MATH 1310

Session 2

MATH 1310

Solving Linear Equations

Definition: To solve an equation in the variable x using the **algebraic method** is to use the rules of algebra to isolate the unknown x on one side of the equation.

Definition: To solve an equation in the variable x using the **graphical method** is to move all terms to one side of the equation and set those terms equal to y. Sketch the graph to find the values of x where $y = 0$.

Solving Linear Equations

Example 1: Solve the following equation **algebraically**.

$5y + 6 = -18 \cancel{-} y$

$$\begin{aligned} +y \quad & \quad +y \\ 6y + \cancel{6} &= -18 \\ -\cancel{6} \quad & \quad -6 \\ \frac{\cancel{6}y}{\cancel{6}} &= \frac{-24}{6} \\ y &= -4 \end{aligned}$$

Check:

$5(-4) + 6 = -18 - (-4)$

$-20 + 6 = -18 + 4$

$-14 = -14$ ✓

→ Move all variables to one side of equation (opposite operation)

→ Move all constant away from the side with the variable

→ Multiply or divide by the constant with ~~th~~ the variable

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Example 2: Solve following equation **algebraically**.

$7 + 2(3 - 8x) = 4 - 6(1 + 5x)$

$7 + 6 - 16x = 4 - 6 - 30x$

$$\begin{aligned} 13 - 16x &= -2 - \cancel{30x} \\ +30x \quad & \quad\;\; \cancel{+30x} \end{aligned}$$

$$\begin{aligned} \cancel{13} + 14x &= -2 \\ -\cancel{13} \quad\quad & \quad -13 \end{aligned}$$

$$\frac{\cancel{14}x}{\cancel{14}} = \frac{-15}{14}$$

$$x = {}^{-15}/_{14}$$

→ Distribute on each side separately

→ Combine like terms on each side

→ Follow previous steps

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Check

$7 + 2(3 - 8x) = 4 - 6(1 + 5x)$

$x = {}^{-15}/_{14}$

$7 + 2\left(3 - \overset{4}{\cancel{8}} \cdot \frac{-15}{\cancel{14}\,7}\right) = 4 - 6\left(1 + 5 \cdot \frac{-15}{14}\right)$

$7 + 2\left(3 + \frac{60}{7}\right) = 4 - 6\left(1 - \frac{75}{14}\right)$

$7 + 2\left(\frac{21}{7} + \frac{60}{7}\right) = 4 - 6\left(\frac{14}{14} - \frac{75}{14}\right)$

$7 + 2\left(\frac{81}{7}\right) = 4 - \overset{3}{\cancel{6}}\left(\frac{-61}{\cancel{14}\,7}\right)$

$7 + \frac{162}{7} = 4 + \frac{183}{7}$

$\frac{49}{7} + \frac{162}{7} = \frac{28}{7} + \frac{183}{7}$

$\frac{211}{7} = \frac{211}{7}$ ✓

Example 3: Solve following equation **algebraically**.

$$\frac{2}{5}\frac{(x-1)}{1}=\frac{7}{3}$$

$$\frac{2x-2}{5} = \frac{7}{3}$$

$$3(2x-2) = 5(7)$$

$$6x - 6 = 35$$

$$\cancel{+6} \quad +6$$

$$\frac{\cancel{6}x}{\cancel{6}} = \frac{41}{6}$$

$$\boxed{x = 41/6}$$

Check

$$\frac{2}{5}\left(\frac{41}{6} - 1\right) = \frac{7}{3}$$

$$\frac{2}{5}\left(\frac{41}{6} - \frac{6}{6}\right) = \frac{7}{3}$$

$$\frac{\cancel{2}^{1}}{\cancel{5}_{1}}\left(\frac{\cancel{35}\,7}{\cancel{6}\,3}\right) = \frac{7}{3}$$

$$\frac{7}{3} = \frac{7}{3} \checkmark$$

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Example 4: Solve following equation **algebraically**. Method1

$$\left(\frac{3}{3}\right) -\frac{3}{8x}+\frac{1}{12x}\left(\frac{2}{2}\right) = 2$$

→ Common denominator

→ combine

→ cross multiply

$$\frac{-9}{24x}+\frac{2}{24x}=2$$

$$\frac{-7}{24x}=\frac{2}{1}$$

$$1(-7)=24x(2)$$

$$\frac{-7}{48}=\frac{\cancel{48}x}{\cancel{48}}$$

$$\frac{-7}{48}=x$$

Method2

→ Multiply both sides by bigger denominator

→ simplify

→ repeat

$$-\frac{3}{8x}+\frac{1}{12x}=2$$

$$12x\left(\frac{-3}{8x}\right)+12x\left(\frac{1}{12x}\right)=12x\cdot 2$$

$$\frac{\cancel{-36x}^{\,-9}}{\cancel{8x}_{\,2}}+\frac{\cancel{12x}}{\cancel{12x}}=24x$$

$$\cancel{2}\cdot\frac{-9}{\cancel{2}}+2\cdot 1=24x\cdot 2$$

$$-9+2=48x$$

$$\frac{-7}{48}=\frac{\cancel{48}x}{\cancel{48}}$$

$$x=\frac{-7}{48}$$

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(y=0) (x=0)

Example 5: Find the x-intercept and y-intercept of the following equation. Express the answers in coordinate point form.

a. $-7x + 8y - 63 = 0$

x-int:

$-7x + 8(0) - 63 = 0$

$-7x + 0 - 63 = 0$

$-7x - 63 = 0$

$\frac{-7x}{-7} = \frac{63}{-7}$

$x = -9$

$(-9, 0)$

y-int:

$-7(0) + 8y - 63 = 0$

$0 + 8y \cancel{-63} = 0$

$\cancel{+63} \quad +63$

$\frac{8y}{8} = \frac{63}{8}, \; y = \frac{63}{8}$

$(0, \frac{63}{8})$

b. $x^2 - y - 16 = 0$

x-int:

$x^2 - 16 = 0$

$x^2 = 16$

$x = \pm\sqrt{16} = \pm 4$

$(\pm 4, 0)$

or

$(-4, 0), (4, 0)$

y-int:

$-y \cancel{-16} = 0$

$\cancel{+16} \quad +16$

$\frac{-y}{-1} = \frac{+16}{-1}$

$y = -16$

$(0, -16)$

c. $4x^2 - y^2 - 81 = 0$

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c. $4x^2 - y^2 - 81 = 0$

x-int:

$4x^2 - 81 = 0$

$4x^2 = 81$

$x^2 = \frac{81}{4}$

$x = \pm\sqrt{\frac{81}{4}}$

$x = \pm\frac{9}{2}$ $\left(\pm\frac{9}{2}, 0\right)$

Y-int:

$-y^2 - 81 = 0$

$-y^2 = 81$

$y^2 = -81$

$y = \pm\sqrt{-81}$

No Y-intercept

Applications

Using Modeling to Solve Problems
Step 1: Define variables.
Step 2: Express each unknown quantity in terms of one variable.
Step 3: Write the equation in one variable which models the situation given.
Step 4: Solve the equation.
Step 5: Answer the question(s) posed, including appropriate units

Turning a presented scenario into
one or several equations

Example 1: Find three consecutive even integers whose sum is 222.

"is" $\rightarrow$ $=$

First integer: $x = 72$

Second integer: $x+2 = 74$

third integer: $x+4 = 76$

$$(x) + (x+2) + (x+4) = 222$$

$$3x + \cancel{6} = 222$$

$$\underline{-\cancel{6} \quad -6}$$

$$\frac{\cancel{3}x}{\cancel{3}} = \frac{216}{3}$$

$$x = 72$$

$$\boxed{72,\ 74,\ 76}$$

Example 2: If the first and third of three consecutive odd integers are added, the result is 63 less than five times the second integer. Find the third integer.

First integer: $x = 19$

second integer: $x+2 = 21$

third integer: $x+4 = 23$

$$(x) + (x+4) = 5(x+2) - 63$$

$$2x+4 = 5x+10-63$$

$$\cancel{2x}+4 = 5x-53$$

$$-\cancel{2x} \qquad -2x$$

$$4 = 3x - \cancel{53}$$

$$+53 \qquad +\cancel{53}$$

$$\frac{57}{3} = \frac{\cancel{3}x}{\cancel{3}}$$

$$19 = x$$

$$\boxed{23}$$

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Example 3: The length of a rectangular garden is 20 feet more than the width. The perimeter is 140 feet. What are the dimensions of the garden?

length $= \ell = w + 20$

width $= w$

$\boxed{w = 25\text{ ft}}$

$\ell = w + 20$

$\boxed{\ell = 45\text{ ft}}$

$P = 140$

$2\ell + 2w = 140$

$2(w+20) + 2w = 140$

$2w + 40 + 2w = 140$

$4w + \cancel{40} = 140$

$\quad\ \ \cancel{-40} \quad -40$

$\frac{\cancel{4}w}{\cancel{4}} = \frac{100}{4}$

$w = 25$

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Example 4: When the sides of a square are each increased by 2cm, the area increases by 14 cm^2. Find the length of a side in the original square. $A = s^2$

original side: x

increased side: $x+2$

original area: x^2

increased area: $(x+2)^2$

: x^2+14

$$(x+2)^2 = x^2+14$$

$$(x+2)(x+2) = x^2+14$$

$$x^2+2x+2x+4 = x^2+14$$

$$\cancel{x^2}+4x+4 = \cancel{x^2}+14$$

$$-\cancel{x^2} \qquad -\cancel{x^2}$$

$$4x+4 = 14$$

$$4x+\cancel{4} = 14$$

$$-\cancel{4} \qquad -4$$

$$\frac{\cancel{4}x}{\cancel{4}} = \frac{10}{4}$$

$x = 5/2$ or 2.5

$\boxed{2.5\text{cm}}$

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Example 5: Two sides of a triangle have the same length. The third side is 15 cm longer than each of the equal sides. The perimeter is no less than 90 cm. What are the smallest possible lengths of the sides of the triangle.

no less than = more than or equal to

smallest possible → equal

(equal)
congruent sides: $x = 25$

Non-congruent side: $x + 15 = 40$
(unequal)

$P = 90$

$S_1 + S_2 + S_3 = 90$

$x + x + (x + 15) = 90$

$3x + 15 = 90$

$$3x + \cancel{15} = 90$$
$$\underline{\cancel{-15} \quad -15}$$
$$\frac{\cancel{3}x}{\cancel{3}} = \frac{75}{3}$$
$$x = 25$$

$$\boxed{\begin{array}{l} 25\text{cm} \\ 25\text{cm} \\ 40\text{cm} \end{array}}$$

Solving 2 x 2 Linear Systems

2 unknown variables
2 equations

To solve a system of two linear equations

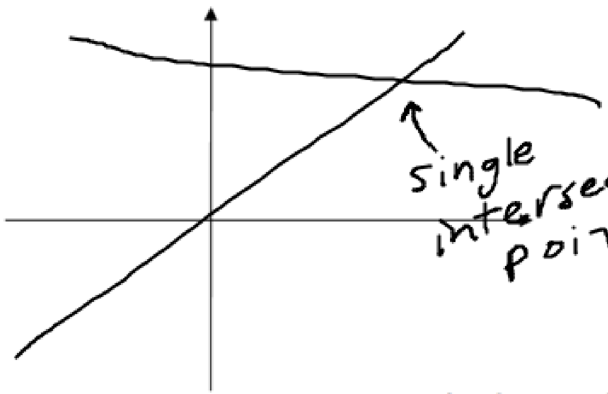
$$\begin{Bmatrix} ax + by = c \\ dx + ey = f \end{Bmatrix}$$

means to find values for x and y that satisfy both equations.
The system will have exactly one solution, no solution, or infinitely many solutions.

1 2 3

Solving 2 x 2 Linear Systems

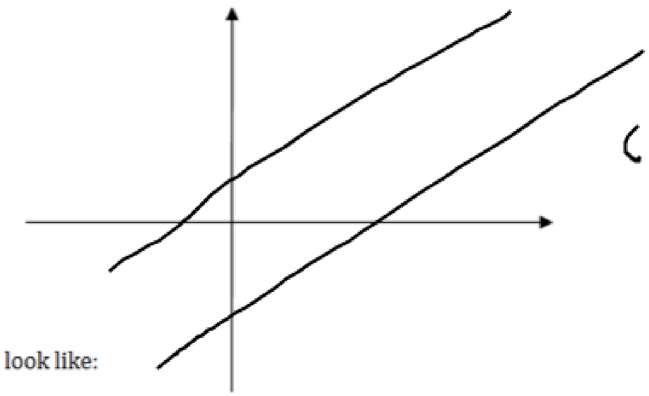
1. Exactly one solution, will look like:



single intersection point

2. No solution, will look like:

2 parallel lines



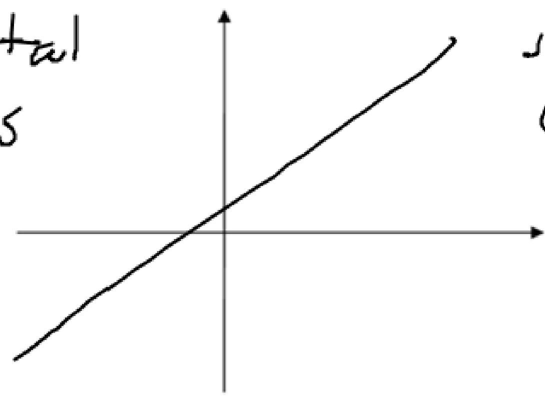
(No inters. point)

3. Infinitely many solutions, will look like:

coincidental equations

$5x+3y=15$

$\frac{x}{3}+\frac{y}{5}=1$



single line (with two equations)

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Example 1: Solve the following systems of linear equations by the substitution method.

$2x - y = 5$
$5x + 2y = 8$

$\rightarrow$ $\cancel{2x} - y = 5$
$\underline{-\cancel{2x} \qquad -2x}$

$\dfrac{-y}{-1} = \dfrac{-2x}{-1} + \dfrac{5}{-1}$

$y = 2x - 5$

$5x + 2y = 8$

$5x + 2(2x - 5) = 8$

$5x + 4x - 10 = 8$

$9x - 10 = 8$

$9x = 18, \; x = 2$

$2x - y = 5$

$2(2) - y = 5$

$4 - y = 5$

$-y = 1$

$y = -1$

$\boxed{(2, -1)}$

→ Solve one equation for one of the variables (pick the easiest option)

→ substitute that answer for the variable in the other equation

→ Solve the remaing equation

→ Back-substitute and solve.

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Example 2 : Solve the following systems of linear equations by the substitution method

$x - 2y = 3$
$2x - 4y = 7$

$$x - \cancel{2y} = 3$$
$$\underline{\;+\cancel{2y} \quad +2y\;}$$
$$x = 2y + 3$$

$$2x - 4y = 7$$
$$2(2y+3) - 4y = 7$$
$$\cancel{4y} + 6 - \cancel{4y} = 7$$
$$6 \neq 7$$

Not True

No Solution
or
$\{\ \}$: Empty Set
$\emptyset$: Null Set

(x, equation)
(equation, y)

If all the variables cancel out:
- You are left with a true statement: Infinitely Many Solutions
- You are left with a false statement: No solutions

Example 3: Solve the following systems by the Elimination Method.

$10(2x + 3y = -16)$
$3(5x - 10y = 30)$

$\rightarrow$

$$\begin{aligned} 20x + \cancel{30y} &= -160 \\ 15x - \cancel{30y} &= 90 \\ \hline \frac{\cancel{35}x}{\cancel{35}} &= \frac{-70}{35} \end{aligned}$$

$$x = -2$$

$$2x + 3y = -16$$

$$2(-2) + 3y = -16$$

$$\cancel{-4} + 3y = -16$$
$$\cancel{+4} \qquad +4$$

$$3y = -12$$

$$y = -4$$

$$\boxed{(-2, -4)}$$

→ Multiply 1 or both equations to make one set of coefficients opposite

→ Add the equations together

→ Solve for the remaining variable

→ Back-substitute and solve

Check

$5(-2) - 10(-4) = 30$

$-10 + 40 = 30$

$30 = 30$ ✓

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Popper 2:

Example 4: Solve the following system

$$x + 4y = 10$$
$$\frac{1}{2}x + 2y = 5$$

Question 1: Rewrite the system:

a. $\begin{array}{c} x + 4y = 10 \\ -x - 4y = -10 \end{array}$

b. $\begin{array}{c} \frac{1}{2}x + 2y = 5 \\ \frac{-1}{2}x - 4y = -5 \end{array}$

c. $\begin{array}{c} y = \frac{1}{4}x + \frac{5}{2} \\ y = \frac{1}{4}x + \frac{5}{2} \end{array}$

d. Any of these choices are correct

Question 2: Combine the equations:

a. $0 = 0$ b. $x = 20$ c. $y = \frac{5}{8}x$ d. None of the above.

Question 3: How many answers are possible:

a. No Possible Solutions b. Infinitely Many Solutions c. One Solution d. Cannot be determined.

Question 4: Give the answer:

a. { } b. All points of the form: (x, $\frac{1}{4}x + \frac{5}{2}$) c. (20, 5) d. (0, 0}

Quadratic Equations

In this section, you'll learn three methods for solving quadratic equations. A quadratic equation is an equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$. Equations of this form can have two solutions, one solution or no solutions. The three methods are:

1. Factoring
2. Completing the Square
3. Quadratic Formula

Quadratic Equations

Factoring

This method makes use of the Zero Product Property, that is, if $ab = 0$, then either $a = 0$ or $b = 0$. The equation must be written in the form $ax^2 + bx + c = 0$, so you may have to rewrite the equation in this form before you get started. Next you'll factor the left hand side of the equation. Once in factored form, you'll use the Zero Product Property to solve the equation.

Example 1: $x^2 - 4x - 5 = 0$

$(x-5)(x+1)=0$

$x-5=0$	$x+1=0$
$+\cancel{5}\quad +5$	$-\cancel{1}\quad -1$
$x=5$	$x=-1$

$\{-1, 5\}$

(Multiply to last number. Add to middle number)

→ Move everything to one side of the equal sign

→ Factor

→ Set each factor equal to zero and solve.

Check $(x=-1)$

$(-1)^2-4(-1)-5=0$

$1+4-5=0$

$0=0$ ✓

$(x=5)$

$(5)^2-4(5)-5=0$

$25-20-5=0$

$0=0$ ✓

Example 2: $x^2 + 5x - 36 = 0$

$$(x-4)(x+9)=0$$

$$x-4=0 \qquad x+9=0$$
$$+4 \quad +4 \qquad -9 \quad -9$$
$$x=4 \qquad x=-9$$

-36		SUM
-1	36	35
-2	18	16
-3	12	9
-4	9	5
-6	6	0

$\{-9, 4\}$

Check $(x=4)$

$$(4)^2+5(4)-36=0$$
$$16+20-36=0$$
$$0=0 \checkmark$$

$(x=-9)$

$$(-9)^2+5(-9)-36=0$$
$$81-45-36=0$$
$$0=0 \checkmark$$

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Example 3: $3x^2 - 22x - 16 = 0$

Multiply to ac
Add to b

$$3 \times -16 = -48$$

$$(3x^2 - 24x) + (2x - 16) = 0$$

$$3x(x - 8) + 2(x - 8) = 0$$

$$(x - 8)(3x + 2) = 0$$

$x \cancel{-8} = 0$	$3x \cancel{+2} = 0$
$\cancel{+8} \quad +8$	$\cancel{-2} \quad -2$
$x = 8$	$\frac{\cancel{3}x}{\cancel{3}} = \frac{-2}{3}$
	$x = -2/3$

-48		SUM
1	-48	-47
2	-24	-22
3	-16	-13
4	-12	-8
6	-8	-2

Example 3: $3x^2 - 22x - 16 = 0$

Check: $(x = 8)$

$3(8)^2 - 22(8) - 16 = 0$

$3(64) - 22(8) - 16 = 0$

$192 - 176 - 16 = 0$

$0 = 0$ ✓

$(x = -2/3)$

$3\left(-\frac{2}{3}\right)^2 - 22\left(-\frac{2}{3}\right) - 16 = 0$

$\cancel{3}^{1}\left(\frac{4}{\cancel{9}_{3}}\right) - 22\left(\frac{-2}{3}\right) - 16 = 0$

$\frac{4}{3} + \frac{44}{3} - \frac{48}{3} = 0$

$0 = 0$

$\left\{-\frac{2}{3}, 8\right\}$

Completing the Square

You'll start with an equation of the form $ax^2 + bx + c = 0$. Move the constant to the right hand side. Next, you'll need to factor so that the coefficient of x^2 is 1. Then you can complete the square.

Example 4: $4x^2 + 16x = 20$

→ x-terms on the left
constant term on right

→ factor out the a-value

→ Locate b-term

$b =$, $\frac{b}{2} =$, $\left(\frac{b}{2}\right)^2 =$

→ Add $\left(\frac{b}{2}\right)^2$ into parenthesis
Add $a\left(\frac{b}{2}\right)^2$ to the right

→ simplify and factor

→ sq. root and solve

$4(x^2 + 4x) = 20$

$b = 4$

$\frac{b}{2} = 2$

$\left(\frac{b}{2}\right)^2 = 4$

$4(x^2 + 4x + 4) = 20 + 4(4)$

$\frac{\cancel{4}(x+2)^2}{\cancel{4}} = \frac{36}{4}$

$(x+2)^2 = 9 \rightarrow x + 2 = \pm 3 \rightarrow x = -2 \pm 3 = 1, -5$

$\{-5, 1\}$

Completing the Square

Example 5: $x^2 + 8x \cancel{-20} = 0$

$$\cancel{+20} \; +20$$

$$x^2 + 8x = 20$$

$$b = 8$$

$$\frac{b}{2} = 4$$

$$\left(\frac{b}{2}\right)^2 = 16$$

$$x^2 + 8x + 16 = 20 + 16$$

$$(x+4)^2 = 36$$

$$\sqrt{(x+4)^2} = \pm\sqrt{36}$$

$$x + 4 = \pm 6$$

$$x = -4 \pm 6 = 2, -10$$

$$\{-10, 2\}$$

check $(x = -10)$

$$(-10)^2 + 8(-10) - 20 = 0$$

$$100 - 80 - 20 = 0$$

$$0 = 0 \checkmark$$

$(x = 2)$

$$2^2 + 8(2) - 20 = 0$$

$$4 + 16 - 20 = 0$$

$$0 = 0 \checkmark$$

Example 6: $x^2 + 7x \cancel{- 3} = 0$

$$\cancel{+3} \quad +3$$

$$x^2 + 7x = 3$$

$$b = 7$$

$$\frac{b}{2} = \frac{7}{2}$$

$$\left(\frac{b}{2}\right)^2 = \frac{49}{4}$$

$$x^2 + 7x + \frac{49}{4} = 3 + \frac{49}{4} = \frac{12}{4} + \frac{49}{4}$$

$$\left(x + \frac{7}{2}\right)^2 = \frac{61}{4}$$

$$\sqrt{\left(x + \frac{7}{2}\right)^2} = \pm\sqrt{\frac{61}{4}}$$

$$x + \cancel{\frac{7}{2}} = \frac{\pm\sqrt{61}}{2}$$

$$-\cancel{\frac{7}{2}} \quad -\frac{7}{2}$$

$$x = \frac{-7}{2} \pm \frac{\sqrt{61}}{2} = \boxed{\frac{-7 \pm \sqrt{61}}{2}}$$

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The Quadratic Formula

The third method for solving quadratic equations is the quadratic formula. Here's the formula. You need to memorize it:

For a quadratic equation $ax^2 + bx + c = 0,\ a \neq 0$, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Example 7: $x^2 - 5x + 3 = 0$

$a = 1 \quad b = -5 \quad c = 3$

$$x = \frac{5 \pm \sqrt{(-5)^2 - 4(1)(3)}}{2(1)}$$

$$x = \frac{5 \pm \sqrt{25 - 12}}{2}$$

$$x = \boxed{\frac{5 \pm \sqrt{13}}{2}}$$

The Quadratic Formula

Example 8: $3x^2 + 2x + 2 = 0$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$a=3 \quad b=2 \quad c=2$

$$x = \frac{-2 \pm \sqrt{2^2-4(3)(2)}}{2(3)}$$

$$x = \frac{-2 \pm \sqrt{4-24}}{6}$$

$$x = \frac{-2 \pm \sqrt{-20}}{6}$$

There is no solution.

You cannot have square root of a negative number.

$\{\ \}$: Empty Set

Example 9: $4x^2 + 2x - 4 = 0$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(4)(-4)}}{2(4)}$$

$$x = \frac{-2 \pm \sqrt{4 + 64}}{8}$$

$$x = \frac{-2 \pm \sqrt{68}}{8} = \frac{-2 \pm 2\sqrt{17}}{8} = \boxed{\frac{-1 \pm \sqrt{17}}{4}}$$

$\sqrt{68}$

$\sqrt{4}\ \sqrt{17}$

$2\sqrt{17}$

$$\begin{array}{r} 17 \\ 4\,\overline{)\,68} \\ \underline{4} \\ 28 \end{array}$$

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Note: The **discriminant** of the equation $ax^2 + bx + c = 0$ ($a \neq 0$) is given by

$$D = b^2 - 4ac$$

If D > 0, then the equation $ax^2 + bx + c = 0$ has two distinct real solutions. b^2-4ac is perf sq : Rational

Not : Irrational

If D = 0, then the equation $ax^2 + bx + c = 0$ has exactly one real solution.

If D < 0, then the equation $ax^2 + bx + c = 0$ has no real solution (The roots of the equation are complex numbers and appear as complex conjugate pairs.)

Inside of the square root

$(b^2 - 4ac)$

Number and Nature of the Roots