Nonexistence of Riemann solutions and Majda-Pego instability

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Abstract. We study Riemann problems for the shallow water equations. We consider weak self-similar Riemann solutions consisting of constant states, rarefaction waves and/or jump discontinuities that satisfy the viscous profile entropy criterion, with a positive definite, symmetric viscosity matrix. We prove that for a "generic" symmetric, positive definite viscosity matrix there is an open set of Riemann initial data for which a weak self-similar Riemann solution does not exist. We show that this happens for the hyperbolic initial data that is unstable in the sense studied by Majda and Pego. We prove that such initial data always exist for positive definite, symmetric, nondiagonal viscosity matrices. In the work that follows [16] we show that in the situations presented in this paper, measure-value solutions exhibiting continuously generated oscillations take place. The results of the present paper provide a new insight into the role of the viscous profile entropy criterion and the Majda-Pego instability in the existence of Riemann solutions for nonlinear conservation laws.

1 Introduction

In this work we study Riemann problems for a system of conservation laws for the shallow water equations

$$U_t + F(U)_x = 0,$$

where

$$U = \begin{bmatrix} v \\ \phi \end{bmatrix} \quad \text{and} \quad F(U) = \begin{bmatrix} v^2/2 + \phi \\ v\phi \end{bmatrix}.$$

We consider weak self-similar solutions that consist of constant states, rarefaction waves and/or jump discontinuities that satisfy the viscous profile entropy criterion

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with a general symmetric, positive definite viscosity matrix. Although many of the Riemann solutions' features appear when a multiple of the identity matrix is used, this is a nongeneric situation and, to capture physically relevant solutions, viscosity matrices other than the identity need to be considered.

The goal of this work is to investigate whether there are symmetric, positive definite viscosity matrices and hyperbolic Riemann initial data that lead to nonexistence of weak self-similar solutions. Indeed, we show that for a "generic" symmetric, positive definite viscosity matrix there exists an open set of Riemann initial data for which we prove that a weak self-similar solution does not exist. Furthermore, we associate this behavior with the presence of the, so called, Majda-Pego unstable region of hyperbolic states, as studied in Refs. [14] and [2]. We show that a nontrivial Majda-Pego unstable region always exists for every symmetric, nondiagonal, positive definite viscosity matrix. Thus, nonexistence of Riemann solutions will always be present in those situations.

The behavior studied in this work is not isolated to the shallow water equations. In Ref. [1] nonexistence of a weak self-similar solution was established for a three-phase flow model arising in oil reservoir modeling, with a fixed viscosity matrix and for particular Riemann initial data. The viscosity matrix in the three-phase flow model derives from capillary pressures and is symmetric, nondiagonal, positive definite. The results of that work were obtained as a combination of analytical methods and numerical simulations. The main difference is that, in contrast with the shallow water equations, the three-phase flow model admits nonclassical (transitional) waves and the construction of Riemann solutions was impossible without the help of numerical simulations. In the present paper we were able to obtain the nonexistence of weak Riemann solutions by using only analytical methods. Furthermore, we prove this result for a general symmetric, positive definite matrix and an open set of Riemann initial data. And finally, we associate this behavior with the presence of an unstable Majda-Pego region. To our knowledge, this is the first time that nonexistence of Riemann solutions in which all shocks are required to satisfy the viscous profile entropy criterion, has been proved.

The main ingredient in the proof of the nonexistence result is the study of shock waves that do not possess a viscous profile due to the presence of a limit cycle in the dynamical system associated with the viscous profile entropy criterion. We identify such shock waves by studying the universal unfolding of a Bogdanov-Takens bifurcation that takes place at the boundary of the region in state space consisting of the points that are stable in the sense studied by Majda and Pego [14]. We then show that it is impossible to use other waves arising in this model to solve the Riemann problem with such shock initial data.

Existence of viscous profiles has been studied by various authors [3, 4, 6, 15]. Most works assume genuine nonlinearity or $D(U) = I$. An exception is a work by Mock [15] in which a broad class of viscosity matrices is identified that ensures existence of connecting orbits for compressive shock waves in the large. Conley and Smoller [3] presented an example of a constant, symmetric, positive definite matrix which is inadmissible for the p-system of the isentropic gas dynamics equations in the sense that for any Riemann initial data, standing wave solutions of the associated dynamical system cannot converge to any shock wave solution of that Riemann problem. This is somewhat in the lines of the present work. Namely, we note that the shallow water equations that we study in the paper at hand, are equivalent to the isentropic equations of gas dynamics with $\gamma = 2$, and so it is reasonable to expect the behavior presented in this work to occur in gas dynamics as well.

To understand what kind of a solution exists in the situations studied in this work, in Ref. [16] we studied solutions of the inviscid conservation law obtained as a singular limit of viscous regularizations. We showed that the solutions that exist in the situations when a weak self-similar solution does not exist, satisfy the system of conservation laws in a measure-valued sense, and exhibit continuously generated oscillations that increase in frequency, but stay uniformly bounded. Such solutions have already been observed, for example, in models for phase transitions, in particular, the formation of a band-like structure in a crystal of memory alloy during austenitic-martensitic transformation. In the form of a conservation law oscillatory solutions for phase transitions were studied in Ref. [7]. Oscillations of this type have also been observed by Majda and DiPerna in incompressible flow [5].

The main question is whether such solutions are physical or not, and whether a new model neeeds to be considered in those situations. We believe that the answer depends on the particular model at hand.

We summarize the main results and techniques of this work. The most important results of this paper are:
(1) the proof of nonexistence of a weak self-similar solution for an open set of model parameters, and
(2) the relationship to the presence of a nontrivial Majda-Pego region.

We present these results in the three sections that follow. We first describe the model equations and Riemann initial data in Section 2. In Section 2 we also talk about admissible waves that can be used in this model to construct a weak self-similar solution. In Section 3 we study the region of Majda-Pego unstable points. We show that a nontrivial Majda-Pego unstable region occurs for every symmetric, nondiagonal, positive definite matrix. At the boundary of the Majda-Pego region a Bogdanov-Takens bifurcation takes place. We calculate the universal unfolding of the Bogdanov-Takens bifurcation and show that there is a region in the two-dimensional parameter plane that consists of points for which the associated dynamical system has a limit cycle surrounding one of the critical points. The dynamical system is derived from the viscous profile entropy criterion. The Riemann initial data (U_L, U_R) that we consider in this paper correspond to the critical points that arise in the universal unfolding of a Bogdanov-Takens bifurcation and have the property that either U_L or U_R is surrounded by a limit cycle. In addition, U_L and U_R satisfy the Lax characterictic inequalities. The presence of a limit cycle prevents a shock wave from U_L to U_R from possessing a viscous profile. In Section 4 we show that for such initial data it is impossible to construct a Riemann solution using the waves at our disposal (the rarefaction waves and viscous profilable shock waves).

2 Description of the problem

2.1 The model equations

We study the shallow water equations

$$\begin{bmatrix} v \\ \phi \end{bmatrix}_t + \begin{bmatrix} v^2/2+\phi \\ v\phi \end{bmatrix}_x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{1}$$

where v is the velocity and ϕ is the depth of water times a constant gravitational acceleration. This is a strictly hyperbolic system of conservation laws for $\phi > 0$. The coincidence line, *i.e.*, the line along which the eigenvalues of the Jacobian of the flux function coincide, corresponds to the zero depth, $\phi = 0$.

We study Riemann solutions consisting of constant states, rarefaction waves and/or jump discontinuities satisfying the viscous profile entropy criterion, described below, with a positive definite viscosity matrix of the form

$$D = \begin{bmatrix} a & -b \\ -b & c \end{bmatrix}, \tag{2}$$

where $ac - b^2 > 0$. In Ref. [16] an application of the results of this paper is used with the parameters $a = 13.0116$, $b = 5.9144$ and $c = 3.1429$. We will keep these parameters in mind throughout the entire paper and present examples of our general calculations in terms of these values.

We consider Riemann data

$$U(x,0) = \begin{cases} U_L, & x < 0 \\ U_R, & x > 0 \end{cases} \tag{3}$$

with the property that U_L and U_R form a Lax-admissible shock wave [13] that does not admit a viscous profile due to the presence of a limit cycle in the associated dynamical system. We describe the admissibility criteria and the dynamical system below.

2.2 Admissibility of shock waves: viscous profile entropy criterion

We consider a shock wave *admissible* if it satisfies the *viscous profile entropy criterion* [9] with a positive definite viscosity matrix given in (2). A shock between U_- and U_+ traveling with speed s, (U_-, U_+, s), is considered admissible if it can be obtained in the limit, as $\epsilon \to 0$, of traveling wave solutions $\widetilde{U}(\xi) = \widetilde{U}((x-st)/\epsilon)$ of the parabolic system

$$U_t + F(U)_x = \epsilon(D(U)U_x)_x, \tag{4}$$

with the asymptotic states $\lim_{\xi\to\pm}\widetilde{U} = U_\pm$. Such a traveling wave corresponds to a connecting orbit in the dynamical system

$$U' = D^{-1}[-s(U-U_-) + F(U) - F(U_-)] \tag{5}$$

between the critical points U_- and U_+, traversed in the direction from U_- to U_+. A complete list of admissible shock waves that may comprise a weak Riemann solution is the following:

- a compressive repellor→ saddle shock wave, or a 1-shock wave (which corresponds to a connecting orbit from a repellor U_- to a saddle U_+, with the characteristic structure consisting of three converging characteristics and one outgoing characteristic),
- a compressive saddle→ attractor shock wave, or a 2-shock wave (which corresponds to a connecting orbit from a saddle U_- to an attractor U_+, with the characteristic structure consisting of three converging characteristics and one outgoing characteristic),
- a saddle→ saddle shock wave, or a transitional shock wave (which corresponds to a connecting orbit from a saddle U_- to a saddle U_+; thus the characteristic structure consists of two converging and two outgoing characteristics),
- a repellor→attractor shock wave, or an overcompressive shock wave (which corresponds to a connecting orbit from a repellor U_- to an attractor U_+; thus the characteristic structure consists of four converging characteristics).

We note that if D is different from a multiple of the identity, a critical point that is a repellor in the case when $D = I$ may become an attractor for $D \neq I$, and similarly, an attractor for $D = I$ may become a repellor for $D \neq I$. (Since D is positive definite, saddle points for $D = I$ stay saddle points for $D \neq I$.) One consequence of this property is that there are repellor to saddle shock waves and saddle to attractor shock waves whose characteristic structure consists of three outgoing and one incoming characteristic. These expansive or rarefaction-shock waves have been studied in Ref. [3]. Their presence is tied with the existence of a nontrivial "Majda-Pego-region of linearized instability", presented in Section 3.1. Based on the long term stability considerations we disregard expansive shock waves. This is because expansive shock waves were shown to be unstable under small perturbations as solutions of the parabolic PDE (4). See Ref. [17]. This is because the Majda-Pego instability occurs in the outgoing characteristic field and thus can never get swallowed by the shock.

In the following section we describe the Majda-Pego region, calculate the universal unfolding of a Bogdanov-Takens bifurcation occurring at the boundary of the Majda-Pego region, and discuss the Riemann initial data considered in this work.

3 Majda-Pego region and the Bogdanov-Takens unfolding

3.1 Majda-Pego region

In Ref. [14] Majda and Pego studied criteria for a viscosity matrix D to be strictly stable in the sense that for every state U, the Cauchy problem for the linearized equation

$$V_t + F'(U)V_x = \epsilon D(U)V_{xx}$$

is uniformly well-posed in L^2 as $\epsilon \to 0$. For a 2×2 system they showed that the stable viscosity matrix criterion is

$$l_j(U)D(U)r_j(U) \geq 0, \quad j = 1, 2,$$

where $l_j(U)$ and $r_j(U)$ are the left and the right eigenvectors of $F'(U)$. In Ref. [2] Čanić and Plohr viewed this condition as a condition on the state U being stable or not in the sense studied by Majda and Pego. Stable points are called the Majda-Pego points. The following result from Ref. [2] will be used below.

PROPOSITION 1 *A state U is a Majda-Pego point for the viscosity matrix D if and only if $tr[-s + F'(U)]$ and $tr\big(D^{-1}[-s + F'(U)]\big)$ have the same (nonzero) sign for each eigenvalue s of $F'(U)$. In particular, the boundary of the set of Majda-Pego points is contained in the union of the coincidence locus and the Bogdanov-Takens locus occurring for $U_L = U_R$.*

Using this proposition we calculate the Majda-Pego boundary for the model in consideration. For the first eigenvalue we have

$$[-\lambda_1(U) + F'(U)] = \begin{bmatrix} \sqrt{\phi} & 1 \\ \phi & \sqrt{\phi} \end{bmatrix}$$

and

$$D^{-1}[-\lambda_1(U) + F'(U)] = \frac{1}{ac - b^2} \begin{bmatrix} c & b \\ b & a \end{bmatrix} \begin{bmatrix} \sqrt{\phi} & 1 \\ \phi & \sqrt{\phi} \end{bmatrix},$$

and for the second

$$[-\lambda_2(U) + F'(U)] = \begin{bmatrix} -\sqrt{\phi} & 1 \\ \phi & -\sqrt{\phi} \end{bmatrix}$$

and

$$D^{-1}[-\lambda_2(U) + F'(U)] = \frac{1}{ac - b^2} \begin{bmatrix} c & b \\ b & a \end{bmatrix} \begin{bmatrix} -\sqrt{\phi} & 1 \\ \phi & -\sqrt{\phi} \end{bmatrix}.$$

For U to be a Majda-Pego point we must have that

$$d_{11} \equiv \frac{tr\left(D^{-1}[-\lambda_1(U) + F'(U)]\right)}{tr[-\lambda_1(U) + F'(U)]}$$

and

$$d_{22} \equiv \frac{tr\left(D^{-1}[-\lambda_2(U) + F'(U)]\right)}{tr[-\lambda_2(U) + F'(U)]}$$

have the same positive sign. In our case

$$d_{11} = \frac{(c+a)\sqrt{\phi} + b(1+\phi)}{2\sqrt{\phi}} \quad \text{and} \quad d_{22} = \frac{-(c+a)\sqrt{\phi} + b(1+\phi)}{-2\sqrt{\phi}}.$$

Theorem 1 *The region of Majda-Pego stable points consists of all points (v, ϕ) with $\phi > 0$ such that*

$$(c+a)\sqrt{\phi} + b(1+\phi) > 0 \tag{6}$$

and

$$-(c+a)\sqrt{\phi} + b(1+\phi) < 0. \tag{7}$$

The only situation in which the entire state space $\phi > 0$ is stable in the sense of Majda and Pego is when D is a diagonal, positive definite matrix. In all other cases, namely when $b \neq 0$, there is a nontrivial subregion of the state space consisting of the unstable Majda-Pego points.

Proof: From the definitions of d_{11} and d_{22} we see that in order for a point (v, ϕ) to be Majda-Pego stable we need that $(c+a)\sqrt{\phi} + b(1+\phi) > 0$ and $-(c+a)\sqrt{\phi} + b(1+\phi) < 0$.

If $b = 0$ and D is positive definite we have that both inequalities are satisfied, and so the entire state space is stable.

If $b \neq 0$ there are several cases to consider.

Case 1. Let $b > 0$. We have two subcases, depending on whether the discriminant $(a+c)^2 - 4b^2$ of the equations $(c+a)\sqrt{\phi} + b(1+\phi) = 0$ and $-(c+a)\sqrt{\phi} + b(1+\phi) = 0$ is positive or negative. Let $a + c > 2b$ (for example, D is diagonally dominant). Then the zeros of the first equation are given by

$$\sqrt{\phi} = (-(c+a) \pm \sqrt{(c+a)^2 - 4b^2})/(2b).$$

Since $a + c > 2b > 0$ the two zeros are both negative, and since $b > 0$, the corresponding parabola in $\sqrt{\phi}$ is positive for all $\phi > 0$. Thus, the first inequality (6) is always satisfied. The second inequality (7) is satisfied for all the points ϕ inside the interval (ϕ_1^*, ϕ_2^*) where $\phi_1^* = (c + a - \sqrt{(c+a)^2 - 4b^2})^2/4b^2$ and $\phi_2^* = (c + a + \sqrt{(c+a)^2 - 4b^2})^2/4b^2$. Since $a + c > 2b > 0$ both ϕ_1^* and ϕ_2^* are positive. Therefore, there is a nontrivial (unbounded) region of the state space containing unstable Majda-Pego points.

Now assume that $a + c < 2b$. Then the discriminant of both equations is negative and since $b > 0$, the first inequality is always satisfied but the second inequality is never true. Thus in this case the entire hyperbolic state space is Majda-Pego unstable.

CASE 2. Let $b < 0$. If $a + c > 2b$ the discriminant of both equations $(c + a)\sqrt{\phi} + b(1 + \phi) = 0$ and $-(c + a)\sqrt{\phi} + b(1 + \phi) = 0$ is positive, and since $b < 0$, we have that $(c + a)\sqrt{\phi} + b(1 + \phi) < 0$ for all $\phi > 0$ such that $\sqrt{\phi} > (-(c + a) + \sqrt{(c + a)^2 - 4b^2})/2b$. Thus, there is an unbounded region of state space that is Majda-Pego unstable. Similarly, if $a + c < 2b$ the discriminant of both equations is negative, and since $b < 0$, the first inequality (6) is never satisfied. Thus the entire state space is unstable in the sense of Majda and Pego. ■

Assume that $a, b, c > 0$ and that $a + c > 2b$. Then the region of Majda-Pego stable points is defined by

$$\frac{1}{2b}\left(c + a - \sqrt{(c + a)^2 - 4b^2}\right) \leq \sqrt{\phi} \leq \frac{1}{2b}\left(c + a + \sqrt{(c + a)^2 - 4b^2}\right). \tag{8}$$

Notice that both bounds are strictly greater than zero. Figure 1 shows the Majda-Pego stable region for the parameters $a = 13.0116$, $b = 5.9144$ and $c = 3.1429$. Following Proposition 1, at the boundary of the Majda-Pego region, a Bogdanov-Takens bifurcation takes place. In the following subsection we calculate the universal unfolding corresponding to the Bogdanov-Takens bifurcation taking place at the boundary where $\sqrt{\phi} = \sqrt{\phi^*} = \frac{1}{2b}\left(c + a - \sqrt{(c + a)^2 - 4b^2}\right)$.

3.2 THE UNFOLDING OF THE BOGDANOV-TAKENS BIFURCATION

To show the main result of this paper it is sufficient to calculate the locus of points corresponding to the Hopf bifurcation and the locus of points corresponding to the homoclinic loop bifurcation and show that there is a nontrivial region of points between these two loci whose dynamical systems contain a limit cycle surrounding one of the states U_L or U_R in the initial data. This will prevent the existence of a connecting orbit from U_L to U_R and thus make the initial shock wave inadmissible. In Section 4 we will show that it is impossible to use other waves to construct a solution between U_L and U_R for such initial data.

We study the dynamical system associated with the shallow water equations and a fixed viscosity matrix of the form (2)

$$\begin{bmatrix} \dot{v} \\ \dot{\phi} \end{bmatrix} = D^{-1} \begin{bmatrix} -s(v - v_-) + \phi - \phi_- + \frac{1}{2}(v + v_-)(v - v_-) \\ -s(\phi - \phi_-) + v(\phi - \phi_-) + \phi_-(v - v_-) \end{bmatrix}. \tag{9}$$

Let $x = v - v_-$ and $y = \phi - \phi_-$. The above system is equivalent to

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = D^{-1} \begin{bmatrix} -sx + y + \frac{1}{2}(v + v_-)x \\ -sy + vy + \phi_- x \end{bmatrix}.$$

Using $v - s = x - (s - \lambda_2) + v_- - \lambda_2$, and denoting

$$\mu = s - \lambda_2$$

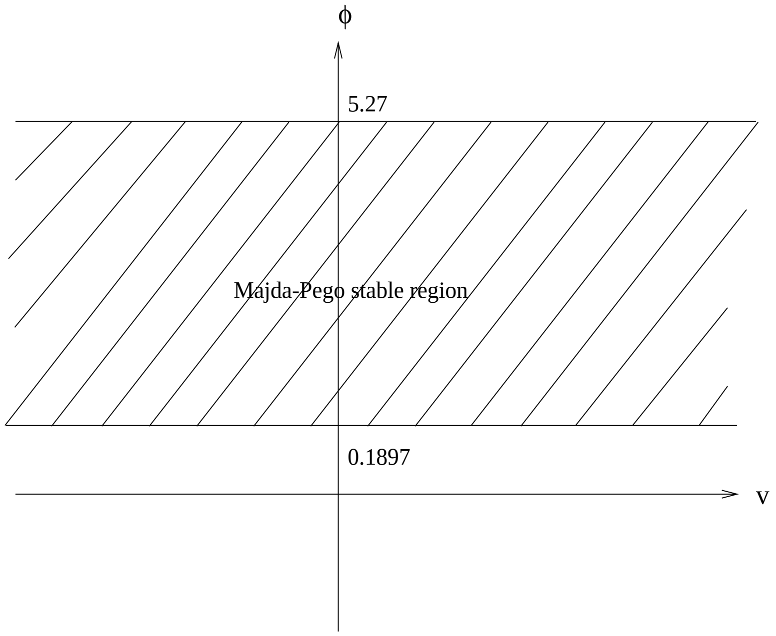

FIGURE 1: Majda-Pego stable region for the viscosity matrix (2) with $a = 13.0116$, $b = 5.9144$ and $c = 3.1429$. The figure shows that the region of stable points lies between $\phi = 0.1897$ and $\phi = 5.27$.

we obtain

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = D^{-1} \begin{bmatrix} (-\mu - \sqrt{\phi_-})x + y + \frac{1}{2}x^2 \\ \phi_- x + (-\mu - \sqrt{\phi_-})y + xy \end{bmatrix}.$$

Taking into account the form of the viscosity matrix we get

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \frac{1}{\det D} \begin{bmatrix} \left(c(-\mu - \sqrt{\phi_-}) + b\phi_-\right) x + \left(c + b(-\mu - \sqrt{\phi_-})\right) y \\ \left(b(-\mu - \sqrt{\phi_-}) + a\phi_-\right) x + \left(b + a(-\mu - \sqrt{\phi_-})\right) y \end{bmatrix} + Q(x,y), \tag{10}$$

where

$$Q(x,y) = \frac{1}{\det D} \begin{bmatrix} \frac{1}{2}cx^2 + bxy \\ \frac{1}{2}bx^2 + axy \end{bmatrix}.$$

This is a two-parameter family of dynamical systems in the parameters μ and ϕ_-. We will denote the vector field on the right-hand side by $X(x, y; \mu, \phi_-)$, and its Jacobian $X'(x, y; \mu, \phi_-)$. A simple calculation shows that there are generically three critical points of this vector field. They are given by $(0, 0)$, (x_1, y_1) and (x_2, y_2) where

$$\begin{aligned} x_i &= \frac{1}{2}\left(3(\mu + \sqrt{\phi_-}) \pm \sqrt{\mu^2 + 2\mu\sqrt{\phi_-} + 9\phi_-}\right), \quad i = 1, 2 \\ y_i &= \mu(\mu + 2\sqrt{\phi_-}) - \frac{5}{2}(\mu + \sqrt{\phi_-})x_i, \quad i = 1, 2. \end{aligned}$$

The critical point (x_1, y_1) has the property that it belongs to the elliptic region, namely, the value of y_1 is always less than $-\phi_-$ which means that the corresponding ϕ is negative. Thus there are only two critical points that belong to the hyperbolic region of the state space. This will be used in Section 4.

Vector field (10) undergoes a Bogdanov-Takens bifurcation at $x = 0, y = 0$ for the parameter values

$$\mu = 0, \phi_- = \phi_-^* = \left[\frac{1}{2b}\left(c + a - \sqrt{(c+a)^2 - 4b^2}\right)\right]^2. \tag{11}$$

Indeed, at $(0, \phi_-^*)$ the Jacobian $X'(0,0;0,\phi_-^*)$ is given by

$$X'(0,0;0,\phi_-^*) = \begin{bmatrix} -c\sqrt{\phi_-^*} + b\phi_-^* & c - b\sqrt{\phi_-^*} \\ -b\sqrt{\phi_-^*} + a\phi_-^* & b - a\sqrt{\phi_-^*} \end{bmatrix}.$$

This matrix is of the form

$$X'(0,0;0,\phi_-^*) = \begin{bmatrix} \alpha & \beta \\ -\alpha^2/\beta & -\alpha \end{bmatrix},$$

where

$$\alpha = \frac{1}{2b}\left(ac + a^2 - a\sqrt{(a+c)^2 - 4b^2}\right) - b, \text{ and} \tag{12}$$

$$\beta = \frac{1}{2}\left(c - a + \sqrt{(a+c)^2 - 4b^2}\right). \tag{13}$$

Its trace and the determinant are both equal to zero, which defines a Bogdanov-Takens bifurcation. If we employ a transformation by the nonsingular matrix

$$R = \begin{bmatrix} \alpha & 0 \\ -\alpha^2/\beta & 1 \end{bmatrix}$$

we obtain the Jordan form

$$R^{-1}X'(0,0;0,\phi_-^*)R = \begin{bmatrix} 0 & \frac{\beta}{\alpha} \\ 0 & 0 \end{bmatrix}.$$

For R to be nonsingular we need $\alpha \neq 0$ and for the existence of the Jordan form we need $\beta \neq 0$. The universal unfolding of a generic Bogdanov-Takens bifurcation contains the Hopf bifurcation locus and the homoclinic bifurcation locus [10] which we now calculate.

Hopf bifurcation locus

The Hopf locus consists of points (μ, ϕ_-) such that the corresponding dynamical system has the property that at a nonhyperbolic critical point U_{cr} the following holds

$$\text{discrm}\left\{D^{-1}\left[-s + F'(U_{cr})\right]\right\} < 0$$

and

$$\operatorname{tr}\left\{D^{-1}\left[-s+F'(U_{cr})\right]\right\}=0.$$

We are interested in the Hopf bifurcation taking place at $U_{cr}=U_-$ which corresponds to $x=y=0$. The trace of the Jacobian evaluated at zero is given by

$$Tr=\frac{1}{\det D}\left((a+c)(-\mu-\sqrt{\phi_-})+b(1+\phi_-)\right),$$

and the discriminant is equal to $Tr^2-4\ Det$ where the determinant Det of the Jacobian evaluated at zero is given by

$$Det=(ac-b^2)\mu(\mu+2\sqrt{\phi}).$$

Thus, the Hopf locus is defined by

$$(a+c)(-\mu-\sqrt{\phi_-})+b(1+\phi_-)=0 \tag{14}$$

in the region where the discriminant is negative.

For the parameters $a=13.0116$, $b=5.9144$ and $c=3.1429$ the Hopf locus is shown in Figure 2.

Homoclinic bifurcation locus

We use the Melnikov's integral analysis (see, for example, Ref. [10]), to calculate the locus tangent to the homoclinic bifurcation locus.

We first write the system (10) as a perturbation in μ and ϕ_- around the Bogdanov-Takens bifurcation point $\mu=0$, $\phi_-=\phi_-^*$ determined by the left Majda-Pego boundary given in (8).

We employ the transformation of coordinates

$$\begin{bmatrix} x \\ y \end{bmatrix}=R\begin{bmatrix} w \\ z \end{bmatrix}$$

and write system (10) as a perturbation around the Bogdanov-Takens point, using Taylor's expansion to approximate $\sqrt{\phi_-}$ around $\sqrt{\phi_-^*}$. We obtain the dynamical system of the form

$$\begin{bmatrix} \dot{w} \\ \dot{z} \end{bmatrix}=\frac{1}{\det D}\left(\begin{bmatrix} 0 & \frac{\beta}{\alpha} \\ 0 & 0 \end{bmatrix}+\mu\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}+\widetilde{\phi_-}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right)\begin{bmatrix} w \\ z \end{bmatrix}+\tilde{Q}(w,z), \tag{15}$$

where

$$\tilde{Q}(w,z)=\frac{1}{\det D}\begin{bmatrix} w(\alpha_1 w+\beta_1 z) \\ w(\alpha_2 w+\beta_2 z) \end{bmatrix},$$

and

$$\widetilde{\phi_-}=\phi_--\phi_-^*.$$

The coefficients a_{ij}, b_{ij}, α_i and β_i are given in the Appendix.

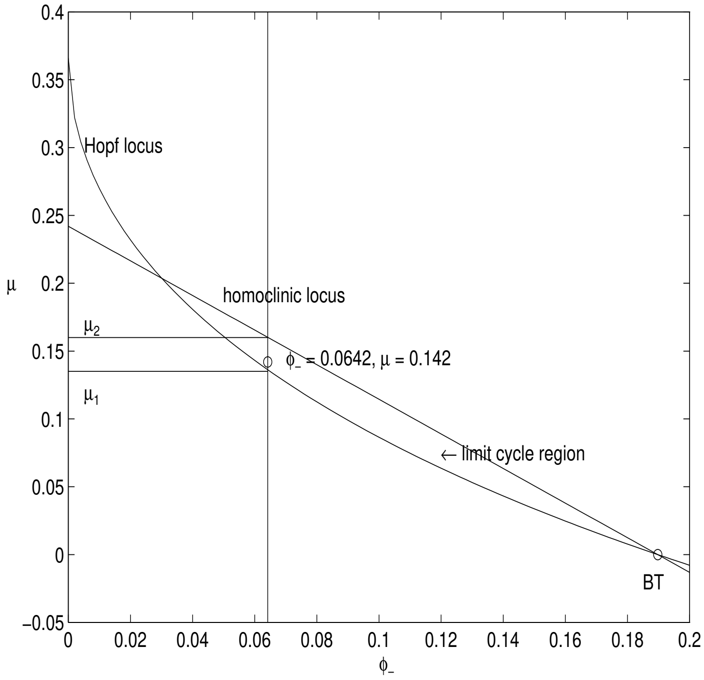

Figure 2: Universal unfolding of the Bogdanov-Takens bifurcation for the shallow water equations. The figure shows the Hopf and the homoclinic bifurcation loci in the unfolding of the Bogdanov-Takens bifurcation calculated for the viscosity parameters $a = 13.0116, b = 5.9114$ and $c = 3.1429$. The Bogdanov-Takens bifurcation, denoted by BT, occurs at the boundary of the Majda-Pego region determined by $\phi_-^* = 0.1897$. The region between the Hopf and the homoclinic locus contains the points for which the corresponding dynamical system has a limit cycle. One such point, $(\phi_-, \mu) = (0.0642, 0.142)$, is shown in this figure. The associated phase space portrait showing the limit cycle is depicted in Figure 3. The points on the vertical line $\phi_- = 0.0642$ define dynamical systems along the Hugoniot curves (2-branch) through (v_-, ϕ_-) where $\phi_- = 0.0642$. Figure 4 shows a portion of such a Hugoniot curve for $(v_-, \phi_-) = (v_R, \phi_R) = (-0.1840, 0.0642)$.

Next we introduce a small parameter ϵ and use it to rescale the dynamical system so that for $\epsilon = 0$ the dynamical system is Hamiltonian and has a homoclinic loop. The rescaling by ϵ "blows up" the Bogdanov-Takens critical point into a saddle and a spiral which is surrounded by a homoclinic loop to the saddle point obtained by the blow-up. The rescaling by ϵ is given by

$$w = \epsilon^2 \overline{w}, \quad z = \epsilon^3 \overline{z}, \quad \mu = \epsilon^2 \overline{\mu}, \quad \widetilde{\phi_-} = \epsilon^2 \overline{\widetilde{\phi_-}}, \quad t = \epsilon(\det D)\tau. \tag{16}$$

For simplicity, we will drop the bar over the new variables, and use the same notation further in the text. From this point on only the rescaled variables appear and define the tangent to the homoclinic bifurcation locus. After rescaling we obtain a system of the form

$$\begin{bmatrix} \dot{w} \\ \dot{z} \end{bmatrix} = f(w,z) + \epsilon g(w,z) + O(\epsilon^2),$$

where

$$f(w,z) = \begin{bmatrix} \frac{\beta}{\alpha} z \\ (\mu a_{21} + \widetilde{\phi_-} b_{21})w + \alpha_2 w^2 \end{bmatrix}$$

and

$$g(w,z) = \begin{bmatrix} (\mu a_{11} + \widetilde{\phi_-} b_{11})w + \alpha_1 w^2 \\ (\mu a_{22} + \widetilde{\phi_-} b_{22})z + \beta w z \end{bmatrix}.$$

The Hamiltonian function defined by f is given by

$$H(w,z;\mu,\widetilde{\phi_-}) \equiv \frac{z^2}{2} + \gamma_1 \frac{w^2}{2} + \gamma_2 w^3$$

where

$$\gamma_1 = \frac{\alpha}{\beta}\left(-a_{21}\mu - \widetilde{\phi_-} b_{21}\right), \quad \gamma_2 = \frac{\alpha}{\beta}\left(\frac{-\alpha_2}{3}\right).$$

The homoclinic loop corresponds to the level curve $H(w,z;\mu,\widetilde{\phi_-}) = 0$. The Melnikov function [10] is defined by

$$M(\mu,\widetilde{\phi_-}) \equiv \int_{intH_0} \mathrm{div} g(w,z;\mu,\widetilde{\phi_-}),$$

where

$$H_0 \equiv \{(w,z) | H(w,z;\mu,\widetilde{\phi_-}) = 0\}.$$

It measures the separation of the stable and unstable manifolds on a cross-section transverse to the homoclinic orbit. The curve $M(\mu,\widetilde{\phi_-}) = 0$ determines the homoclinic manifold up to $O(\epsilon^2)$ in the neighborhood of the Bogdanov-Takens bifurcation point. A calculation gives that the tangent locus is given by

$$7\gamma_2 \left[\mu(a_{11} + a_{22}) + \widetilde{\phi_-}(b_{11} + b_{22})\right] + 2\gamma_1(2\alpha_1 + \beta_2) = 0. \tag{17}$$

Figure 2 shows the homoclinic locus obtained for the parameters $a = 13.0116$, $b = 5.9144$ and $c = 3.1429$.

Thus, we have proved the following theorem.

THEOREM 2 *Suppose that the coefficients of the matrix D are such that $\alpha \neq 0$ and $\beta \neq 0$. Furthermore, assume that the Bogdanov-Takens point $(0, \phi_-^*)$ is non-degenerate in the sense that $\alpha_2 \neq 0$. Then the Hopf bifurcation locus and a tangent to the homoclinic bifurcation locus that arise in the universal unfolding*

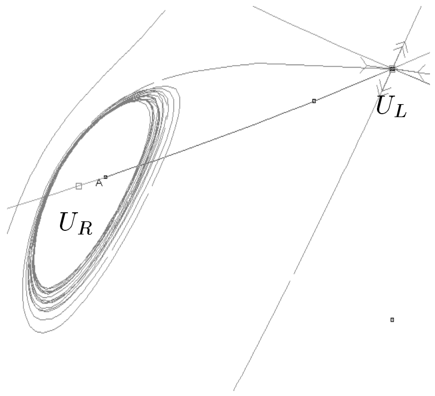

FIGURE 3: The figure shows a portion of the phase space portrait corresponding to $(\phi_-, \mu) = (0.0642, 0.142)$ from Figure 2. Shown is the state $U_L = (0, 0.12)$ which is a saddle point, and the state $U_R = U_- = (-0.1840, 0.0642)$ which is an attractor surrounded by a repelling limit cycle. Also shown in this figure is a part of the Hugoniot locus through U_R containing U_L. This figure was obtained using the Riemann solver package [11].

of the Bogdanov-Takens point are given, respectively, by the following:

$$(a+c)(-\mu - \sqrt{\phi_-}) + b(1+\phi_-) = 0, \tag{18}$$

$$7\gamma_2 \left[\mu(a_{11}+a_{22}) + (\phi_- - \phi_-^*)(b_{11}+b_{22})\right] + 2\gamma_1(2\alpha_1 + \beta_2) = 0 \tag{19}$$

in the region where the discriminant is negative.

The Bogdanov-Takens unfolding for the values $a = 13.0116$, $b = 5.9144$ and $c = 3.1429$ is shown in Figure 2. All the points that lie in the region between the Hopf and the homoclinic locus have the property that their corresponding dynamical systems have a limit cycle surrounding the critical point $U = U_-$. One such point is shown in Figure 2. It corresponds to the values $\mu = 0.142, \phi_- = 0.0647$. The corresponding dynamical system is shown in Figure 3.

In the next section we will show that it is impossible to construct a Riemann solution with Riemann data corresponding to the limit cycle region. More precisely, if U_L is a saddle point and $U_R = U_-$ is an attractor which has the property that there is a limit cycle surrounding the attractor, then this Riemann data corresponds to a Lax 2-shock wave that does not admit a viscous profile. We will show that it is impossible to use other waves to construct a solution of this class of Riemann problems. The same proof holds for the case when U_R is a saddle point and $U_L = U_-$ is a repellor surrounded by a limit cycle.

4 Nonexistence of weak solutions

In this section we discuss nonexistence of scale-invariant weak solutions consisting of constant states, rarefaction waves, and/or viscous admissible shock waves, for the Riemann problem described in the previous section.

The waves that could be used to construct a scale-invariant solution consisting of constant states, rarefaction waves and jump discontinuities are 1- and 2-rarefaction waves, viscous admissible compressive 1 and 2-shock waves, composite waves (adjoint rarefaction waves and shock waves of the same family), viscous admissible transitional shock waves, transitional rarefaction waves, and overcompressive shock waves. We will show that there cannot be transitional rarefaction waves, transitional shock waves, overcompressive waves or composite waves, which leaves only "classical" waves: compressive 1- and 2-shock waves, and 1- and 2-rarefaction waves.

Transitional rarefaction waves occur when a 2-rarefaction wave is joined on its right by a 1-rarefaction wave. See Ref. [12]. At the point of connection, the eigenvalues of $F'(U)$ must coincide. Since

$$F'(U) = \begin{bmatrix} v & 1 \\ \phi & v \end{bmatrix}$$

the eigenvalues of $F'(U)$ are $\lambda_1 = v - \sqrt{\phi}$ and $\lambda_2 = v + \sqrt{\phi}$. Note that the strictly hyperbolic region is the open half-plane where $\phi > 0$, and the eigenvalues coincide only at $\phi = 0$. Hence for a transitional rarefaction wave to occur, λ_2 must be increasing as it approaches $\phi = 0$. However, $\nabla\lambda_2 \cdot \vec{n} > 0$, where $\vec{n} = [0, 1]$ is the normal to the boundary of the elliptic region. So λ_2 is decreasing as it approaches the line of coincidence. Thus we have the following proposition.

Proposition 2 *Transitional rarefaction waves do not occur in this model.*

The following lemma shows that neither transitional shock waves nor overcompressive shock waves occur in our model. Recall that in Section 3 we showed that the dynamical system associated with the viscous profile entropy criterion has at most two critical points that lie in the strictly hyperbolic region of the state space, $\phi > 0$. We will show that for any positive definite viscosity matrix D one of these two critical points is always a saddle and the other an anti-saddle (a node or a spiral). For this we need the Rankine-Hugoniot conditions, which are

$$\begin{aligned} -s(v - v_-) + v^2/2 + \phi - v_-^2/2 - \phi_- &= 0, \text{ and} \qquad (20)\\ -s(\phi - \phi_-) + v\phi - v_-\phi_- &= 0. \qquad (21) \end{aligned}$$

Along the Hugoniot curve through U_- that is tangent to the first eigenvector $r_1(U)$ of $F'(U)$ we have that

$$v = v_- - \frac{\phi - \phi_-}{\sqrt{\phi}}, \quad \text{and} \quad s = v_- - \frac{\phi}{\sqrt{\phi}}, \qquad (22)$$

where $\overline{\phi} = \frac{1}{2}(\phi + \phi_-)$. Similarly, along the Hugoniot curve tangent to $r_2(U)$ we have

$$v = v_- + \frac{\phi - \phi_-}{\sqrt{\overline{\phi}}}, \quad \text{and} \quad s = v_- + \frac{\phi}{\sqrt{\overline{\phi}}}. \tag{23}$$

We will use these to show the following

LEMMA 1 *For any D positive definite, it is impossible to have either two saddle points or two anti-saddle points (nodes or spirals) as critical points in the dynamical system* (10) *that lie in the hyperbolic state space determined by $\phi > 0$.*

PROOF: Let $X(U; s, U_-) = D^{-1}\{-s(U - U_-) + F(U) - F(U_-)\}$. If U_{cr} is a critical point that is a saddle we have that at U_{cr} $\det(X'(U_{cr}; s, U_-)) < 0$. Similarly, at an anti-saddle point U_{cr}, $\det(X'(U_{cr}; s, U_-)) > 0$. Since D is positive definite we have that the sign of $\det(X') = \text{sign}((v-s)^2 - \phi)$. We will show that, given $U_- = (v_-, \phi_-)$, there does not exist a (v, ϕ) satisfying the Rankine-Hugoniot equations, such that $\left((v_- - s)^2 - \phi_-\right)\left((v-s)^2 - \phi\right) > 0$. This will imply that one of the critical points has to be a saddle and the other an anti-saddle.

Using equations (22) and (23) we obtain that

$$(v - s)^2 = \frac{2{\phi_-}^2}{\phi + \phi_-} \quad \text{and} \quad (v_- - s)^2 = \frac{2\phi^2}{\phi + \phi_-}.$$

Assume that $(v - s)^2 - \phi < 0$ and $(v_- - s)^2 - \phi_- < 0$. This corresponds to having saddle points at (v_-, ϕ_-) and (v, ϕ). Taking into account that ϕ and ϕ_- are positive we obtain that

$$\begin{aligned} (v-s)^2 - \phi \quad &< \quad 0 \iff 2{\phi_-}^2 - \phi\phi_- - \phi^2 < 0, \text{ and} \\ (v_- - s)^2 - \phi_- \quad &< \quad 0 \iff 2\phi^2 - \phi_-\phi - {\phi_-}^2 < 0. \end{aligned}$$

The first inequlity is true if $\phi > \phi_- > 0$, and the second if $\phi_- > \phi > 0$ which leads to a contradiction. Similarly, if we assume that $(v - s)^2 - \phi > 0$ and $(v_- - s)^2 - \phi_- > 0$ we obtain that the first inequality holds if $\phi_- > \phi > 0$ and the second if $\phi > \phi_- > 0$, which is a contradiction. ■

Thus, we have shown that neither transitional nor overcompressive shock waves occur in this model.

To study solutions consisting of classical waves we use wave curve analysis. For this purpose we define the forward and the backward i-wave curves. We define an *i-wave curve* through a fixed state U_0 to be the set of all states U which can be connected to U_0 by a wave of the i-th characteristic family (an i-rarefaction, an i-shock, or an i-composite wave). If U_0 is on the left of the i-wave and U is on the right, the i-wave curve is called the *forward i-wave curve*. Otherwise, we call it the *backward i-wave curve*. To construct a Riemann solution between two states U_L and U_R consisting of only 1- and 2- waves, we find the forward 1-wave curve through U_L and the backward 2-wave curve through U_R. If the two curves intersect at some point U_M, the Riemann solution consists of two waves: a 1-wave from U_L to the intermediate state U_M and a 2-wave from U_M to U_R.

In our model a simple calculation shows that the forward 1-wave curve through a state $U_0 = (v_0, \phi_0)$ consists of the 1-rarefaction part

$$v = v_0 - 2\sqrt{\phi} + 2\sqrt{\phi} \quad \text{for} \quad \phi < \phi_0,$$

and the 1-shock part for $\phi > \phi_0$. The 1-shock part consists of the points $U = (v, \phi)$ that belong to the Hugoniot curve (22) through U_0 and admit a viscous profile from the repellor U_0 to a saddle U. Similarly, a backward 2-wave curve through $U_0 = (v_0, \phi_0)$ consists of the 2-rarefaction part

$$v = v_0 + 2\sqrt{\phi} - 2\sqrt{\phi} \quad \text{for} \quad \phi < \phi_0,$$

and the 2-shock part for $\phi > \phi_0$. The 2-shock part consists of the points $U = (v, \phi)$ that belong to the Hugoniot curve (23) through U_0 and admit a viscous profile from a saddle U to the attractor U_0. This structure of the wave curves is global. In partucular, note that the wave curves do not have any composite wave parts. If they did, the composite segment would have to start at the inflection point (see, for example Ref. [8]). At this point $\nabla\lambda_i(U) \cdot \vec{r_i}(U) = 0$, where the $\vec{r_i}$'s are the eigenvectors of $F'(U)$. Since $\vec{r_1} = [1, -\sqrt{\phi}]^T$, $\vec{r_2} = [1, \sqrt{\phi}]^T$, $\lambda_1 = v - \sqrt{\phi}$, and $\lambda_2 = v - \sqrt{\phi}$, we have $\nabla\lambda_i \cdot \vec{r_i} = 3/2$. Therefore, $\nabla\lambda_i \cdot \vec{r_i} \neq 0$. Thus there are no composite waves with a shock on the right.

We also cannot have composite waves with a shock on the left, called composite waves sonic on the left. Composite waves sonic on the left occur only when the i-shock speed, s, equals λ_i at some point on the i-shock curve. Consider, for example, the shock branch of the 1-wave curve through U_0

$$v = v_0 - \frac{\phi - \phi_0}{\sqrt{\phi}} \tag{24}$$

with $\phi > \phi_0$. Using s from the Rankine-Hugoniot equation (21) and $\lambda_1 = v - \sqrt{\phi}$, we see that $s = \lambda_1$ only when

$$v = v_0 + \frac{\sqrt{\phi}(\phi - \phi_0)}{\phi_0}. \tag{25}$$

Obviously, the curves in equations (24) and (25) do not intersect for $\phi > \phi_0$ and so there are no composite waves sonic on the left.

The only waves left to construct a standard Riemann solution are the "classical" compressive 1- and 2-shock waves and 1- and 2-rarefaction waves. To do this, we construct the forward 1-wave curve through U_L and the backward 2-wave curve through U_R. Recall that the initial data that we consider in this paper are the shock initial data, *i.e.*, U_L and U_R satisfy the Rankine-Hugoniot conditions (and the Lax characteristic entropy criterion) but there is no viscous profile from U_L to U_R due to the presence of a limit cycle surrounding either U_L or U_R.

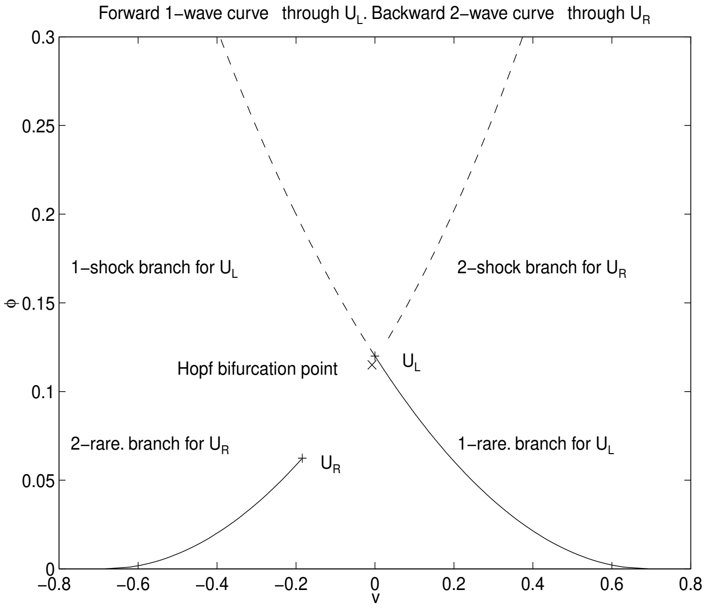

Figure 4: The forward 1-wave curve through U_L and the backward 2-wave curve through U_R. The figure shows that these two curves do not intersect. The Hugoiot curve through U_R contains U_L. However, this part of the Hugoniot curve does not belong to the backward 2-wave curve of U_R because there is no viscous profile from U_L to U_R due to a limit cycle that was born in a nearby Hopf bifurcation point and dies at a homoclinic loop bifurcation point. The Hopf bifurcation point is also shown in this figure. The corresponding dynamical systems are determined by the parameters $\phi_- = 0.0642$ and $\mu_1 \leq \mu \leq \mu_2$, shown in Figure 2. The dynamical system having U_L and U_R as critical points is determined by $\phi_- = 0.642, \mu = 0.142$. The segment of the Hugoniot curve through U_R between U_R and the Hopf bifurcation point is inadmissible because it contains the points U_* for which U_* is a saddle point and U_R a repellor. The corresponding dynamical systems are determined by $\phi_- = 0.0642$ and $0 \leq \mu \leq \mu_1$, see Figure 2.

To fix ideas, assume that (U_L, U_R) correspond to a Lax 2-shock. U_L is a saddle point and U_R an attractor surrounded by a limit cycle. The associated dynamical system is determined by the parameters (ϕ_-, μ) that belong to the limit cycle region in the universal unfolding of a Bogdanov-Takens bifurcation. See Figure 2. For this kind of initial data U_R lies on the 2-shock branch of the Hugoniot curve through U_L, or, equivalently, U_L lies on the backward 2-shock branch through U_R. A typical situation of the structure of the Hugoniot curves is depicted in Figure 4. This figure shows a forward 1-wave curve through U_L and a backward 2-wave curve through U_R. A point of intersection of these two curves would give a solution. However, the gap in the backward 2-wave curve, due to viscous inadmissibility, keeps the curves from crossing. The part of the shock curve between the Hopf bifurcation point and the homoclinic bifurcation point, *which includes* U_L, is inadmissible due to a limit cycle in the dynamical system keeping the saddle point from connecting to the attractor at U_R. (The rest of the inadmissible points U_* fail due to the dynamical system having a *repellor* at U_R (c.f. Figure 2), which makes an admissible 2-shock from U_* to U_R impossible.)

The 1-wave curve through U_L and the backward 2-wave curve through U_R never intersect because the 1-shock branch through U_L has the property that

$$\frac{dv}{d\phi} = -\frac{\sqrt{2}/2\phi + 3\sqrt{2}/2\phi_L}{(\phi + \phi_L)^{3/2}}$$

which is negative for all ϕ, and the backward 2-shock branch through U_R satisfies

$$\frac{dv}{d\phi} = \frac{\sqrt{2}/2\phi + 3\sqrt{2}/2\phi_R}{(\phi + \phi_R)^{3/2}}$$

which is always positive. Hence these curves never meet. They also will not intersect the 1- or 2-rarefaction branches given by

$$v = v_L - 2\sqrt{\phi} + 2\sqrt{\phi_L} \text{ and } v = v_R + 2\sqrt{\phi} - 2\sqrt{\phi_R}$$

respectively.

Note that we could repeat the same reasoning for the situation when U_L and U_R correspond to a viscous inadmissible Lax 1-shock wave. The same conclusion holds. Thus we have proved the following theorem.

THEOREM 3 *Let Riemann initial data* (3) *be such that the left state U_L and the right state U_R form a Lax admissible shock wave which does not satisfy the viscous profile entropy criterion due to the presence of a limit cycle surrounding either U_L or U_R. Then the Riemann problem for the shallow water equations* (1) *does not have a scale-invariant solution in which all shock waves admit viscous profiles with the viscosity matrix given by* (2).

Note that we have proved that whenever there is a gap in the wave curves due to shock inadmissibility, there will be nonexistence of solutions in which all shock waves comply with the viscous profile entropy criterion. In Section 3 we have shown that such gaps in the wave curves occur naturally if a nondiagonal, positive definite, symmetric viscosity matrix is considered. A study of what kind of solution exists in those situations is presented in Ref. [16]. The results of that work show that measure-valued solutions exhibiting continuously generated oscillations take place.

APPENDIX

The coefficients a_{ij}, b_{ij}, α_i and β_i that are used in Section 3 and first appear in equation (15) are given by the following (here α and β are given by (12) and (13) and ϕ_-^* by (11)):

$$
\begin{aligned}
a_{11} &= -c + \frac{\alpha}{\beta} b, \\
a_{12} &= -\frac{b}{\alpha}, \\
a_{21} &= \frac{\alpha}{\beta^2}\left(-\beta\alpha c - b\beta^2 + \alpha^2 b + a\alpha\beta\right) \\
a_{22} &= -\frac{\alpha b + a\beta}{\beta} \\
b_{11} &= \frac{1}{2\sqrt{\phi_-^*}} a_{11} + b, \\
b_{12} &= \frac{1}{2\sqrt{\phi_-^*}} a_{12}, \\
b_{21} &= \frac{1}{2\sqrt{\phi_-^*}} a_{21} + \frac{\alpha}{\beta}(\alpha b + a\beta), \\
b_{22} &= \frac{1}{2\sqrt{\phi_-^*}} a_{22}, \\
\alpha_1 &= \frac{c\alpha\beta - 2\alpha^2 b}{2\beta}, \\
\alpha_2 &= \frac{\alpha}{2\beta^2}(c\beta\alpha^2 - 2b\alpha^3 + \beta^2\alpha b - 2a\alpha^2\beta), \\
\beta_1 &= b, \\
\beta_2 &= \frac{\alpha}{2\beta^2}(2\alpha\beta^2 + 2\alpha\beta b).
\end{aligned}
$$

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