ON THE INFLUENCE OF VISCOSITY ON RIEMANN SOLUTIONS

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ABSTRACT. We show how existence and uniqueness of Riemann solutions are affected by the precise form of viscosity which is used to select shock waves admitting a viscous profile. We study a complete list of codimension–1 bifurcations that depend on viscosity and distinguish between Lax shock waves with and without a profile. These bifurcations are the saddle–saddle heteroclinic bifurcation, the homoclinic bifurcation, and the nonhyperbolic periodic orbit bifurcation. We prove that these influence existence and uniqueness of Riemann solutions, and affect the number and type of waves comprising a Riemann solution. We present "generic" situations in which viscous Riemann solutions differ from Lax solutions.

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1. INTRODUCTION

In this paper we present a new approach to studying the role of viscosity in the construction, existence and uniqueness of Riemann solutions for a 2×2 system of conservation laws that changes type. We study Riemann problems for a nonlinear system of conservation laws

$$U_t + F(U)_x = 0,$$

with shock initial data, *i.e.*, the left and the right states in the initial data (U_L and U_R) satisfy the shock (Rankine-Hugoniot) conditions.

A shock between U_L and U_R, traveling with speed s, admits a viscous profile if there is a connecting orbit from U_L to U_R in a vector field defined by

$$U' = D^{-1}(U)\left\{-s\,(U - U_L) + F(U) - F(U_L)\right\}, \tag{1}$$

where $D(U)$ is a fixed viscosity matrix. Small perturbations in U_L and s may result in a loss of viscous profile. Structural stability, *i.e.*, continuous dependence, of viscous profilable shock waves with respect to changes of the state U_L on the left side of the shock wave and of the shock speed s was studied in Refs. [5] and [8]. It is crucial to our study and we summarize the main results in Section 3.2.

To parametrize vector fields associated with the viscosity admissibility criterion we use the framework of the fundamental wave manifold, $\mathcal{W}$, introduced in Ref. [17], described in Section 3. Each point in $\mathcal{W}$ corresponds to a single shock wave. To determine a subregion of $\mathcal{W}$ containing admissible shock points, we first determine its boundary by studying structurally unstable vector fields (1); see Refs. [5] and [8]. From a complete list of codimension-1 bifurcations that determine the "admissibility boundary", only the heteroclinic saddle–saddle connection bifurcation, the homoclinic bifurcation, and the nonhyperbolic periodic orbit bifurcation distinguish between Lax shocks with and without a viscous profile. Since these loci depend on the precise form of the viscosity matrix, we obtain the following observation, central to this paper. Whether a Lax shock between two fixed states U_L and U_R in a system of conservation laws with a flux function F admits a profile or not, depends entirely on the form of the viscosity matrix D. In this paper we show how the bifurcations that affect the profilability of Lax shock waves influence existence, uniqueness, and number of waves comprising "viscous" Riemann solutions, thereby elucidating how the precise form of viscosity affects the structure, existence and uniqueness of Riemann solutions. Here, by the viscous Riemann solutions we mean solutions of the inviscid equations in which all shock waves are required to have viscous profiles. We obtain our results in the context of general 2×2 systems of conservation laws. To illustrate the results we present a couple of examples involving quadratic models. We follow the list of bifurcations that depend on the precise form of viscosity, and study their role in the construction of Riemann solutions. The main results can be summarized as follows.

Heteroclinic saddle-saddle connection bifurcation. We show that heteroclinic saddle–saddle connection bifurcation gives rise to solutions which contain at least one transitional wave for shock Riemann data. We present a construction of a "generic" viscous Riemann solution near a point on the heteroclinic saddle–saddle bifurcation locus in $\mathcal{W}$. The solution consists of a 1-shock, followed by a transitional shock wave, and a 2-rarefaction wave. An interesting feature of this solution is that the left and right states in the Riemann initial data satisfy the Rankine–Hugoniot condition and the Lax admissibility criterion. This means that the single shock between the initial left and right state is the Lax admissible solution. Thus the viscous Riemann solution (containing three waves as described above) differs from the Lax solution (single shock). Details are presented in Section 4..

If this shock also admits a profile, we have **nonuniqueness** of viscous Riemann solutions. In Section 4 we derive the conditions under which this happens.

Homoclinic loop and periodic orbit bifurcations. These bifurcations belong to the "admissibility boundary" because they give rise to hyperbolic periodic orbits. The presence of a periodic orbit surrounding one of the states in a shock wave makes the shock inadmissible and may, therefore, lead to **nonexistence** of a viscous Riemann solution. Indeed, in Section 5.2 we present an example of a quadratic model (which derives from steady gas dynamics) and an open set of viscosity matrices, for which we prove nonexistence of a self–similar viscous Riemann solution due to the presence of a periodic orbit which was "born" in a homoclinic loop bifurcation.

In some situations, transitional waves can be used to avoid nonexistence. This is, for example, the case for systems of conservation laws for which the associated family of vector fields (1) admits the universal unfolding of a (codimension–2) double saddle–saddle connection bifurcation point. In Section 4 we show that in this case a viscous Riemann solution containing at least two transitional waves can be constructed for shock initial data for which the presence of a periodic orbit surrounding one of the states prevents the existence of a viscous profile. The solution in the example we present (Section 5) consists of a 1–shock, two transitional waves, and a 2–rarefaction wave. This solution differs from the Lax-admissible solution which consists of a single shock. It is an open question whether in models where transitional waves are present, nonexistence of classical Riemann solutions can always be resolved by using transitional waves.

2. Preliminaries

We study nonlinear systems of conservation laws

$$U_t + F(U)_x = 0\,, \quad x \in \mathbf{R}, \quad t > 0\,, \tag{2}$$

where the state variable $U \in \mathcal{U} \subset \mathbf{R}^2$, and the flux function $F : \mathcal{U} \to \mathbf{R}^2$ is a smooth function of U. System (2) is strictly hyperbolic if the Jacobian $F'(U)$ has distinct real eigenvalues $\lambda_1(U) < \lambda_2(U)$. In this paper we shall assume that systems (2) change type from hyperbolic to elliptic. The set of all states $U(x,t) \in \mathcal{U}$ for which the Jacobian $F'(U)$ has complex conjugate pair of eigenvalues will be called the *elliptic region* and we shall denote it by $\mathcal{E} \subset \mathcal{U}$. We assume that the elliptic region is a bounded subset of $\mathcal{U}$. The boundary of the elliptic region will be called the *coincidence locus*.

We study *Riemann initial value problems*, *i.e.*, initial value problems with piecewise constant initial data, separated by a jump discontinuity:

$$U(x,0) = \begin{cases} U_L, & x < 0\,, \\ U_R, & x > 0\,. \end{cases}$$

Because of the scale-invariant nature of the problem (the equations as well as the initial data are invariant under the scalings $(x,t) \mapsto (\alpha x, \alpha y)$, $\quad \forall \alpha > 0$), it is natural to look for scale-invariant solutions. These consist of wave groups separated by constant states. A wave group consists of elementary waves that are shock waves or rarefaction waves. A *rarefaction wave* solution of a system of conservation laws is a differentiable function $U(x,t) = \tilde{U}(\frac{x}{t})$, defined on a bounded interval $\xi_0 \le \xi \equiv x/t \le \xi_1$, such that for $t > 0$

$$\left[-\xi + F'(\tilde{U}(\xi))\right] \tilde{U}'(\xi) = 0, \quad \text{where } \xi = \frac{x}{t}\,.$$

A *shock wave* solution is a jump discontinuity

$$U(x,t) = \begin{cases} U_-, & x < st\,, \\ U_+, & x > st\,, \end{cases}$$

between two states, U_- and U_+, traveling at constant speed s, which satisfies (2) in weak sense, *i.e.*, in the form of the Rankine-Hugoniot condition:

$$-\,s\,[U_+ - U_-] + F(U_+) - F(U_-) = 0 \tag{3}$$

which embodies the conservation principle across the discontinuity. For a fixed state U_-, the set of states $U \in \mathcal{U}$ such that

$$-\,s\,[U - U_-] + F(U) - F(U_-) = 0 \tag{4}$$

for some $s \in \mathbf{R}$, is called the *Hugoniot locus* of U_-.

It is well known that not all shock wave solutions are physical. In order to obtain the physical weak solution several selection criteria have been proposed. The best known are the Lax admissibility criterion and the viscosity admissibility criterion.

The *Lax admissibility criterion* requires that the characteristics of precisely one family approach a shock wave from both sides. For a 2×2 system of conservation laws this means that a Lax admissible 1-shock wave between the states U_- and U_+, traveling at speed s, must satisfy

$$s < \lambda_1(U_-), \quad \lambda_1(U_+) < s < \lambda_2(U_+)\,,$$

whereas a Lax admissible 2-shock wave satisfies

$$\lambda_1(U_-) < s < \lambda_2(U_-), \quad \lambda_2(U_+) < s\,.$$

The *viscosity admissibility criterion* is based on taking into account physical effects, such as viscosity and diffusion, that have been neglected in the conservation law formulation. It derives from a parabolic system of equations

$$U_t + F(U)_x = \epsilon\,(D(U)U_x)_x\,, \quad x \in \mathbf{R}, \quad t > 0, \tag{5}$$

where $\epsilon > 0$, and $D(U)$ is called the viscosity matrix. It states that a shock wave solution between U_- and U_+, traveling at constant speed s, is *admissible* if it can be recovered in the singular limit, as $\epsilon \to 0$, of traveling wave solutions $U(x,t) = \tilde{U}(\dfrac{x-st}{\epsilon}) = \tilde{U}(\xi)$ of the parabolic equation (5), such that

$$\lim_{\xi\to\pm\infty} \tilde{U}(\xi) = U_\pm, \quad \lim_{\xi\to\pm\infty} \tilde{U}'(\xi) = 0.$$

We shall work with viscosity matrices that satisfy the following assumption.

> ASSUMPTION: The viscosity matrix is independent of U and has eigenvalues with positive real parts.

The viscosity admissibility criterion is equivalent to requiring that $\widetilde{U}(\xi)$ be an orbit in the vector field defined by

$$\widetilde{U}' = X(\widetilde{U}; s, U_-) := D^{-1}(\widetilde{U})\left\{-s(\widetilde{U} - U_-) + F(\widetilde{U}) - F(U_-)\right\}. \quad (6)$$

Equation (3) implies that U_- and U_+ are singular points of (6), *i.e.*, they satisfy $X(\widetilde{U}; s, U_-) = 0$. In fact, all singularities of (6) lie on the Hugoniot locus of U_-.

We consider the following four main types of admissible shock waves:

- a *1-wave*, which corresponds to a heteroclinic orbit from a repellor U_- to a saddle point U_+;
- a *2-wave*, which corresponds to a heteroclinic orbit from a saddle point U_- to an attractor U_+;
- a *transitional wave* or an *undercompressive wave*, which corresponds to a heteroclinic orbit from a saddle point U_- to another saddle U_+, or to a homoclinic orbit in case U_- and U_+ are the same saddle point;
- an *overcompressive wave*, which corresponds to a heteroclinic orbit from a repellor U_- to an attractor U_+.

This list can be extended by the possibilities that involve a heteroclinic (or a homoclinic) orbit between nonhyperbolic singularities. Here by a nonhyperbolic singularity we mean a singularity, U_s, of a vector field, $X(U; s, U_-)$, for which at least one of the eigenvalues of the Jacobian $X'(U_s; s, U_-)$ has a vanishing real part.

For $D = I$ the Lax admissibility criterion can be stated as follows.

DEFINITION 1 *A shock between U_- and U_+ is a Lax admissible 1-shock if U_- is a repellor and U_+ a saddle point in the vector field defined by (6). Similarly, a Lax 2-shock corresponds to U_- being a saddle and U_+ an attractor.*

This can be generalized to $D \neq I$. Notice that in this case the type of singularity at U_- or U_+ may change.

The two admissibility criteria are equivalent only for weak shocks [10]. Their relationship for strong shocks was studied in detail in Refs. [4], [5], [7], [8], [19], and [20]. In many situations both criteria provide unique weak solutions. Nevertheless, there are examples of nonstrictly hyperbolic systems for which Lax and viscous Riemann solutions differ due to the loss of profilability of some strong Lax shocks, see Refs. [15], [18], [22], [24], and [25]. In those cases transitional waves were used to fill the gap. Unfortunately, transitional waves might cause nonuniqueness of Riemann solutions. Examples of such situations are presented in Refs. [2] and [3]. When this happens additional conditions are needed to single out the physical solution.

Thus far the question of global existence and uniqueness of viscous Riemann solutions for nonstrictly hyperbolic systems of conservation laws is still open. In an attempt to capture global structure of Riemann solutions, in Ref. [17] Isaacson, Marchesin, Palmeira and Plohr introduced a framework called the fundamental wave manifold, $\mathcal{W}$, which we adopt to analyze the influence of viscosity on Riemann solutions. We describe its construction in the following section.

3. GEOMETRIC FRAMEWORK AND THE ADMISSIBILITY REGION

3.1 FUNDAMENTAL WAVE MANIFOLD $\mathcal{W}$

The fundamental wave manifold $\mathcal{W}$ provides a global geometric framework for studying waves that comprise Riemann solutions of $n \times n$ systems of nonlinear conservation laws. Its construction is based on the study of the zero set of the Rankine-Hugoniot condition

$$\mathcal{H}(U_-, U_+, s) := -s\,[U_+ - U_-] + F(U_+) - F(U_-) = 0 \tag{7}$$

as a subset of the ambient space $\mathcal{P} := \mathcal{U} \times \mathcal{U} \times \mathbf{R}$. We define a solution point $(U_-, U_+, s) \in \mathcal{P}$ of (7) to be either:

- a *shock point*, if $U_- \neq U_+$,
- a *rarefaction point* , if $U_- = U_+ =: \overline{U}$ and s an eigenvalue of $F'(\overline{U})$; or
- a *trivial point*, otherwise (i.e., $U_- = U_+$ and s not an eigenvalue of $F'(\overline{U})$).

The solution set is regular everywhere except at the rarefaction points where the derivative $d\mathcal{H}$ is rank deficient. This singularity is removed by "blowing up" the diagonal $\{(U_-, U_+, s) \in \mathcal{P} \,|\, U_- = U_+\}$ by using polar coordinates $R \in \mathbf{R}$ and $\Omega \in S^{n-1}$. In these coordinates we have $\Delta U := U_+ - U_- = Rr(\Omega)$, where $r(\Omega)$ corresponds to unit vectors in the tangent space of $\mathcal{U}$ at the midpoint $\overline{U} := \frac{1}{2}(U_- + U_+)$ (for $n = 2$, $r(\Omega) = (\cos\Omega, \sin\Omega)^T$). Thus, we can recover the orientation of a vector ΔU even in the limit $\Delta U = 0$. Given $(\overline{U}, R, \Omega, s)$ we define $U_- := \overline{U} - \frac{1}{2}Rr(\Omega)$ and $U_+ := \overline{U} + \frac{1}{2}Rr(\Omega)$. Since the same (U_-, U_+, s) is recovered if the antipodal point $(\overline{U}, -R, -\Omega, s)$ is used, we define $\mathcal{P}^*$ to be the space obtained from $\{(\overline{U}, R, \Omega, s) \,|\, (\overline{U}, R, \Omega, s) \in \mathcal{U} \times \mathbf{R} \times S^{n-1} \times \mathbf{R}\}$ after identification of antipodal points. The space $\mathcal{P}^*$ is a $(2n+1)$-dimensional manifold called the blow-up of $\mathcal{P} = \mathcal{U} \times \mathcal{U} \times \mathbf{R}$ over its diagonal.

For $n = 2$ we can restrict Ω to the segment $[0, \pi]$ and have $(R, \Omega) \in \mathbf{R} \times [0, \pi]$ serve as coordinates of $\mathcal{P}^*$, identifying pairs $(-R, \pi)$ and $(R, 0)$. Then $\mathcal{P}^* = \mathcal{U} \times M_2 \times \mathbf{R}$, where M_2 is a Möbius strip.

We now study the Rankine-Hugoniot solution set as a subset of $\mathcal{P}^*$. First notice that the blow-up manifold $\mathcal{P}^*$ of $\mathcal{P}$ was constructed in such a way that the singularity of the solution set $\mathcal{H}(U_-, U_+, s) = 0$ should be resolved if studied as a subset of $\mathcal{P}^*$. To see this we write

$$F(U_+) - F(U_-) = RA(\overline{U}, R, \Omega)r(\Omega),$$

where A is a matrix-valued function, for example,

$$A(\overline{U}, R, \Omega) := \int_{-\frac{1}{2}}^{\frac{1}{2}} F'(\overline{U} + \xi Rr(\Omega))d\xi\,. \tag{8}$$

Now, the Rankine-Hugoniot condition defined on $\mathcal{P}^*$ becomes

$$R\,\left[-s + A(\overline{U}, R, \Omega)\right] r(\Omega) = 0\,. \tag{9}$$

By dividing this equation by R, we remove the singularity in the Rankine-Hugoniot solution set. We now study the zero set of the mapping

$$\mathcal{F}(\overline{U}, R, \Omega, s) := \left[-s + A(\overline{U}, R, \Omega)\right] r(\Omega)\,.$$

Notice that this mapping is not well-defined on $\mathcal{P}^*$ since $\mathcal{F}(\overline{U}, R, \Omega, s) = -\mathcal{F}(\overline{U}, -R, \Omega + \pi, s)$. Nonetheless, the zero set is well-defined after the identification of antipodal points. Therefore, we have the following.

Definition 2 *[17] The fundamental wave set $\mathcal{W} \subset \mathcal{P}^*$ is the image of the zero set of $\mathcal{F}$ under the identification of antipodal points.*

Under some weak assumptions on the flux function F which imply that zero is a regular value of $\mathcal{F}$, the following theorem holds.

Theorem 1 *[17] The fundamental wave set $\mathcal{W}$ is an $(n+1)$-dimensional submanifold of $\mathcal{P}^*$.*

Shock waves and rarefaction waves are described by the points of $\mathcal{W}$. *Rarefaction points* are the points $(\overline{U}, 0, \Omega, s)$ such that

$$\left[-s + A(\overline{U}, R, \Omega)\right] r(\Omega) = 0\,.$$

The $R = 0$ slice of $\mathcal{W}$ that contains rarefaction points is called the *characteristic manifold*, $\mathcal{C}$. A *rarefaction curve* in $\mathcal{W}$ is an integral curve of the rarefaction line field which is the line field that contains vectors $(\dot{\overline{U}}, 0, \dot{\Omega}, \dot{s})$ tangent to $\mathcal{C}$, such that $\dot{\overline{U}}$ is parallel to $r(\Omega)$. A *rarefaction curve* in $\mathcal{U}$ is a projection to the state space of a rarefaction curve in $\mathcal{W}$ via the projection map $\Pi_{\overline{U}} : (\overline{U}, 0, \Omega, s) \mapsto \overline{U}$. This projection is singular at the points $(\overline{U}, 0, \Omega, s) \in \mathcal{C}$ for which s is a multiple eigenvalue of $F'(\overline{U})$ (see Ref. [17]). The set of all such points in $\mathcal{C}$ is called the *coincidence locus* and it is characterized by the following.

LEMMA 1 *[17] If s is a real eigenvalue of $F'(\overline{U})$, then s has algebraic multiplicity greater than 1 if and only if the $n \times n$ matrix $(\mathcal{F}_\Omega, \mathcal{F}_s)$ is singular at $(\overline{U}, 0, \Omega, s)$.*

If zero is a regular value of the function $\det(\mathcal{F}_\Omega, \mathcal{F}_s)$, then the coincidence locus is a one-dimensional submanifold of $\mathcal{C}$ (see Ref. [17]).

Shock points are the points $(\overline{U}, R, \Omega, s)$, $R \neq 0$, such that the Rankine-Hugoniot condition holds, i.e.,

$$\left[-s + A(\overline{U}, R, \Omega)\right] r(\Omega) = 0\,.$$

For a fixed state U_- (U_+), the set of all shock points $(\overline{U}, R, \Omega, s) \in \mathcal{W}$ such that $\overline{U} - \dfrac{1}{2} R r(\Omega) = U_-(U_+)$ is called a fixed *U_-–shock curve* (*U_+–shock curve*).

The set of all points (U, s) which satisfy the Rankine-Hugoniot condition (4) for a fixed state U_-, form a curve in a neighborhood of every solution if the derivative of the left hand-side of the Rankine–Hugoniot condition has full rank, *i.e.*, if

$$-\left[U - U_-\right] ds + \left[-s + F'(U)\right] dU$$

has rank n. If $U \neq U_-$ this happens if the matrix

$$(U - U_-, -s + F'(U)) \tag{10}$$

has full rank. The situation in which the matrix (10) has rank less than n is called *secondary bifurcation*. It was first identified by Wendroff in [28].

Definition 3 *A shock point $S \in \mathcal{W}$ is called a secondary bifurcation point if, for the corresponding U_- and U_+, the matrix*

$$(U_+ - U_-, -s + F'(U_+))$$

has rank less than n. The set of all secondary bifurcation points in $\mathcal{W}$ is called the secondary bifurcation locus, $\mathcal{B}$.

At a secondary bifurcation point a shock curve typically has either a cross structure, or it is an isolated point. We shall make use of the following.

Proposition 1 *Suppose that S is a secondary bifurcation point and that the corresponding $U_+ \in \mathcal{U}$ is in the strictly hyperbolic region. Then the right eigenvector of the second family is transverse to the 1-shock curve through U_- at the point U_+.*

Proof: The proof follows by contradiction. Suppose that the right eigenvector of the second family, r_2, is tangent to the 1-shock curve through U_- at U_+. Since U_+ is in the strictly hyperbolic region, $l_1 \cdot r_2 = 0$ at U_+, where l_1 denotes the left eigenvector of the first family. Therefore, $l_1 \cdot \dot{U} = 0$ at U_+. From the Rankine–Hugoniot condition we have

$$- \dot{s}\,[U - U_-] + [-s + F'(U)]\,\dot{U} = 0\,. \tag{11}$$

Multiplying equation (11) by the left eigenvector l_1 and taking into account that $l_1 \cdot \dot{U} = 0$ at U_+^s, we get

$$-\dot{s}\; l_1\; (U_+^s - U_-) = [s - \lambda_1]\;\; l_1\; \dot{U} = 0.$$

This is equivalent to saying that the matrix $(U_+ - U_-, -s + F'(U_+))$ is rank deficient. ∎

Shock points which satisfy the Lax admissibility criterion are bounded by the loci of points for which the shock speed equals one of the characteristic speeds at either U_- or U_+. These points form the *left sonic locus*, $\mathcal{S}_L := \{(\overline{U}, R, \Omega, s) \in \mathcal{W} \,|\, s = \lambda_i(U_-)\}$, or the *right sonic locus*, $\mathcal{S}_R := \{(\overline{U}, R, \Omega, s) \in \mathcal{W} \,|\, s = \lambda_i(U_+)\}$.

In order to determine shock points that satisfy the viscosity admissibility criterion we study the $n+1$-dimensional family of vector fields

$$U' = X(U; U_-, s) = D^{-1}\left(-s\,[U - U_-] + F(U) - F(U_-)\right), \tag{12}$$

for $U \in \mathcal{U} \subset \mathbf{R}^n$, where $(U_-, s) \in \mathcal{U} \times \mathbf{R}$ are regarded as parameters, and D is a fixed, positive definite matrix. We would like to determine the set of all points $(\overline{U}, R, \Omega, s) \in \mathcal{W}$ such that there is a connecting orbit between the corresponding singularities U_- and U_+.

DEFINITION 4 *[8] A point of $\mathcal{W}$ is admissible if there exists a connecting orbit from the singularity U_- to the singularity U_+ in the corresponding vector field (12). Admissible points form a subset $\mathcal{A} \subset \mathcal{W}$ called the admissibility region; the (topological) boundary $\partial\mathcal{A}$ of this region is the admissibility boundary.*

Notice that a fixed point of $\mathcal{W}$ determines a vector field X of the form (12), two singular points U_- and U_+ that serve as the left and right states in a shock wave, and a shock speed s. We will make use of the following definition.

DEFINITION 5 *Two points of $\mathcal{W}$ are equivalent if their corresponding vector fields are the same. Let $\mathcal{X}$ denote the set of all planar vector fields. The equivalence class [S] containing a point $S \in \mathcal{W}$, is the preimage of the image of S under the mapping $\mathcal{W} \to \mathcal{X}$ that associates a vector field to each point of $\mathcal{W}$.*

Therefore, two equivalent points differ because of a different choice of two singularities as U_- and U_+.

For quadratic models there is a particularly convenient choice of global coordinates for the fundamental wave manifold that simplifies the analysis of admissible shock points. The admissibility boundary for a 2×2 system of quadratic conservation laws was studied in detail in Refs. [4], [5], [7], and [8]. We summarize these results in the following subsection to demonstrate how the heteroclinic saddle-saddle connection bifurcation, the homoclinic bifurcation, and the nonhyperbolic periodic orbit bifurcation influence the admissibility boundary.

3.2 ADMISSIBILITY REGION IN $\mathcal{W}$

In this section we restrict our attention to 2×2 systems of conservation laws with quadratic flux functions $F(U)$. We shall assume that the elliptic region is a bounded subset of $\mathcal{U}$.

The fundamental wave manifold $\mathcal{W}$ for a 2×2 system of conservation laws with a quadratic flux function has a global topological structure of $M_2 \times \mathbf{R}$ ([21]). Moreover, since the Jacobian $F'(U)$ is affine linear in U, it is easy to see that the matrix valued function $A(\overline{U}, R, \Omega)$, defined by Eq. (8), is equal to $F'(\overline{U})$. Therefore, the function $\mathcal{F}$ now takes the form

$$\mathcal{F}(\overline{U}, R, \Omega, s) = \left[-s + F'(\overline{U})\right] r(\Omega). \tag{13}$$

We shall make use of a distinguished set of global coordinates for $\mathcal{W}$, denoted by (R, κ, Ω), where R and Ω are the polar coordinates described in the previous section, and κ parametrizes the "distance" from the midpoint $\overline{U}$ to the coincidence locus. For more details see Ref. [8].

We use global coordinates (R, κ, Ω) to analyze the region of admissible shock waves $\mathcal{A}$. For a fixed choice of U_- and s, equation (12) determines a vector field with at most four singularities, one of which is U_-. There are $2\binom{4}{2}$ points of $\mathcal{W}$ that correspond to the same vector field with different choices of singularities to serve as U_- and U_+. In the sense of Definition 5 this means that for a quadratic system of conservation laws, a class of equivalence of $S \in \mathcal{W}$ consists of at most $2\binom{4}{2}$ points of $\mathcal{W}$.

In order to determine the admissibility boundary $\partial\mathcal{A}$, we study a set of regularly structurally unstable vector fields X. More precisely, let $\mathcal{X}_n$ denote the complete metric space of all polynomial vector fields of degree n endowed with the usual metric on $\mathbf{R}^{(n+1)(n+2)}$. Let $\mathcal{P}(X)$ denote a Poincaré compactification of $X \in \mathcal{X}_n$ obtained by the unique analytic extension of the vector field on the sphere S^2 obtained by central projection of $\mathbf{R}^2$ into S^2.

Definition 6 *[26]*
(1) A vector field $X \in \mathcal{X}_n$ is structurally stable if there is a neighborhood $V \subset \mathcal{X}_n$ and a map $h : V \to \mathrm{Hom}(S^2, S^1)$ (homeomorphisms of S^2 that preserve S^1) such that $h(X) = Id$, and $h(Y)$ maps orbits of $\mathcal{P}(X)$ onto orbits of $\mathcal{P}(Y)$ for every $Y \in V$.
(2) A vector field X is regularly structurally stable if, in addition, the homeomorphism h, restricted to every periodic orbit of X, is continuously differentiable along every smooth curve in V passing through X.

The set of regularly structurally stable polynomial vector fields in the plane is characterized by a theorem of Sotomayor and Paterlini in Ref. [26] in the following way:

Theorem 2 *A vector field $X \in \mathcal{X}_n$ is regularly structurally stable if and only if:*

- *All singularities of $\mathcal{P}(X)$ are hyperbolic,*
- *All saddle-saddle connecting orbits of $\mathcal{P}(X)$ are contained in the equator S^1, and*
- *All periodic orbits of $\mathcal{P}(X)$ are hyperbolic.*

We follow this list to determine the corresponding codimension–1 submanifolds of $\mathcal{W}$. We then focus only on the bifurcations that influence the connection between the singularities U_- and U_+. These determine $\partial\mathcal{A}$.

Before we pursue this approach we have to assume that quadratic vector fields (12) are generic in the sense that there are no nonhyperbolic singularities of $\mathcal{P}(X)$ in S^1. It can be shown that the set of all such vector fields is open and dense in the metric of $\mathcal{X}_2$, see Ref. [27]. The genericity of X imposes some slight conditions on the second order coefficients of the flux function F and the viscosity matrix D. A precise formulation of these conditions can be found in Ref. [6].

1. Nonhyperbolic Singularities. A singularity U_s such that the Jacobian $\tilde{X}'(U_s; R, \kappa, \Omega)$ has an eigenvalue with a vanishing real part is called nonhyperbolic. A nonhyperbolic singularity U_s is called:

- a *saddle-node* if the Jacobian has exactly one eigenvalue equal to zero, or
- a *Hopf singularity* if the Jacobian has a pair of purely imaginary eigenvalues.

Definition 7 *The sonic locus, S, comprises points of $\mathcal{W}$ for which the corresponding vector fields contain a nonhyperbolic singularity of a saddle-node type. The Hopf locus, HF, comprises points of $\mathcal{W}$ corresponding to the Hopf bifurcation.*

The points on the sonic locus for which U_s equals U_- or U_+ form the left and right sonic loci $\mathcal{S}_L$ and $\mathcal{S}_R$. In the same manner we obtain the left and right Hopf loci HF_L and HF_R. In case when both U_- and U_+ are involved in a saddle-node bifurcation ($U_- = U_+ = U_s$) the locus of such points in $\mathcal{W}$ corresponds to the characteristic manifold $\mathcal{C}$.

Since the Jacobian $\tilde{X}'(U_s; R, \kappa, \Omega)$ is linear in R and κ, the left and right sonic loci and the left and right Hopf loci are ruled surfaces in $\mathcal{W}$, *i.e.*, they all take the form

$$A(\Omega)R + B(\Omega)\kappa + C(\Omega) = 0.$$

Precise formulae for the coefficients A, B and C, can be found in Ref. [8]. Crossing these loci transversally at a generic point, a shock between U_- and U_+ becomes inadmissible due to the change of a singularity type at U_- or U_+. Notice that by Definition 1 these loci determine a boundary of Lax admissible shock waves in $\mathcal{W}$.

2. Saddle-Saddle Connecting Orbits. We consider the loci of $\mathcal{W}$ that correspond to the vector fields with either a homoclinic orbit or a heteroclinic orbit between two saddle points.

DEFINITION 8 *The heteroclinic locus HET comprises points of $\mathcal{W}$ for which the corresponding vector field has two saddle points joined with a heteroclinic orbit. The homoclinic locus HOM comprises points of $\mathcal{W}$ for which the corresponding vector field has a saddle point involved in a homoclinic loop bifurcation.*

The points on the heteroclinic locus for which one of the saddles involved in the heteroclinic bifurcation is U_- or U_+ will be denoted by HET_L or HET_R. We can similarly define HOM_L and HOM_R. The points of HET for which U_- and U_+ are the two saddles connected by a heteroclinic orbit will be denoted by HET_0. Notice that these points correspond to transitional shock waves.

We shall further distinguish between the heteroclinic loci corresponding to straight line heteroclinic orbits, $HET^{straight}$, and the ones with curved heteroclinic orbits, HET^{curved}. The straight line saddle-saddle heteroclinic orbits occur at angles which are associated with the invariant lines of a quadratic vector field (12) (at most three). They are called the *viscosity angles* because they depend on the viscosity matrix D as well as the flux function F. If F is homogeneous quadratic, and D is positive definite, all saddle-saddle heteroclinic orbits are straight lines, [9]. Under certain nondegeneracy assumptions (that a viscosity angle Ω_* is nonexceptional, see Ref. [8]), the following theorem was proved in Ref. [8].

THEOREM 3 *Let Ω_* be a nonexceptional viscosity angle. Then the straight line heteroclinic locus $HET_0^{straight}$ is contained in the union of the planes $\Omega = \Omega_*$. Moreover, for each Ω_* the straight line heteroclinic loci $HET_L^{straight}$ and $HET_R^{straight}$ are contained in ruled surfaces within $\mathcal{W}$, namely those defined by*

$$\begin{aligned} \hat{A}(\Omega;\Omega_*)R + \hat{B}(\Omega;\Omega_*)\kappa + \hat{C}(\Omega;\Omega_*) &= 0\,, \\ -\hat{A}(\Omega;\Omega_*)R + \hat{B}(\Omega;\Omega_*)\kappa + \hat{C}(\Omega;\Omega_*) &= 0\,. \end{aligned}$$

For definitions of $\hat{A}$, $\hat{B}$ and $\hat{C}$ see Ref. [8].

Explicit formulae describing the locus HET^{curved} cannot be expected in general. In certain cases it is possible to obtain a tangent locus to HET^{curved} by unfolding a higher codimension point of $\mathcal{W}$ and by using the Melnikov integral analysis, [4], [8]. In general, we used numerical simulations performed by Riemann Solver Package [16] to obtain the loci HET_L^{curved} and HET_R^{curved}. Heteroclinic loci are displayed in Figure 1.

To determine the homoclinic locus in $\mathcal{W}$ in Ref. [8] we studied unfoldings of the Bogdanov–Takens points of $\mathcal{W}$.

DEFINITION 9 *[8] The Bogdanov-Takens locus BT consists of points of $\mathcal{W}$ for which one of the singularities U_s of the corresponding vector field satisfies*

$$\det\{D^{-1}[-s+F'(U_s)]\} = 0\,,$$

$$\mathrm{tr}\{D^{-1}[-s+F'(U_s)]\} = 0\,.$$

The following theorem, describing the homoclinic loci, was proved in Ref. [8]:

THEOREM 4 *Consider a quadratic model with a nontrivial elliptic region. Suppose that $(0,\kappa_0,\Omega_0)\in BT\cap\mathcal{C}$ and assume that Ω_0 is not a viscosity angle. Then in a neighborhood of $(0,\kappa_0,\Omega_0)$, the homoclinic loci HOM_L and HOM_R form two-dimensional manifolds whose boundaries are the Bogdanov-Takens points. Furthermore, the ruled surfaces of the form*

$$\widetilde{A}(\Omega)R+\widetilde{B}(\Omega)\kappa+\widetilde{C}(\Omega)=0$$

are tangent to HOM_L and HOM_R along $BT\cap\mathcal{C}$.

For definitions of $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$ see Ref. [8]. Homoclinic loci (obtained by numerical computations) are displayed in Figure 1. Figure 2 shows how the homoclinic and heteroclinic loci comprise the admissibility boundary.

3. NONHYPERBOLIC PERIODIC ORBITS. The hyperbolicity of periodic orbits is determined by the derivative of a Poincaré map, or the *return map*, P. If γ is a periodic orbit and p a point on γ, then a classical formula of Poincaré ([1]) gives that for a planar vector field X, $\log P'(p)=\int_\gamma \mathrm{div}X$. A periodic orbit γ is hyperbolic if p is a hyperbolic fixed point of the Poincaré map, i.e., if $P'(p)\neq 1$.

DEFINITION 10 *The nonhyperbolic periodic orbit locus NPO comprises points of $\mathcal{W}$ for which the corresponding vector fields undergo a nonhyperbolic periodic orbit bifurcation.*

A nonhyperbolic periodic orbit arises when two or more periodic orbits disappear (or appear). Bifurcations that give rise to two periodic orbits in a quadratic vector field are ([12]):

- Hopf bifurcation of codimension-2,
- homoclinic loop of codimension-2,
- double saddle-saddle heteroclinic connection (a two-cycle),
- simultaneous occurrence of a Hopf bifurcation of codimension-1 and a homoclinic loop of codimension-2, and

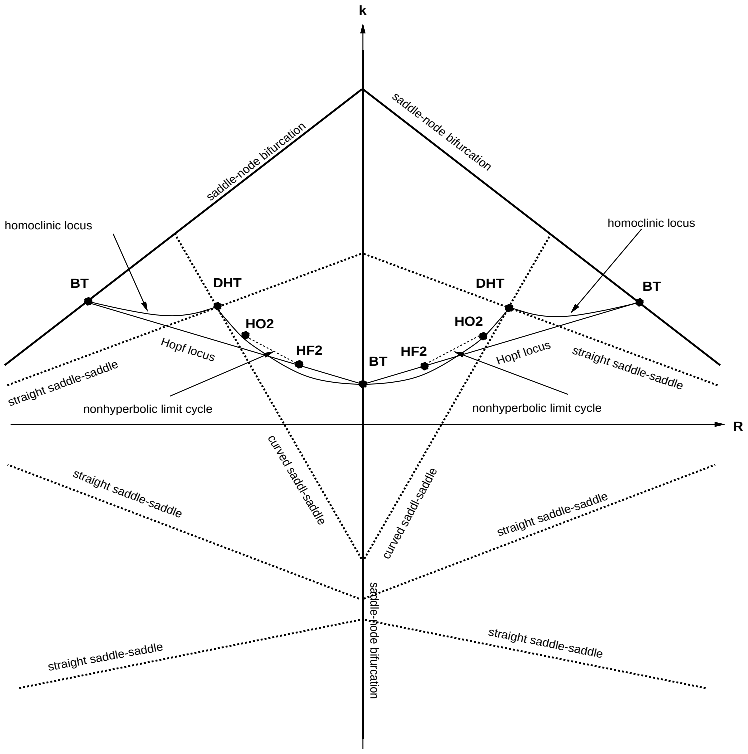


FIGURE 1: The figure shows a bifurcation diagram for a generic slice $\Omega = const.$ of the fundamental wave manifold for a quadratic system of conservation laws.

- nonhyperbolic periodic orbit.

When crossing the locus NPO transversally, the connection between the singularities that serve as U_- and U_+ breaks if a nonhyperbolic periodic orbit surrounds either U_- or U_+. A study of the role of the NPO locus in the admissibility of shock waves was presented in Ref. [5]. Figures 1 and 2 show how the NPO locus is contained in the admissibility boundary.

In contrast with the loci associated with bifurcations of singular points (saddle-node or Hopf bifurcation), crossing the loci HET, HOM, or NPO transversally, the type of a singularity involved in a bifurcation does not change. This means that these bifurcations do not influence the Lax admissibility of shock waves, but only the existence of a profile. The positions of the loci HET, HOM, and NPO depend on the coefficients of the vector field (12) which involve the viscosity matrix D. Whether a Lax shock between two fixed states U_- and U_+ in a system of conservation laws with a fixed flux function F admits a profile or not, depends entirely on the form of the viscosity matrix D.

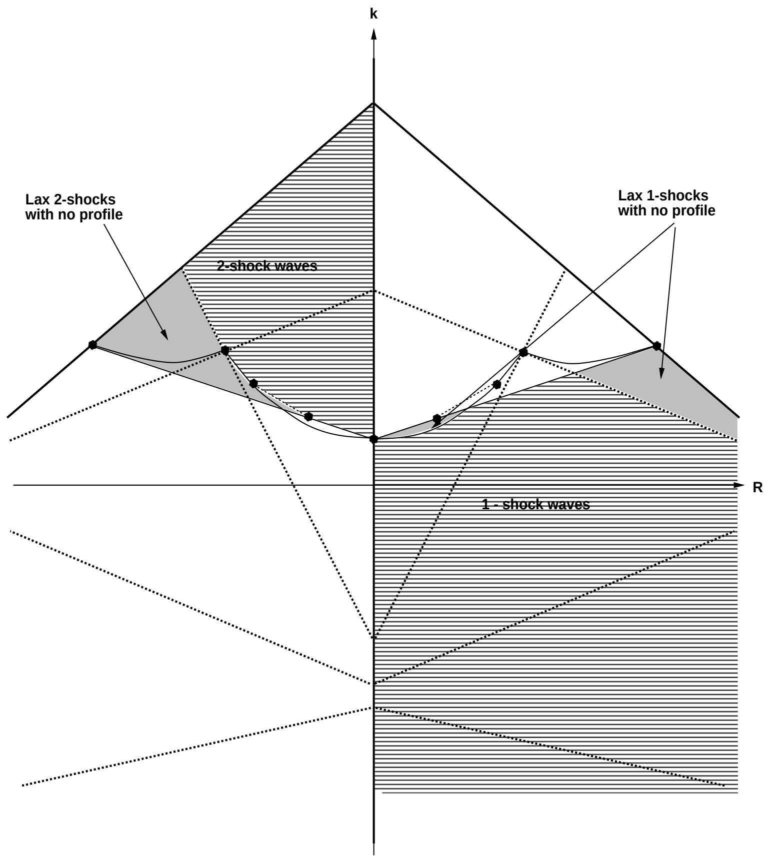


Figure 2: Displayed is the admissibility region for a generic two–dimensional slice $\Omega = const.$ of Figure 1. A comparison with Figure 1 shows that the boundary between the regions containing Lax shock waves with and without a profile consists of the heteroclinic locus, the homoclinic locus, and the nonhyperbolic periodic orbit locus. These loci depend on the viscosity matrix D.

In the sections that follow we study the role of the heteroclinic, homoclinic and nonhyperbolic periodic orbit bifurcations in the construction, existence and uniqueness of viscous Riemann solutions for a general 2×2 system of conservation laws. We shall illustrate our results in the framework of quadratic models, described in Section 2.

4. The Heteroclinic Saddle-Saddle Bifurcation Loci

In this section we show how the heteroclinic saddle-saddle bifurcation loci influence the structure, existence and uniqueness of Riemann solutions for

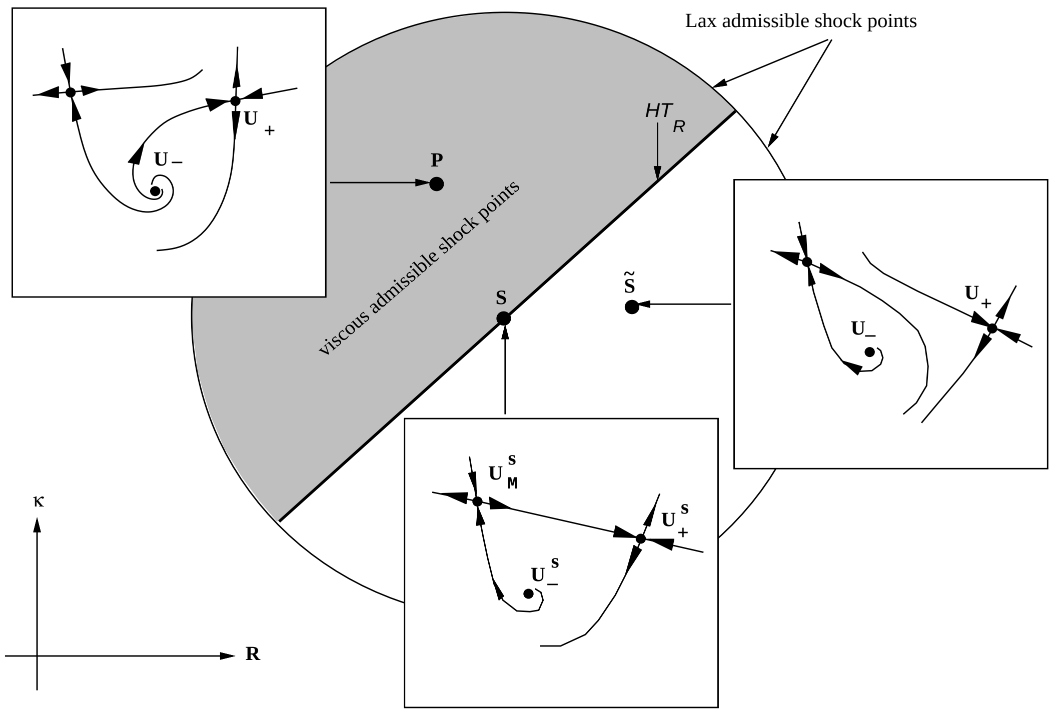


FIGURE 3: The figure shows a slice of $\mathcal{W}$ for $\Omega = const.$ displaying a region of Lax 1-shock points. The shaded subregion contains Lax shock points admitting a viscous profile. The heteroclinic saddle-saddle connection locus HET_R serves as its boundary. The points P, S and $\widetilde{S}$ all belong to the region of Lax 1-shock points. Only the shock between U_- and U_+ corresponding to the point P admits a viscous profile. The phase space portraits associated with P, S and $\widetilde{S}$ are shown in the rectangularly shaped windows.

a general 2×2 system of conservation laws.

4.1 Riemann Solutions Containing One Transitional Wave

We study a codimension–1 point, S, on the heteroclinic locus that divides a region of Lax-admissible shock waves into two parts: one that consists of points of $\mathcal{W}$ for which there is a heteroclinic orbit between U_- and U_+, and the other for which there is no profile from U_- to U_+, see Figure 3. By codimension-1 we mean that there is no other bifurcation taking place in a vector field (12) except for a heteroclinic saddle-saddle connection. Moreover, we assume that S is not a secondary bifurcation point in the sense of Definition 3. In fact, we shall assume that the entire equivalence class $[S]$ is away from the secondary bifurcation locus $\mathcal{B}$. We shall concentrate on the saddle-saddle bifurcation locus HET_R that serves as a boundary of admissible 1-shock waves. The results presented in this section hold as well if HET_R divides a region of Lax 2-shock points. By interchanging the roles of U_- and U_+ the locus HET_R can be replaced by HET_L.

THEOREM 5 *Let $S \in \mathcal{W}$ be a codimension–1 point on HET_R that serves as a boundary of admissible 1-shock waves. Moreover, suppose that all the points in $[S]$ are away from the secondary bifurcation locus. Let U_-^s and U_+^s denote the associated repellor and saddle point that correspond to the left and right states in a shock wave associated with S. Let U_M^s be the other saddle point involved in the saddle-saddle connection such that:*

- *the heteroclinic orbit from U_M^s to U_+^s corresponds to the intersection of the unstable manifold of U_M^s and the stable manifold of U_+^s, and*
- *there exists a heteroclinic connection from U_-^s to U_M^s.*

Then there are points $\widetilde{S} \in \mathcal{W}$ near S for which viscous Riemann solutions between the corresponding states U_- and U_+ involve a transitional shock wave from U_M^s to U_+^s. Lax admissible Riemann solutions at these points consist of a single 1-shock between U_- and U_+.

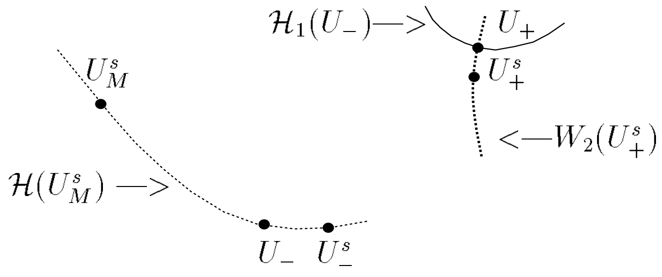
PROOF: Points $\widetilde{S}$ can be obtained by the following construction. Let $\mathcal{U}_- \subset \mathcal{U}$ denote a set of all states U_- near U_-^s such that:

- U_- belongs to the Hugoniot curve through U_M^s, i.e., $U_- \in \mathcal{H}(U_M^s)$,
- the shock speed $s(U_-; U_M^s)$ is less than the shock speed $s(U_-^s; U_M^s)$, and
- there exists a heteroclinic orbit from U_- to U_M^s in the vector field (12) where $s = s(U_-; U_M^s)$.

Since S is of codimension–1, and since all the points in $[S]$ are away from the secondary bifurcation locus, the set $\mathcal{U}_-$ is not empty. More precisely, since the singularity U_-^s is hyperbolic, *i.e.*, $\lambda_i(U_-^s) \neq s(U_M^s; U_-^s)$ for $i = 1, 2$, and since $[S]$ is away from $\mathcal{B}$, by the Bethe–Wendroff theorem, $\dot{s} \neq 0$ at U_-^s. Here $\dot{s}$ denotes the derivative of the shock speed along the Hugoniot curve based at U_M^s. Therefore, s is monotone in a neighborhood of U_-^s, along the Hugoniot locus. Moreover, since there is no other bifurcation taking place at $S \in HET_R$, the intersection between the unstable manifold of U_-^s and the stable manifold of U_M^s is stable under small perturbations.

Let $\mathcal{U}_+ \subset \mathcal{U}$ denote a set of all points U_+ such that the following two requirements hold:

- U_+ belongs to the intersection of the 2–wave curve through U_+^s, $W_2(U_+^s)$, with the 1-shock curve through U_-, $\mathcal{H}_1(U_-)$, for $U_- \in \mathcal{U}_-$,



- the speed of the local 2-wave from U_+^s to U_+ is bigger than the shock speed of the transitional wave from U_M^s to U_+^s.

Since S is not a secondary bifurcation point, the set $\mathcal{U}_+$ is not empty. More precisely, since $S \notin \mathcal{B}$, by Proposition 1, the intersection $W_2(U_+^s) \cap \mathcal{H}_1(U_-^s)$ is transverse at U_+^s. Therefore, for small perturbations $U_- \in \mathcal{U}_-$ of U_-^s, there exists a unique point U_+ which lies in the intersection $W_2(U_+^s) \cap \mathcal{H}_1(U_-)$. Moreover, the speed of a 2-wave from U_+^s to U_+ is greater than that of the transitional wave since the transitional wave from U_M^s to U_+^s satisfies (even for $D \neq I$):

$$\lambda_1(U_+) < s < \lambda_2(U_+), \qquad \lambda_1(U_+^s) < s < \lambda_2(U_+^s).$$

This immediately implies that a rarefaction wave of the second family is faster than the transitional shock. If the 2-wave is a shock, because $\lim\limits_{U_+ \to U_+^s} s_2(U_+^s; U_+) = \lambda_2(U_+^s)$, there exists a neighborhood of U_+^s in $W_2(U_+^s)$ containing the points $U_+ \in W_2(U_+^s)$ such that the speed of a shock from U_+^s to U_+, $s_2(U_+^s; U_+)$, satisfies

$$s_2(U_+^s; U_+) > s(U_M^s; U_+^s).$$

Therefore, the set $\mathcal{U}_+$ is not empty.

Let $\Pi_- : \mathcal{W} \to \mathcal{U}$ denote the U_--projection of a point in $\mathcal{W}$ to the state space $\mathcal{U}$, *i.e.*, $\Pi_-(S)$ is defined to be the left state associated with $S \in \mathcal{W}$. Similarly we define the U_+-projection, Π_+. Let $\mathcal{N}$ denote a set of points $\widetilde{S} \in \mathcal{W}$ such that

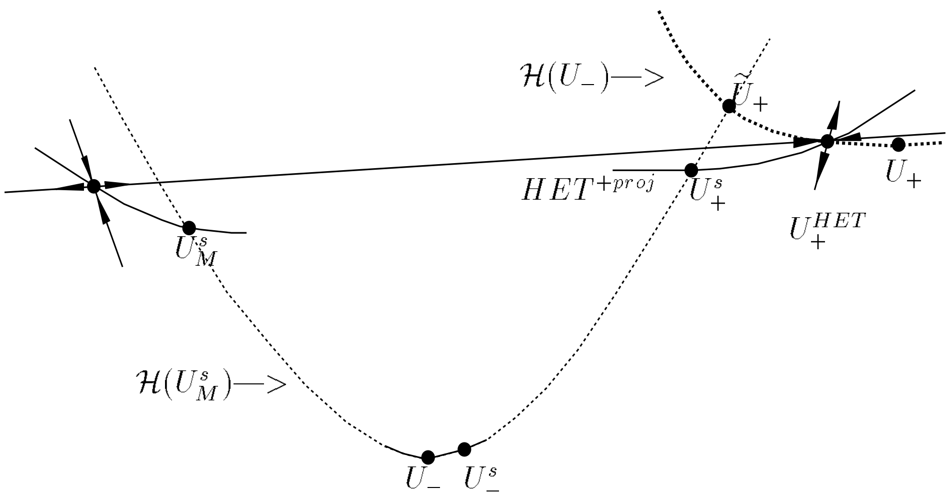
$$\Pi_-(\widetilde{S}) \in \mathcal{U}_- \quad \text{and} \quad \Pi_+(\widetilde{S}) \in \mathcal{U}_+.$$

All the points $\widetilde{S} \in \mathcal{N}$ satisfy the property that a viscous Riemann solution between the corresponding states U_- and U_+ contains a transitional shock wave from U_M^s to U_+^s. In fact, by the above construction, a viscous Riemann solution for Riemann data U_- and U_+ consists of a 1-shock from U_- to U_M^s, a transitional wave from U_M^s to U_+^s, and a 2-wave from U_+^s to U_+. This is in contrast with the corresponding Lax solution which consists of a single shock between U_- and U_+, since U_+ belongs to the 1-shock curve through U_-. ■

Notice that if $\widetilde{S}$ lies in the admissibility region $\mathcal{A}$, then there are two viscous Riemann solutions for the corresponding shock intial data (U_-, U_+): one consisting of a single viscous profilable shock, and another containing a transitional wave described in the proof of Theorem 5. The conditions under which two viscous Riemann solutions occur are given in Proposition 2. In order to state the proposition, we introduce the following notation. Let HET^{+proj} denote a set defined by

$$HET^{+proj} \equiv \{\Pi_+(P) | P \in HET_R \text{ and } \Pi_-(P) \in \mathcal{U}_-\}$$

In other words, HET^{+proj} consists of all points U_+^{HET} which lie on the Hugoniot curves through U_- for $U_- \in \mathcal{U}_-$, such that each U_+^{HET} is connected to another saddle point via a saddle-saddle heteroclinic connection. Define $\widetilde{U}_+$ to be a point near U_+^s such that $\widetilde{U}_+ \in \mathcal{H}(U_-) \cap \mathcal{H}(U_M^s)$. Assuming that $S \in HET_R$ is a codimension–1 point away from the secondary bifurcation locus, the point $\widetilde{U}_+$ is uniquely defined. Let $U_+ = \Pi_+(\widetilde{S})$. The following proposition holds.



PROPOSITION 2 *Let $\widetilde{S} \in \mathcal{W}$ be a point constructed in the proof of Theorem 5. Suppose that U_-^s and U_+^s lie on a connected Hugoniot branch through U_M^s and that there are no singularities between U_M^s and U_+^s other than U_-^s.*

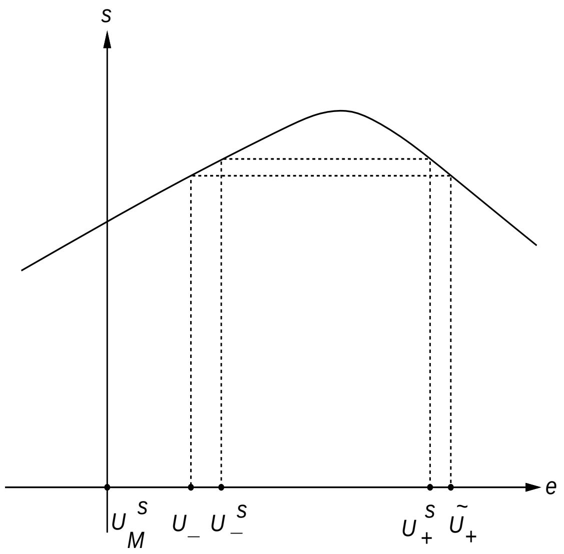


FIGURE 4: The figure shows a shock speed diagram along the Hugoniot curve through U_M^s with the singularities corresponding to the repellors U_- and U_-^s and the saddle points $\widetilde{U}_+$ and U_+^s.

Suppose that HET^{+proj} is transverse to the Hugoniot locus through U_M^s at the point U_+^s. Then $\widetilde{S} \in \mathcal{A}$ if and only if the intersection of HET^{+proj} with the Hugoniot curve through U_- lies between $\widetilde{U}_+$ and U_+, where $\widetilde{U}_+ \in \mathcal{H}(U_-) \cap \mathcal{H}(U_M^s)$.

PROOF: Since U_-^s is a repellor and U_+^s a saddle point, and since there are no singularities between U_M^s and U_+^s other than U_-^s, the graph of the shock speed along the connected branch of the Hugoniot curve has odd number of extrema between U_-^s and U_+^s, see Figure 4. Moreover, since U_-^s is a repellor and U_M^s a saddle point, the shock speed at U_-^s in increasing.

Since U_-^s is the only singularity between U_M^s and U_+^s, and s is continuous along a connected Hugoniot branch, the shock speed at U_+^s has to be decreasing. Choosing U_- so that $s(U_-, U_M^s) < s(U_-^s, U_M^s)$ implies that $\widetilde{U}_+$ lies on the side of HET^{+proj} for which there is no connection between the singularities U_- and $\widetilde{U}_+$. Therefore, the intersection of the Hugoniot curve through U_- with HET^{+proj} lies between $\widetilde{U}_+$ and U_+ if and only if $\widetilde{S} \in \mathcal{A}$.

■

COROLLARY 1 *When crossing the locus HET_R transversally, either the number of viscous Riemann solutions between U_- and U_+ changes, or the*

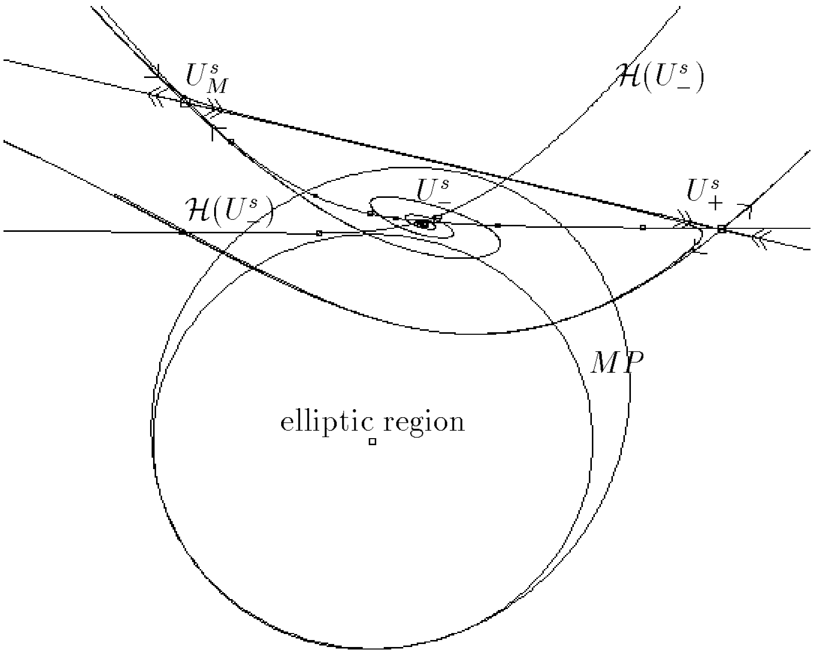


FIGURE 5: The figure shows the projection the phase portrait of a point $S \in HET_R^{upper}$. The singularities involved in the straight saddle-saddle connection correspond to U_M^s and U_+^s in the construction presented above. The repellor corresponds to U_-^s. The fourth singularity is not shown in this figure. All singularities lie in the strictly hyperbolic region. The two circular regions correspond to the elliptic region (inner circle), and to the region of Majda–Pego points (the outer ellipse).

number of waves in a Riemann solution between U_- and U_+ changes.

EXAMPLE 1: We illustrate the above results on an example of a 2×2 system of conservation laws with a quadratic flux function given by

$$F(U) = \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(-u^2+v^2)+0.23v \\ uv - 0.23u \end{pmatrix},$$

and a positive definite viscosity matrix

$$D = \begin{pmatrix} 1.06 & -0.638 \\ -0.07 & 0.709 \end{pmatrix},$$

with the eigenvalues $\delta_1 = 1.16$ and $\delta_2 = 0.6$. The inverse of D is

$$D^{-1} = \begin{pmatrix} 1 & 0.9 \\ 0.1 & 1.5 \end{pmatrix}.$$

This example will also illustrate that even though all the states U_-, U_+ and U_M lie in the strictly hyperbolic region, we may still have a singularity of (12) with complex conjugate eigenvalues. The region of all such states is bounded by the projection of the Bogdanov–Takens locus $BT \cap \mathcal{C}$ to the state space $\mathcal{U}$. We denote this projection by MP. The states bounded by MP give rise to unstable Cauchy problems of the linearized parabolic equations $V_t + F'(U)V_x = \epsilon D V_{xx}$. More precisely, we have the following definition.

Definition 11 *[8] A state U is a Majda -Pego point for the viscosity matrix D provided that $F'(U)$ has distinct real eigenvalues and that the Cauchy problem for $V_t + F'(U)V_x = \epsilon D V_{xx}$ is uniformly well-posed in L^2 as $\epsilon \to 0$.*

In Ref. [8] we proved the following.

Proposition 3 *A state U is a Majda-Pego point for the viscosity matrix D if and only if $\mathrm{tr}\,[-s + F'(U)]$ and $\mathrm{tr}\,\{D^{-1}\,[-s + F'(U)]\}$ have the same (nonzero) sign for each eigenvalue s of $F'(U)$. In particular, the boundary of the set of Majda-Pego points is contained in the union of the projections onto state space of the coincidence locus and the Bogdanov–Takens locus $BT \cap \mathcal{C}$.*

We take a point $S \in HET_R^{straight}$ so that all the corresponding singularities lie in the strictly hyperbolic region, with U_-^s inside the region bounded by MP, see Figure 5. We then choose a state U_- so that $s(U_-, U_M^s) < s(U_-^s, U_M^s)$. The corresponding phase portrait is shown in Figure 6. All the phase plane pictures in this example were obtained by using the Riemann Solver Package [16]. We see that the shock from U_- to U_M^s has a profile.

In Figure 7 we show the 2-wave curve through U_+^s. The "upper" part consists of rarefaction waves, and the "lower" part of 2-shock waves. We draw a Hugoniot curve through U_- and find its intersection with the 2-wave curve. We denote this intersection by U_+. The state U_+ in Figure 7 lies on the rarefaction segment. The speed of the 2-rarefaction is greater than that of the transitional shock from U_M^s to U_+^s. Therefore, there is a solution between U_- and U_+ consisting of a 1-shock from U_- to U_M^s, a transitional wave from U_M^s to U_+^s, followed by a 2-rarefaction from U_+^s to U_+. In addition, since the point U_+ lies on a 1-shock segment of the wave curve through U_-, the Lax solution for the Riemann data (U_-, U_+) consists of a single shock wave between U_- and U_+.

Figure 7 also shows the phase portrait corresponding to the point $\widetilde{S}$ in $\mathcal{W}$ such that $\Pi_-(\widetilde{S}) = U_-$ and $\Pi_+(\widetilde{S}) = U_+$. The left state U_- is a repellor and U_+ a saddle. The phase portrait shows that there is no heteroclinic orbit from U_- to U_+, which implies $\widetilde{S} \notin \mathcal{A}$. Figure 8 shows the (x,t)-diagram of the corresponding solutions.

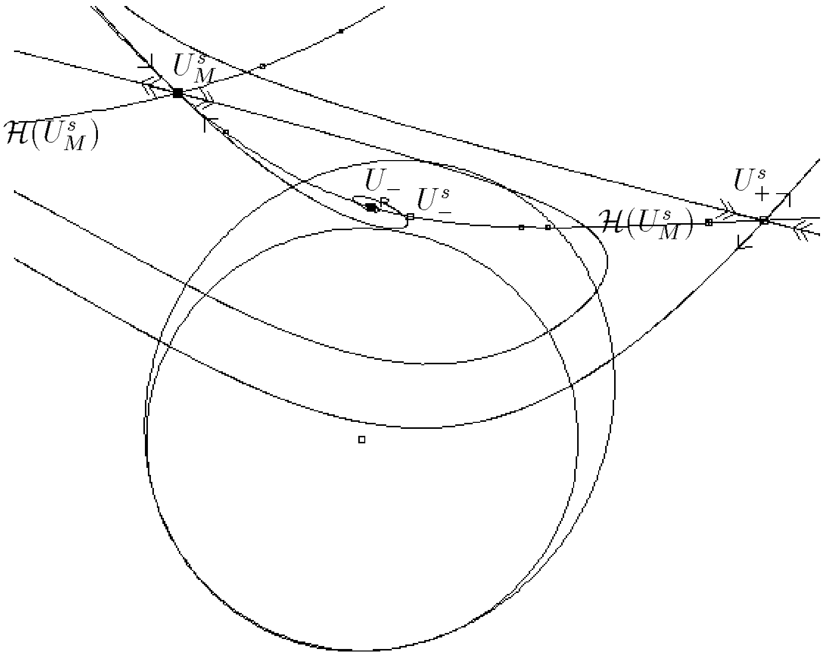


FIGURE 6: The figure shows the phase portrait of a 1-shock between the states U_- and U_M^s such that $s(U_-, U_M^s) < s(U_-^s, U_M^s)$.

4.2 RIEMANN SOLUTIONS CONTAINING TWO TRANSITIONAL WAVES

In this section we construct shock points of $\mathcal{W}$ satisfying the property that a viscous Riemann solution between the associated left and right states contains two transitional waves. Moreover, we show that the existence of such points is tied to the presence of a double saddle-saddle connection locus $DHET \subset \mathcal{W}$.

DEFINITION 12 *The double saddle-saddle heteroclinic bifurcation locus, denoted by $DHET$, comprises points of $\mathcal{W}$ for which the corresponding vector field has two saddle points joined with two heteroclinic orbits.*

The universal unfolding of a point $S \in DHET$ depends on the hyperbolicity ratios of the corresponding saddle points. Namely, for a point $S \in DHET$ let U_M^s and $\widetilde{U}_M^s$ denote the two saddle points involved in the bifurcation (see Figure 10), and let r_1 and r_2 be their corresponding hyperbolicity ratios, *i.e.*,

$$r_1 \equiv \frac{\mu_1(U_M^s)}{\mu_2(U_M^s)}, \quad r_2 \equiv \frac{\mu_1(\widetilde{U}_M^s)}{\mu_2(\widetilde{U}_M^s)},$$

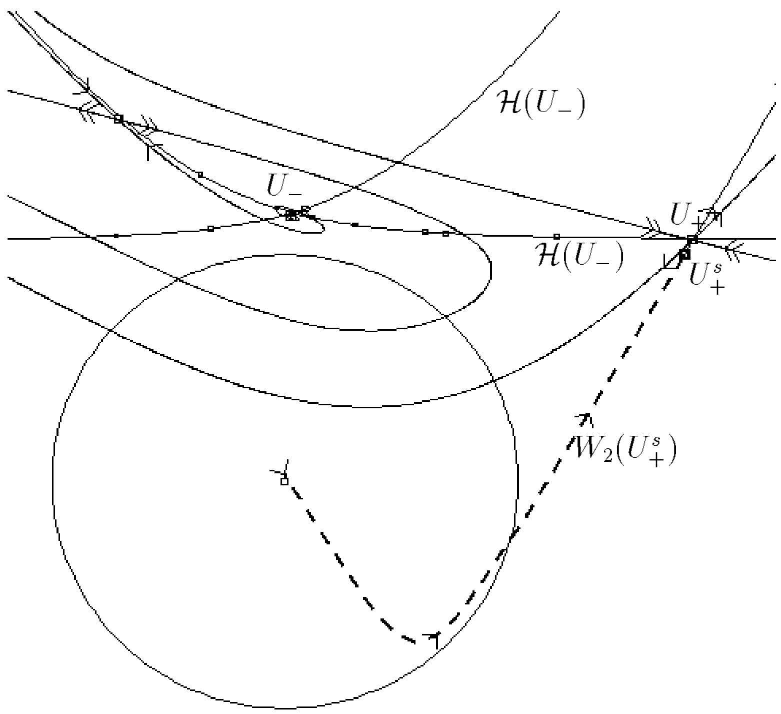


FIGURE 7: The phase portrait in this figure shows that the Lax 1-shock between U_- (repellor) and U_+ (saddle) has no profile. The figure also shows the 2-wave curve through U_+^s, $W_2(U_+^s)$, and the Hugoniot curve through U_-, $\mathcal{H}(U_-)$. The point U_+ lies in the intersection of $\mathcal{H}(U_-)$ and $W_2(U_+^s)$.

where μ_1 and μ_2 denote the eigenvalues of the Jacobian of (12). We shall assume that the two hyperbolicity ratios are both different from 1.

DEFINITION 13 *A point $S \in DHET$ is generic if $r = r_1 r_2 \neq 1$. A generic double saddle-saddle connection is attracting if $r > 1$ and repelling if $r < 1$.*

There are two subcases in each generic case. Without loss of generality we concentrate on the attracting case. A generic double saddle-saddle connection is *strongly attracting* if $r_1 > 1$ and $r_2 > 1$, or *weakly attracting* if $r_1 > 1$ and $r_2 < 1$. The corresponding universal unfoldings are studied in Ref. [11]. The universal bifurcation diagrams are shown in Figure 9.

DEFINITION 14 *A point $S \in DHET \subset \mathcal{W}$ is a codimension–2 point if there is no bifurcation taking place other than a double saddle-saddle connection, and the corresponding hyperbolicity ratio $r \neq 1$.*

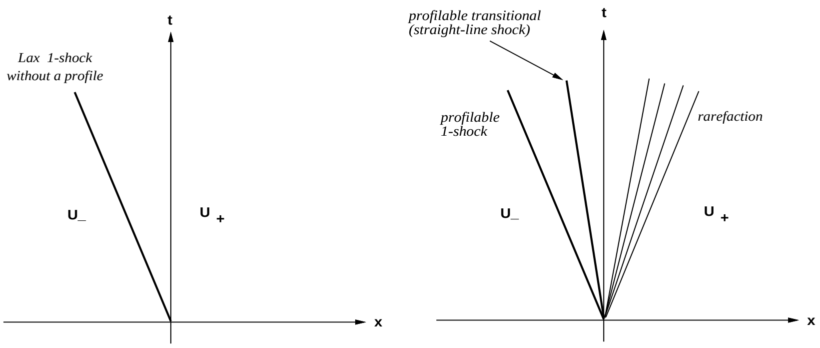


Figure 8: The (x,t)-diagrams in this figure show the waves that comprise the Lax and viscous Riemann solutions between the states U_- and U_+ of Example 1. We see that the Lax solution consists of a 1-shock, whereas the viscous solution consists of a 1-wave, a transitional wave, and a 2-rarefaction wave.

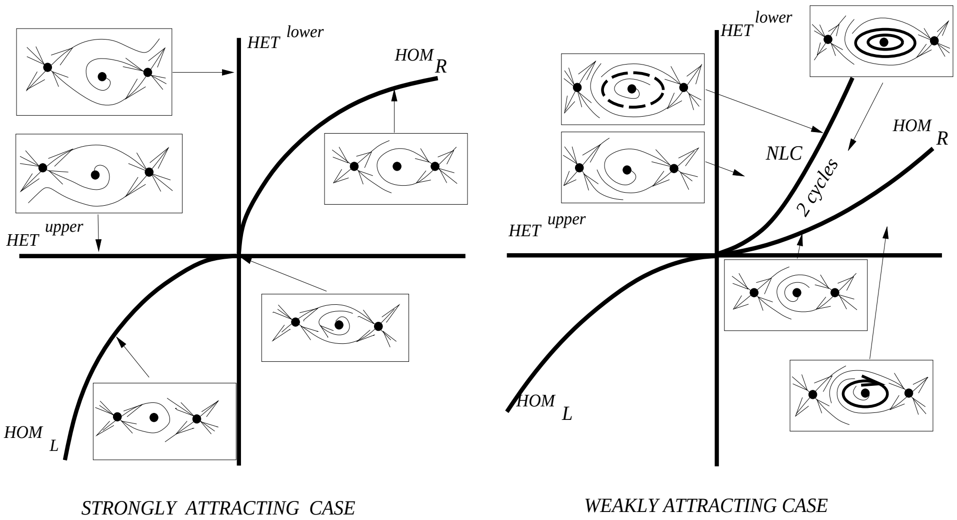


Figure 9: The figure shows the universal unfoldings of a double saddle–saddle connection point, S.

We now show that by using the universal unfolding of a codimension-2 point in $DHET$ we can construct viscous Riemann solutions which contain two transitional waves. The two transitional waves will correspond to small perturbations of the two saddle-saddle heteroclinic connections corresponding to a codimension-2 point in $DHET$.

Without loss of generality let us assume that $S \in DHET$ lies on the boundary of admissible 1-shock waves, and that $DHET = HET_R^{lower} \cap HET_R^{upper}$, as shown in Figure 10. Let U_-^s and U_+^s denote the corresponding left and right states (U_-^s is a repellor and U_+^s a saddle point), and let U_M^s be the third singularity, a saddle point, involved in the bifurcation.

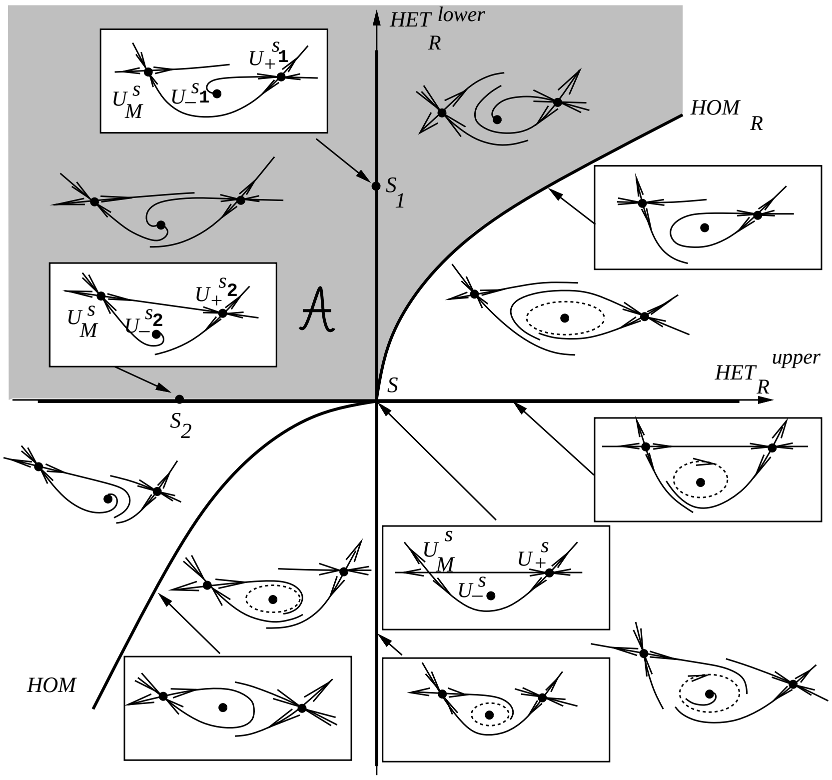


FIGURE 10: The figure shows a generic two–dimensional cross-section of $\mathcal{W}$ associated with a quadratic model displaying a neighborhood of a weakly attracting double saddle-saddle connection point of codimension–2 with its universal unfolding. The shaded region corresponds to admissible 1–shock waves.

Furthermore, let us assume that the fixed U_-–shock curve based at U_M^s intersects both heteroclinic loci HET_R^{upper} and HET_R^{lower} transversally at S. If S is a codimension-2 point and $[S]$ is away from the secondary bifurcation locus, then the Bethe-Wendroff theorem implies that the derivative of the shock speed s along the fixed U_--shock curve is different from zero at S, $\dot{s}(S) \neq 0$. By these two conditions there exists a perturbation of the fixed U_--shock curve in $\mathcal{W}$ which intersects the two heteroclinic loci transversally at points S_1 and S_2 such that the corresponding shock speeds are related by $s(S_1) < s(S_2)$. Without loss of generality suppose that $S_1 \in HET_R^{lower}$ and $S_2 \in HET_R^{upper}$. Notice again that both S_1 and S_2 lie on the same fixed U_--shock curve based on some U_M, a perturbation of U_M^s. By using the two transitional waves corresponding to the points S_1 and S_2 we construct a shock point of $\mathcal{W}$ for which the viscous Riemann solution between the corresponding U_- and U_+ contains the two transitional waves. The following theorem holds.

Theorem 6 *Suppose that*

- *$DHET = HET_R^{lower} \cap HET_R^{upper}$, and $DHET$ belongs to the boundary of admissible Lax 1-shocks,*
- *$S \in DHET$ is a codimension-2 point, with two saddle points U_+^s and U_M^s involved in the bifurcation,*
- *$[S]$ is away from the secondary bifurcation locus,*
- *a fixed U_--shock curve based on U_M^s is transverse to both heteroclinic loci at S,*
- *$s(S_1) < s(S_2)$, where $S_1 \in HET_R^{lower}$ and $S_2 \in HET_R^{upper}$ correspond to the intersections of the perturbed U_--shock curve with the heteroclinic loci, and let $S_1 \in \mathcal{A}$.*

Then there are shock points $\widetilde{S}$ in a neighborhood of S for which the Lax solution of the Riemann problem between the corresponding U_- and U_+ consists of a single shock, and the viscous Riemann solution contains two transitional shock waves corresponding to S_1 and S_2.

Proof: We construct a set $\mathcal{N} \subset \mathcal{W}$ containing the shock points $\widetilde{S}$ for which the viscous Riemann solution consists of a 1-shock, two transitional waves corresponding to S_1 and S_2, and a 2-wave.

We start with a 1-shock. For $S_1 \in HET_R^{lower}$, $U_-^{s_1}$ and $U_+^{s_1}$ denote the corresponding left and right states (a repellor and a saddle). We study a fixed U_--shock curve through $U_+^{s_1}$. Since $S_1 \in \mathcal{A}$ and since S is a codimension–2 point of $DHET$ such that $[S]$ is away from the secondary bifurcation locus, $\dot{s}(U_-^{s_1}) \neq 0$ and therefore there are points U_- on the Hugoniot locus of $U_+^{s_1}$, which are close to $U_-^{s_1}$, such that the shock speed of the 1-shock from U_-

to $U_+^{s_1}$ is less than that of the transitional wave corresponding to S_1. Let us denote the set of all such points by $\mathcal{U}_-$.

The set of all 2-waves following the transitional wave in S_2 can be obtained as follows. Let $W_2(U_+^{S_2})$ denote the 2-wave curve through $U_+^{S_2}$. By the definition of a transitional wave, there exists a neighborhood, $\mathcal{U}_+$, of $U_+^{s_2}$ along $W_2(U_+^{S_2})$ such that the speed of the corresponding 2-wave is less than that of the transitional shock in S_2. Here if the 2-wave is a rarefaction, by the wave speed we mean the slowest characteristic speed in the rarefaction fan.

Let U_+ be a point which lies in the intersection of the 1-shock curve through U_- and the 2-wave curve through $U_+^{s_2}$. Since $[S]$ does not belong to the secondary bifurcation locus, and assuming small perturbations of S, by Proposition 1 this intersection is not empty.

Let $\mathcal{N}$ denote the set of points $\widetilde{S}$ of $\mathcal{W}$ such that the U_--projection of $\widetilde{S}$ is contained in $\mathcal{U}_-$ and the U_+-projection of $\widetilde{S}$ in $\mathcal{U}_+$, i.e.,

$$\Pi_-(\widetilde{S}) \subset \mathcal{U}_-, \quad \text{and} \quad \Pi_+(\widetilde{S}) \subset \mathcal{U}_+.$$

By the above construction $\mathcal{N}$ is not empty. The points in $\mathcal{N}$ have the property that there is a Riemann solution consisting of a 1-shock wave from U_- to $U_+^{s_1}$, a transitional wave from $U_+^{s_1}$ to U_M, another transitional wave from U_M to $U_+^{s_2}$, and a 2-wave from $U_+^{s_2}$ to U_+, with speeds in increasing order. ∎

This concludes our discussion about the role of heteroclinic loci in the construction of Riemann solutions. Notice that, just as in the previous case, if $\widetilde{S}$ lies in $\mathcal{A}$, we have nonuniqueness of viscous Riemann solutions since in this case the single 1-shock from U_- to U_+ has a viscous profile. The role of a double heteroclinic bifurcation locus in nonuniqueness of Riemann solutions is studied in Ref. [3].

5. The Homoclinic and the Nonhyperbolic Periodic Orbit Locus

These loci influence the number of viscous Riemann solutions. Namely, across both loci the number of Riemann solutions between the singularities U_- and U_+ changes due to the presence of at least one periodic orbit surrounding either U_- or U_+. This causes a shock between U_- and U_+ to lose profile and become inadmissible. In Section 5.2 we give an example of a quadratic model in which this phenomenon causes nonexistence of a self–similar viscous Riemann solution for a certain class of initial data. Therefore, crossing these loci transversally, viscous Riemann solutions may cease to exist.

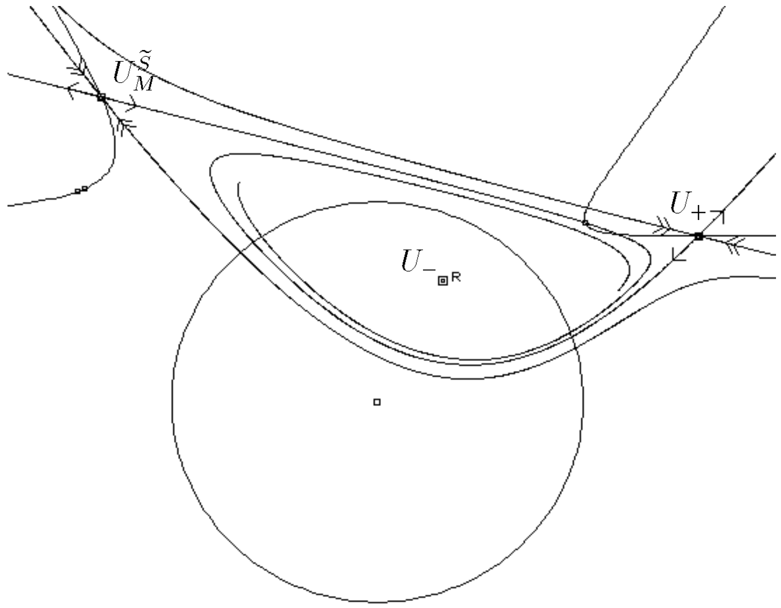


Figure 11: The figure shows a phase portrait corresponding to a point $\widetilde{S} \in \mathcal{W}$ which lies in the region of Lax admissible 1-waves without a profile. The left state U_- (repellor), is surrounded by an attracting periodic orbit. The right state U_+ is a saddle. We see that the unstable manifold of U_+ is trapped between the stable and unstable manifolds of $U_M^{\widetilde{S}}$ and the periodic orbit around U_-.

5.1 Periodic Orbits and Transitional Waves

Nonexistence of Riemann solutions may be avoided in certain cases when transitional waves can be used to construct a solution. This is, for example, the case when HOM or NPO arise in the universal unfolding of an attracting double saddle-saddle heteroclinic bifurcation point (see Figure 9). We shall show that there are "generic" points in a neighborhood of $DHET$ locus for which the corresponding Lax 1-shock becomes inadmissible due to the presence of a periodic orbit but for which there exists a Riemann solution between U_- and U_+ containing two transitional waves. We give an example of such a situation for the quadratic model and the viscosity matrix given in Example 1.

Example 2: Let

$$F(U) = \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(-u^2+v^2) + 0.23v \\ uv - 0.23u \end{pmatrix},$$

and the viscosity matrix

$$D = \begin{pmatrix} 1.06 & -0.638 \\ -0.07 & 0.709 \end{pmatrix}.$$

We take a point $\widetilde{S} \in \mathcal{W}$ which lies in the region of Lax admissible 1-shock waves such that $\widetilde{S} \notin \mathcal{A}$ because of a periodic orbit surrounding U_-. The phase portrait corresponding to $\widetilde{S}$ is shown in Figure 11. The corresponding left and right states are given by

$$U_- = \begin{pmatrix} 0.115382 \\ 0.0970946 \end{pmatrix} \quad \text{and} \quad U_+ = \begin{pmatrix} 0.357128 \\ 0.163145 \end{pmatrix}.$$

Using the Riemann Solver Package [16] we present a geometric construction of a Riemann solution between the states U_- and U_+ containing a profilable 1-shock, a curved transitional wave, a straight line transitional wave, and a 2-rarefaction, with the increasing sequence of corresponding wave speeds.

We construct the 1-wave curve through U_-, $W_1(U_-)$, and the backward 2-wave curve through U_+, $W_2^-(U_+)$ (here the superscript $-$ denotes the backward orientation of the wave curve). Notice that $U_+ \in W_1(U_-)$. We shall find a state $U_+^{s_1}$ on the 1-wave curve through U_- near the point U_+, and a state $U_+^{s_2}$ on the backward 2-wave curve through U_+, near U_+, such that there is a (curved) transitional wave from $U_+^{s_1}$ to some U_M and a (straight) transitional wave from U_M to $U_+^{s_2}$, with the corresponding shock speeds in increasing order, and such that the speed of the first transitional wave is greater than that of the shock from U_- to $U_+^{s_1}$. The construction we present here is reversed from the construction presented in the proof of Theorem 6.

Since $\widetilde{S}$ lies in a neighborhood of $DHET$, there exist points $S_1(\alpha) \in HET_R^{lower}$, $\alpha \in A$, and $S_2(\beta) \in HET_R^{upper}$, $\beta \in B$, such that their U_+-projections lie near U_+, on $W_1(U_-)$ and $W_2^-(U_+)$, respectively.

Let $U_M^{s_1}(\alpha)$, $\alpha \in A$, and $U_M^{s_2}(\beta)$, $\beta \in B$, denote the corresponding saddle singularities involved in the heteroclinic saddle-saddle connection. In other words, if we define the *transitional map* ([18]) $T : \mathcal{U} \to \mathcal{U}$ to be the map that associates to a saddle singularity U_1 another saddle singularity U_2 such that there exists a heteroclinic connecting orbit from U_1 to U_2, then $U_M^{s_1}(\alpha)$ corresponds to the image, under the transitional map, of a point $U_+^{s_1}(\alpha)$. Similarly, $U_M^{s_2}(\beta)$ corresponds to the preimage, under the transitional map, of $U_M^{s_2}(\beta)$. Suppose that there exists an $\alpha^* \in A$ and a $\beta^* \in B$ such that:

- $U_M^{s_1}(\alpha^*) = U_M^{s_2}(\beta^*) = U_M$,
- $s(U_+^{s_1}(\alpha^*), U_M) < s(U_M, U_+^{s_2}(\beta^*))$, and

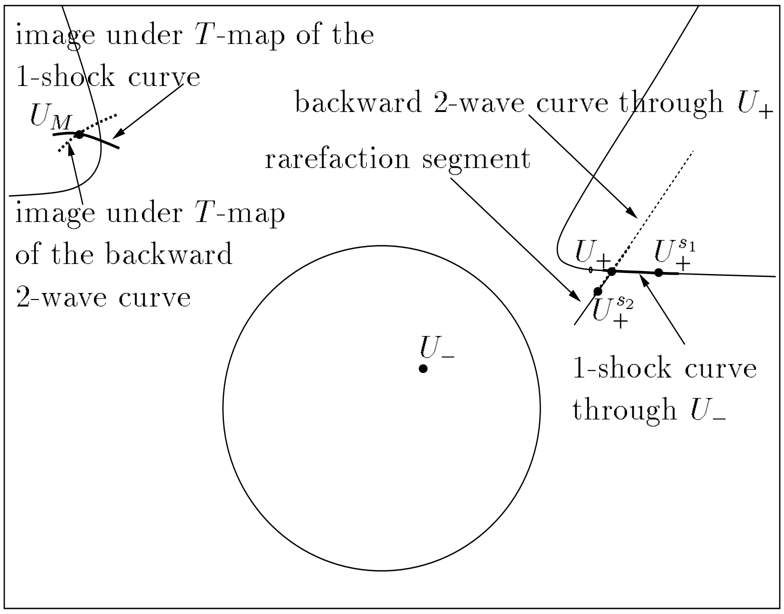


Figure 12: The figure shows the left and the right state U_- and U_+ that correspond to the point $\widetilde{S}$ whose phase portrait is shown in Figure 11. We show a local backward 2-wave curve through U_+ and a shock part of the forward 1-wave curve through U_-. The two curves in the upper left corner of the figure correspond to the image, under the transitional map, of the points on the backward 2-wave curve, and to the preimage of the points on the 1-shock curve through U_-. The two curves intersect at U_M. This means that there is a transitional wave from a point $U_+^{s_1}(\alpha^*)$ to U_M, followed by a transitional wave from U_M to $U_+^{s_2}(\beta^*)$. Their phase portraits are shown in Figures 13 and 14, respectively.

- $s(U_-, U_+^{s_1}(\alpha^*)) < s(U_+^{s_1}(\alpha^*), U_M)$.

Then there exists a Riemann solution between the states U_- and U_+ consisting of a 1-shock from U_- to $U_+^{s_1}(\alpha^*)$, a curved transitional wave from $U_+^{s_1}(\alpha^*)$ to U_M, a straight-line transitional wave from U_M to $U_+^{s_2}(\beta^*)$, and a two-wave from $U_+^{s_2}(\beta^*)$ to U_+. This is what happens in our example. A schematic picture of the construction of the solution is presented in Figure 12. The phase portraits corresponding to the points $S_1(\alpha^*)$ and $S_2(\beta^*)$ are shown in Figures 13 and 14, respectively. The (x,t)–diagrams of the Lax and viscous solution are shown in Figure 15. The diagrams show the wave types and the wave speeds in the Riemann solution associated with the U_- and U_+ from Figure 11.

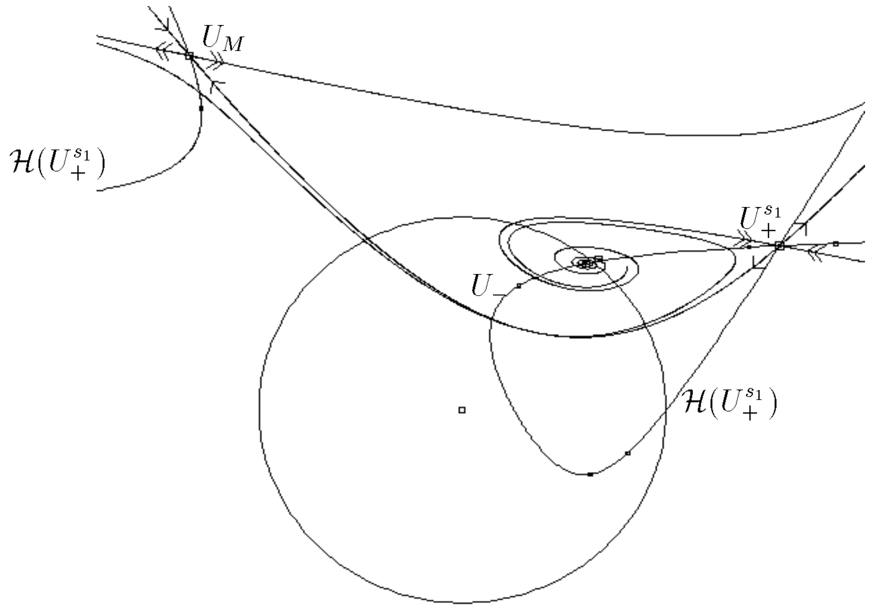


FIGURE 13: The figure shows the phase portrait of the curved transitional wave from $U_+^{s_1}(\alpha^*)$ to U_M. The state U_- lies on the Hugoniot curve through $U_+^{s_1}(\alpha_*)$, also shown in this figure.

5.2 An Example of Nonexistence in a Quadratic Model

In this section we show how the homoclinic bifurcation locus gives rise to nonexistence of a self-similar viscous Riemann solution. For this purpose we study the following quadratic model:

$$\begin{aligned} u_t - v_x &= 0 \\ v_t + \left(\frac{u^2}{2}\right)_x &= 0. \end{aligned} \tag{14}$$

This model derives from steady transonic flow. If we replace t with y, we have the steady transonic small disturbance equation:

$$\begin{aligned} uu_x + v_y &= 0, \\ -v_x + u_y &= 0. \end{aligned}$$

System (14) is elliptic in the right half plane of the state space, i.e., for $u > 0$, and hyperbolic for $u < 0$. The eigenvalues of the Jacobian $F'(U)$

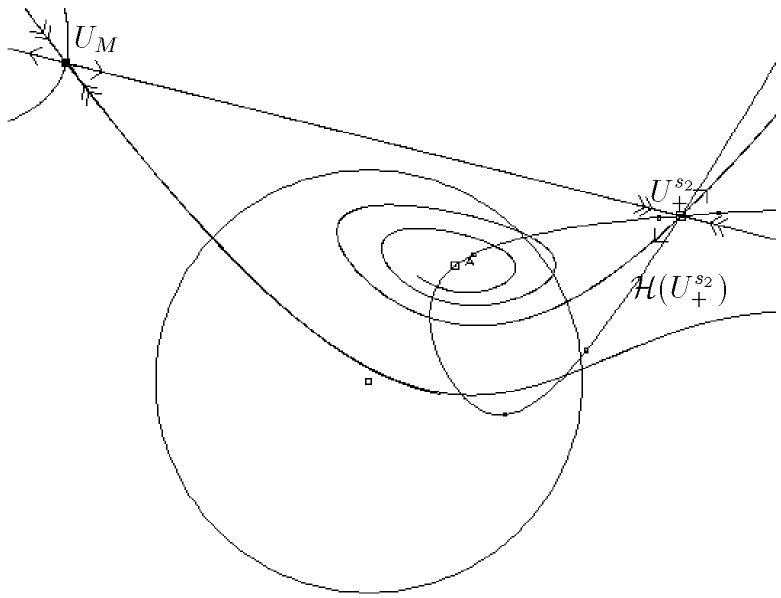


FIGURE 14: The figure shows the phase portrait of the straight-line transitional wave from U_M to $U_+^{s_2}(\beta^*)$. The state U_M lies on the Hugoniot curve through $U_+^{s_2}(\beta^*)$ also shown in this figure.

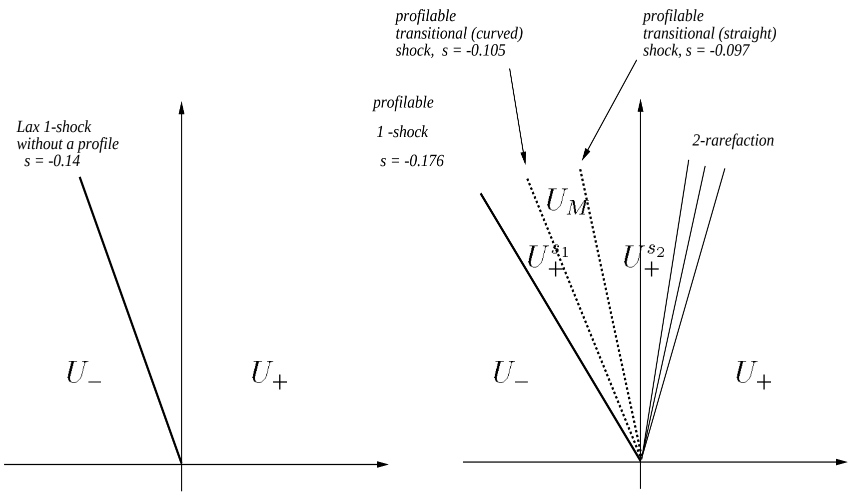


FIGURE 15: The figure shows the (x, t)–diagrams of the Lax and viscous Riemann solutions between the states U_- and U_+ of Figure 11.

are given by $\lambda_{1,2}(U) = \mp\sqrt{-u}$. The Rankine–Hugoniot equations based on $U_- = (u_-, v_-)^T$ read

$$\begin{aligned} -s(u-u_-) + v_- - v &= 0, \\ -s(v-v_-) + \tfrac{1}{2}(u-u_-)(u+u_-) &= 0. \end{aligned} \tag{15}$$

The Rankine–Hugoniot locus through U_- can be written as

$$(u-u_-)^2 \left(\frac{u+u_-}{2}\right) + (v-v_-)^2 = 0. \tag{16}$$

We look for self-similar viscous Riemann solutions associated with a viscosity matrix of the following form

$$D \equiv \begin{pmatrix} 1 & -\delta \\ 0 & 1 \end{pmatrix},$$

where $\delta \in \mathbf{R}$ is fixed. A shock wave between U_- and U_+, traveling at speed s, admits a profile if there is a connecting orbit between the singularities U_- and U_+ in the vector field $X(U; s, u_-, v_-)$ defined by

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = X(U; s, u_-, v_-) \equiv D^{-1} \begin{pmatrix} -s(u-u_-) + v_- - v \\ -s(v-v_-) + \frac{1}{2}(u-u_-)(u+u_-) \end{pmatrix}. \tag{17}$$

Let

$$\begin{aligned} x &= u - u_-, \\ y &= v - v_-, \end{aligned} \tag{18}$$

then (17) is equivalent to

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -y + (\delta u_- - s)\ x - \delta s\ y + \frac{1}{2}\delta\ x^2 \\ u_-\ x - s\ y + \frac{1}{2}\ x^2 \end{pmatrix}. \tag{19}$$

The singularities of this vector field are given by $P_1 = (0,0)$ (left state), and $P_2 = (-2(s^2 + u_-), 2s(s^2 + u_-))$ (right state). The Jacobian of X is given by

$$X'(x, y; s, u_-, v_-) = \begin{pmatrix} \delta u_- - s + \delta x & -1 - \delta s \\ u_- + x & -s \end{pmatrix}.$$

Proposition 4 *There are no transitional waves in this model.*

PROOF: We shall show that it is not possible to have two saddle singularities in the vector field defined by (19). From the form of the determinant of the Jacobian of X:

$$\det\left(X'(x,y;s,u_-,v_-)\right) = s^2 + u_- + x$$

we see that $\det\left(X'(P_1;s,U_-)\right) = -\det\left(X'(P_2;s,U_-)\right) = s^2 + u_-$. Since at a saddle point the determinant must be negative, we see that P_1 and P_2 can never be two saddle points corresponding to the same vector field. ■

Next we shall show that for a certain range of parameters s and u_-, there are Lax shock waves that do not admit a profile due to the presence of a periodic orbit surrounding one of the singularities that serves as either the left or the right state in a shock wave. For this purpose notice that the vector field (19) undergoes a Bogdanov–Takens bifurcation at $(x,y) = (0,0)$ for the parameter values $s = u_- = 0$. Namely, we can write the vector field (19) as:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \delta x^2 \\ x^2 \end{pmatrix} + \begin{pmatrix} \delta u_- - s & -\delta s \\ u_- & -s \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{20}$$

For $u_- = s = 0$ we have a double zero eigenvalue at $(x,y) = (0,0)$. Moreover, $(x,y) = (0,0)$ is always a singular point. A universal unfolding of a Bogdanov–Takens point with this property has been studied in Refs. [8], [14]. The universal unfolding contains transcritical bifurcation, homoclinic and Hopf bifurcation loci. Notice that along the curve $s^2 + u_- = 0$ a transcritical bifurcation takes place. Across the curve $s^2 + u_- = 0$ the roles of U_- and U_+ are interchanged. The left and the right Hopf bifurcation loci are easily calculated from the trace of the Jacobian evaluated at the left or right state. Taking into account that the discriminant of the Jacobian is negative and that the trace of the Jacobian is zero at the Hopf bifurcation loci, we obtain:

PROPOSITION 5 *The left and the right Hopf loci for the vector field (20) are given by*

$$\begin{aligned} &HF_R \;\ldots\; \delta u_- + 2s + 2\delta s^2 = 0, \quad u_- \le 0 \\ &HF_L \;\ldots\; \delta u_- - 2s = 0, \quad u_- \ge 0 \end{aligned}$$

We obtain the homoclinic loci in the parameter space (s, u_-) by using a standard technique based on Melnikov's integral. Namely, the following theorem holds.

THEOREM 7 *The left and right homoclinic loci that arise in the unfolding of the Bogdanov-Takens point $s = u_- = 0$ for the vector field defined by (20) are given by*

$$\mp \frac{5}{7}\frac{\delta}{2} u_- + s = 0.$$

PROOF: We use a rescaling by ξ to "blow-up" the variables u and v, as well as the parameters s and u_-, defining $x = \xi^2 u$, $y = \xi^3 v$, $s = \xi^2 \nu_1$, $u_- = \xi^2 \nu_2$. Rescaling the time variable by $\tau = \xi t$ leads to

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = H(u, v; \nu_2) + \xi g(u, v; \nu_1, \nu_2) + \mathcal{O}(\xi^2),$$

where

$$H(u, v; \nu_2) = \begin{pmatrix} -v \\ \nu_2 u + \frac{1}{2}u^2 \end{pmatrix}$$

is a Hamiltonian vector field and

$$g(u, v; \nu_1, \nu_2) = \begin{pmatrix} \delta\nu_2 - \nu_1)u + \frac{1}{2}\delta u^2 \\ -\nu_1 v \end{pmatrix}.$$

Analysis based on Melnikov's integral ([13]) yields the tangent manifold to the homoclinic bifurcation locus

$$-\text{sign}(\nu_2)\, \frac{\delta}{7}\nu_2 + \frac{2}{5}\nu_1 = 0.$$

In the original variables, depending on the sign of ν_2, we recover the left and right homoclinic loci. ■

The universal unfolding in the (s, u_-)-parameter space, with $\delta = 1$, is shown in Figure 16.

THEOREM 8 *Let (s, u_-) belong to the region which is bounded by the Hopf and homoclinic loci such that there is no connection from the singularity $P_- = (0, 0)$ to $P_+ = (-2(s^2 + u_-), 2s(s^2 + u_-))$, owing to the presence of a periodic orbit. Then there is no viscous Riemann solution between $U_- = (u_-, v_-)$ and $U_+ = (-2s^2 - u_-, 2s(s^2 + u_-) + v_-)$.*

PROOF: Without loss of generality we shall assume that $\delta > 0$, and that $v_- = 0$. The bifurcation diagram shown in Figure 16 implies that if $u_- > 0$, then there is a region bounded by the left Hopf and homoclinic loci containing the points for which the corresponding vector fields have a repellor at

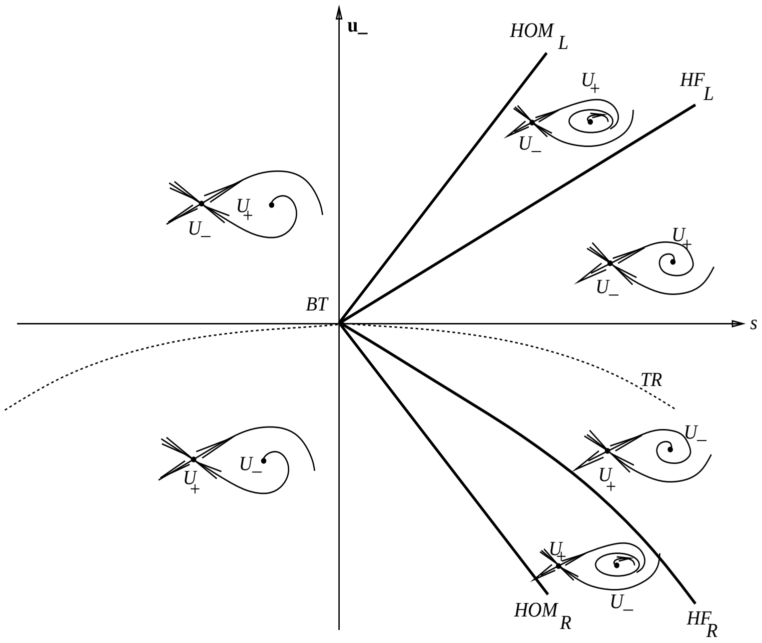


Figure 16: The figure shows the loci arising in the universal unfolding of a Bogdanov–Takens point in the vector field (19) with $\delta = 1$. The curves HOM and HF are the homoclinic and the Hopf bifurcation loci. Along the transcritical bifurcation locus, TR, the singularities that serve as U_- and U_+ exchange their roles.

U_- and a saddle point at U_+ so that there is no connection from U_- to U_+ because of the periodic orbit surrounding U_-. By contradiction, we shall show that there is no viscous Riemann solution between those U_- and U_+.

Assume that there is a viscous Riemann solution between U_- and U_+. Then, by the results of Ref. [23] (Theorem 2.3 (Wave Structure)), the solution consists of a 1-wave group, followed by an arbitrary number of transitional wave groups, followed by a 2-wave group.

By Proposition 4, there are no transitional waves in this model. Therefore, we are looking for a solution that consists of a 1-wave group, followed by a 2-wave group.

We construct the 1-wave curve through U_-. It consists of a 1-shock branch given by (16). Using the viscosity admissibility criterion we find its admissible part, shown in Figure 17.

PROPOSITION 6 *A Riemann solution for Eq. (14) for which U_- lies in the elliptic region has a 1-wave group that consists of at most one shock wave.*

PROOF: All other possibilities involve a 1-shock from U_- to some U_1 followed by a 1-rarefaction from U_1 to U_2, where the first shock is characteristic on the right. Our model does not allow this possibility since in order to have a characteristic shock on the right, the following must hold:

$$s(U_-;U_1) = \lambda_1(U_1), \quad \text{i.e.,} \quad \pm\sqrt{-\frac{1}{2}(u_1+u_-)} = -\sqrt{-u_1}.$$

This is true if and only if $u_1 = u_-$ (or $u_1 = u_- = 0$ if on the upper part of the Rankine-Hugoniot branch). ■

Therefore, if a solution exsits, it has to consist of a 1-shock wave followed by a 2-wave group.

We construct a backward 2-wave curve based at U_+. By a similar argument as in the proof of Proposition 6 one can see that there are no characteristic shock waves in the 2-wave group. The backward 2-wave curve consists of a rarefaction segment given by

$$v - v_+ = 2/3\left[u_+\sqrt{-u_+} - u\sqrt{-u}\right], \quad u_+ \le u \le 0$$

and a 2-shock segment through U_+ given by (16) for $u \le u_+$. Its admissibility can be obtained from the bifurcation diagram in Figure 16. It can easily be verified that the only intersection between the forward 1-wave curve based at U_- and the backward 2-wave curve based at U_+ is at U_+. This concludes the proof of Theorem 8. In Figure 17 we show the wave curves for $u_- = 1$, $v_- = 0$, $\delta = 1$ and $s = 0.5 + \epsilon$, for which U_- is a repellor surrounded by a periodic orbit. ■

This example illustrates that a homoclinic bifurcation locus might lead to nonexistence of a viscous Riemann solution.

6. CONCLUSIONS

In this paper we presented an organized approach to understanding the effects of viscosity in the construction, nonuniqueness and nonexistence of Riemann solutions. We studied the mechanisms that distinguish shock wave solutions of a conservation law that comply with the viscosity admissibility criterion from those that satisfy the Lax admissibility criterion. We showed that "generically" these mechanisms give rise to bifurcations in the number of waves that comprise Riemann solutions, and determined the boundaries

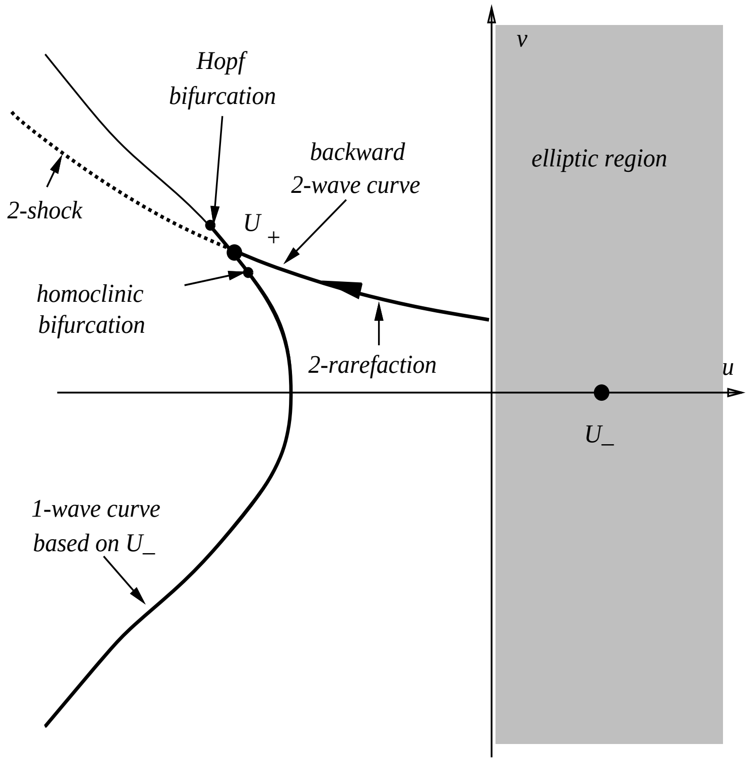


Figure 17: We show the Rankine–Hugoniot curve based at U_- and the backward 2-wave curve through U_+. The part of the 1-wave curve corresponding to Lax admissible shock waves is shown in bold. It is bounded by the Hopf bifurcation point. The part containing profilable 1-shock waves is bounded by the homoclinic bifurcation point. The dashed segment of the backward 2-wave curve through U_+ corresponds to viscous admissible 2-shock waves, and the bold curve denotes 2-rarefaction waves.

of the regions in parameter space across which the number of Riemann solutions changes.

Our results show that knowing the precise form of physical viscosity is crucial in obtaining a physically meaningful Riemann solution. This is especially important in numerical schemes which might provide Riemann solutions with incorrect wave structures since they are based on a particular (generally unknown and often time-dependent) viscosity.

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