present a detailed bifurcation diagram in the vicinity of the point P of Fig. 5.6. Phase portraits for the magnified neighborhood of the point P are shown in Fig. 5.7.

6. CONCLUSION

We have studied shock wave solutions of quadratic conservation laws using the

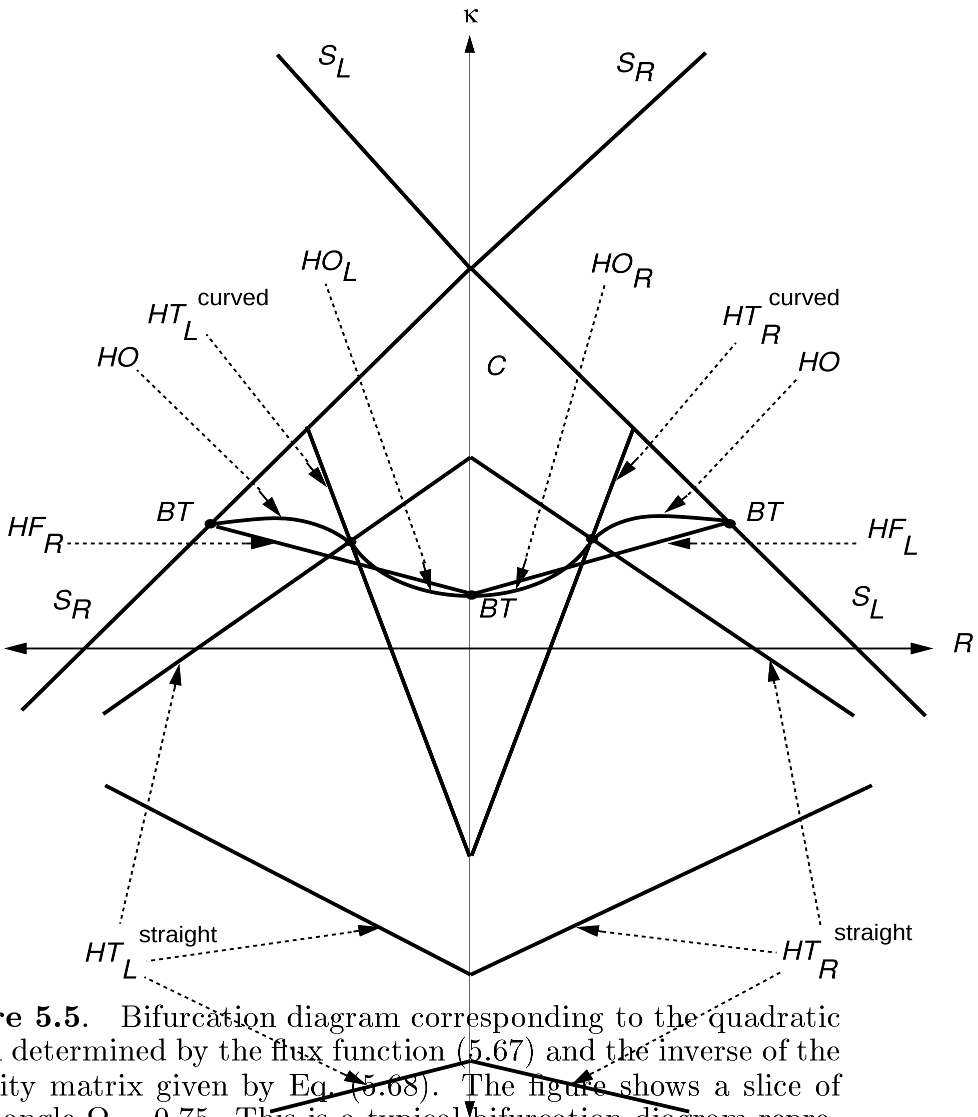

Figure 5.5. Bifurcation diagram corresponding to the quadratic model determined by the flux function (5.67) and the inverse of the viscosity matrix given by Eq. (5.68). The figure shows a slice of $\mathcal{W}$ at angle $\Omega = 0.75$. This is a typical bifurcation diagram representing the class of models for which the corresponding dynamical system has one antisaddle and three saddle points. Shown here are the sonic loci, the Hopf loci, the numerically observed homoclinic loci, the straight-line heteroclinic loci, and the numerically observed curved heteroclinic loci.

viscosity admissibility criterion to distinguish physical waves. The fundamental wave manifold $\mathcal{W}$ was employed to represent admissible shock waves because points of $\mathcal{W}$ parametrize the dynamical systems associated with shock waves. In order to determine the region of admissible shock waves in $\mathcal{W}$, we first determined the admissibility boundary, the (topological) boundary of the region of admissible shock waves. The admissibility boundary consists of regularly structurally unstable dynamical systems, which are responsible for breaking the connecting orbit between the critical points associated with the left and right states of a shock wave, *i.e.*, for the loss of admissibility. The set of all admissible shock wave solutions is an open subset of $\mathcal{W}$ bounded by the admissibility boundary. We have shown that the admissibility boundary consists of portions of the loci corresponding to the heteroclinic bifurcations, limit cycles, homoclinic orbits, Bogdanov-Takens, and Hopf

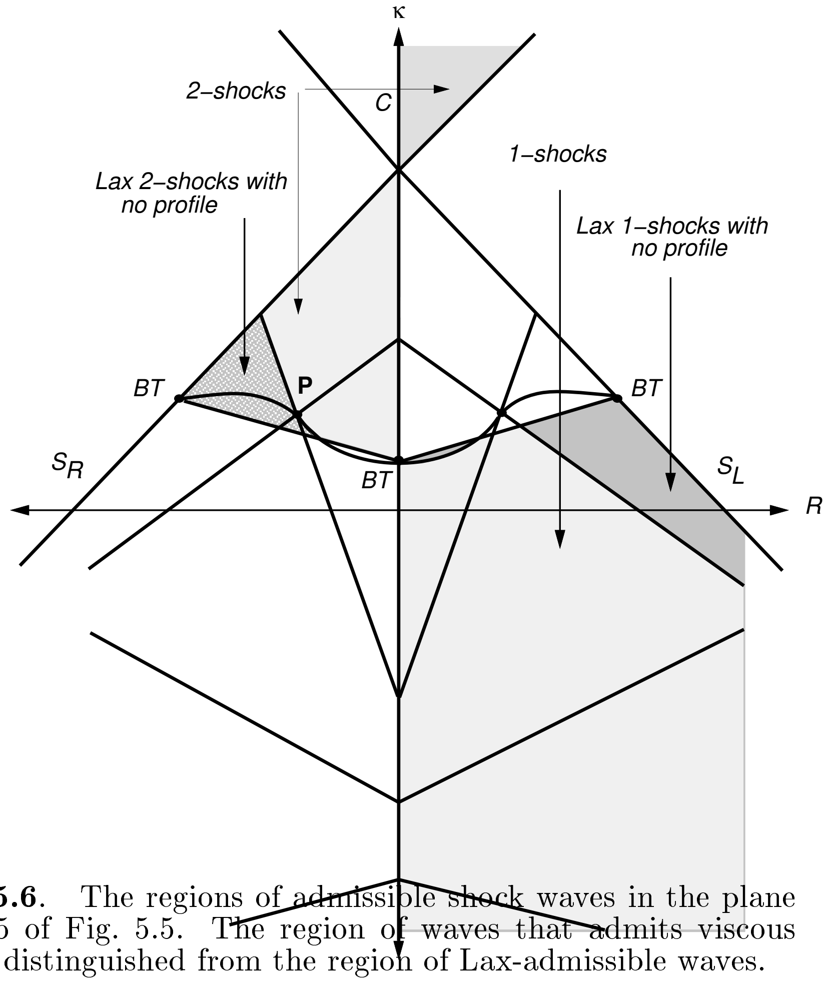

Figure 5.6. The regions of admissible shock waves in the plane $\Omega = 0.75$ of Fig. 5.5. The region of waves that admits viscous profile is distinguished from the region of Lax-admissible waves.

bifurcations. (The influence of nonhyperbolic limit cycles on the structure of the admissibility boundary is a topic of a current research.)

The shock wave admissibility analysis carried out in this work relates to the more general questions of the construction of solutions of Riemann problems and the stability of wave curves. In studying the solution of a Riemann problem, one constructs wave curves, *i.e.*, one-parameter families of Riemann solutions consisting of segments of rarefaction, shock, and composite waves. Solving Riemann problems with wave curves depends upon the stability of wave curves with respect to changes in their initial states. The stability of a wave curve, in turn, reduces to the stability of each constituent segment. A systematic study of wave curve stability was carried out by Furtado [**6**]; assuming the Lax admissibility criterion, he presented a complete list of loci where wave curves bifurcate. The results of this paper represent a step toward understanding the stability of wave curves and Riemann problem solutions when the viscosity admissibility criterion is adopted.

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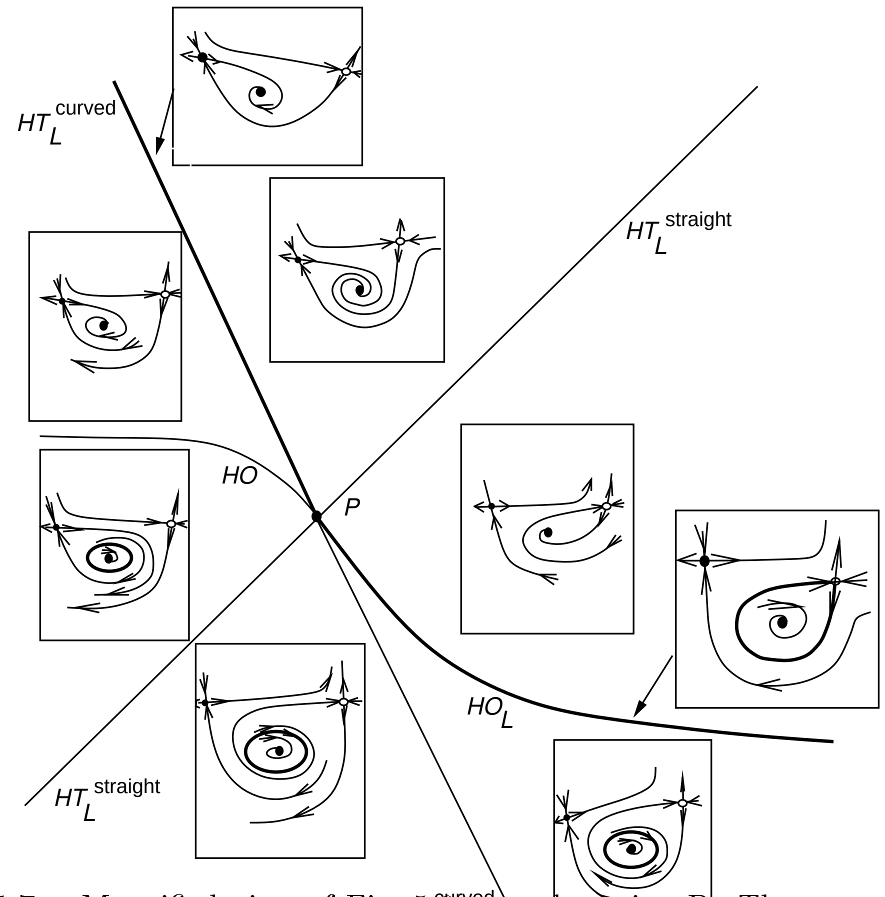

Figure 5.7. Magnified view of Fig. 5.6 near the point P. The phase portraits in the boxes indicate the bifurcations responsible for breaking the viscous profile connection between U_- and U_+.

for pointing out the relevance of Ref. [**9**].

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