MATH 1310

Session 3

Complex Numbers

Definition: A **complex number** is a number that can be written in the form $a + bi$, where a is called the **real part** and bi is called the **imaginary part**. The a and b are real numbers and $i = \sqrt{-1}$

Here are several properties of complex numbers:

Addition of Complex Numbers:

$(a + bi) + (c + di) = (a + c) + (b + d)i$

Add the real parts together and add the imaginary parts together.

Subtraction of Complex Numbers

$(a + bi) - (c + di) = (a - c) + (b - d)i$

Subtract the real parts and subtract the imaginary parts.

Multiplication of Complex Numbers:

Multiply in the same manner as multiplying binomials and remember that $i^2 = -1$

Example 1: Simplify each.

a. $\sqrt{-16}$

b. $\sqrt{-40}$

Example 2: Simplify each of the following and write the answer in form a + bi.

a. $(5 + 4i) + (2 - i)$

c. $-i(-3 + 6i)$

b. $(-6 - 3i) - (-2 + 2i)$

d. $(-1 - i)(2 + 5i)$

Next, you'll need to be able to find various powers of *i*. You'll need to know these 4 powers:

$$i = \sqrt{-1}$$
$$i^2 = -1$$
$$i^3 = i^2 * i = -1 * i = -i$$
$$i^4 = i^2 * i^2 = -1 * -1 = 1$$

For other powers of *i*, divide the exponent by 4 and find the remainder. Your answer will be *i* raised to the remainder power. If the remainder is zero, your answer will be i^4 or 1.

Example 3: Simplify each.

a. i^{15}

c. i^{42}

b. i^{72}

d. i^{313}

Division of Complex Numbers

The **complex conjugate** of the complex number $a + bi$ is the complex number $a - bi$.

To simplify the quotient $\frac{a+bi}{c+di}$ multiply both the numerator and denominator by the complex conjugate of the denominator.

a. $\frac{5+4i}{2-3i}$

b. $\frac{-1-i}{i}$

c. $\frac{1}{4-i}+\frac{4}{4+i}$

d. $\frac{5-2i}{3+4i}$

Complex Roots of Quadratic Equations

Using complex numbers, we can now find all solutions to quadratic equations. We can use any of the techniques from the previous section to solve, but usually, we will just take the square root of both sides of the equation, complete the square or use the quadratic formula.

Example 5: Find all complex solutions of the following equations. Express your answer in form a + b*i*.

a. $x^2 + 100 = 0$

c. $x^2 - 6x = -13$

b. $49x^2 + 36 = 0$

d. $x^2 + 12x + 75 = 0$

e. $4x^2 + 8x + 9 = 0$

Popper #3

Simplify the following:

1. $\sqrt{-50}$

   a. -10i b. $5\sqrt{10}i$ c. $5\sqrt{2}i$ d. $-5\sqrt{2}$

2. (8 + 5i) – (2 – i)

   a. 6 + 6i b. 6 + 4i c. 10 + 6i d. 15

3. (4 + i)(3 + 2i)

   a. 12 – 2i b. 7 + 3i c. 14 + 11i d. 10 + 11i

4. $\frac{4-i}{6+i}$

   a. $\frac{25-10i}{35}$ b. $\frac{23-10i}{37}$ c. $\frac{-2}{3}$ d. $\frac{12i}{2-i}$

Other Techniques for Solving Equations

Solving by Factoring:

Factoring can be used to solve many types of equations. Always begin by Factoring Completely. Then, set each factor equal to zero.

Find all the solutions of $x^3 = x$

Find all solutions to

$x^3 + 3x^2 + 2x + 6 = 0$

Equations involving radicals:

If an equation involves a square root (also called a radical), you must isolate the radical, square both sides, and solve the remaining equation. Be certain to check your answers!

Find all solutions to $\sqrt{x+8}-2=x$

Extraneous Solutions: In a radical solution, you may “create” additional answers that are not correct. These must be rejected!

Find all solutions to $\sqrt{3x+1} - 1 = x$

Solving by Substitution:

When a function looks, "almost" quadratic, you may want to solve it by relating it to another function.

$x^{10} - x^5 - 6 = 0$

If the exponents go "full amount → half amount → nothing" then you can rewrite as a quadratic.

Let $u = x^5$, *then*

$x^{10} - x^5 - 6 = 0 \rightarrow u^2 - u - 6 = 0$

$$x^{1/2} + 2x^{1/4} - 15 = 0$$

In this lesson, you will learn to solve linear inequalities, graph the solution on a real number line and state the solution using interval notation.

You'll solve 4 different types of linear inequalities, involving these four symbols:

$<$ less than (the quantity to the left is less than the quantity to the right)

$\leq$ less than or equal to (the quantity to the left is less than or equal to the quantity to the right)

$>$ greater than (the quantity to the left is greater than the quantity to the right)

$\geq$ greater than or equal to (the quantity to the left is greater than or equal to the quantity to the right)

To solve an inequality containing a variable, find all values of the variable that make the inequality true. In solving linear inequalities, isolate the variable on one side of the inequality symbol by using the following rules.

1. If $A < B$ then $A + C < B + C$.
2. If $A < B$ then $A - C < B - C$.
3. Let $C > 0$. If $A < B$ then $AC < BC$.
4. Let $C < 0$. If $A < B$ then $AC > BC$.

Example 1: Solve each of the following inequalities.

a. $2(7 - 4x) \geq -13 + 8x$

c. $2x + 1 < 7$

b. $-3 \leq 2x + 1$

Next, you'll need to be able to work with interval notation. An interval is a set of real numbers. It can be a line segment, a ray or the entire number line. If it is a line segment, it can include one or both endpoints. If it is a ray it may or may not include the endpoint. We note intervals using brackets, parentheses or a combination.

The interval $[a, b]$ is the line segment from point a to point b, including both endpoints. This corresponds to the inequality $a \leq x \leq b$.

The interval $[a, b)$ is the line segment from point a to point b, including point a but not point b. This corresponds to the inequality $a \leq x < b$.

The interval $[a, \infty)$corresponds to $x \geq a$ and is a ray beginning at (and including) point a and including all real numbers to the right of the point.

The interval $(-\infty, a)$ corresponds to $x < a$ and is a ray beginning at (and not including) point a and including all real numbers to the left of the point.

Example 2: Write each of these inequalities using interval notation.

a. $x \geq -2$

b. $x < 3$

c. $4 < x \leq 7$

Example 3: Solve each inequality. Graph each solution on the real number line. Write your solutions using interval notation.

a. $7x > 35$

b. $-4x \leq 48$

3. $5x - 4 > 2x + 7$

4. $-2(x - 5) < 3(4x - 7) + 12$

You can also solve some compound inequalities. All of the same rules apply to these problems

Solve each inequality. Write your solutions using interval notation. Graph each solution on the real number line.

Example 4: $-2 \leq x + 5 < 7$

Example 5: $-4 < 3 - 2x \leq 9$

Example 6: $-\frac{7}{6} < \frac{-3(-x-1)}{8} < \frac{7}{3}$

Example 7: $35 < 5x - 5(x - 7)/2 \leq 70$

Non-Linear Inequalities

In this section, we will examine how to solve inequalities involving (1) quadratic functions, and (2) rational functions.

In these examples, we will use a method known as the Number Line Test.

Solving a Quadratic Inequality

- Rewrite the inequality as an equation (with an equal sign).
- Solve as done before.
- Test an x-value between the two solutions by plugging into the original inequality.
  - If you get a true statement, your solution is between the two solutions.
  - If you get a false statement, your solution is outside the two solutions.

Try this: $x^2 - x > 6$

Now, try this: $x^2 - x - 2 \leq 0$

Solving a Rational Inequality

- Set the denominator equal to zero and solve.
- Set the numerator equal to zero and solve.
- Plot these points on a number line (denominator is always open dot).
- Use test points between these values to determine the solution set.

Try this: $\frac{x+3}{x-1} > 0$

Now try this: $\frac{x-5}{x+8} \geq 0$

Popper #4:

$x^2 + x - 12 < 0$

$$\frac{x+8}{x-2} > 0$$

Absolute Value Equations

In this lesson, you'll learn to solve absolute value equations and inequalities.

Definition: The **absolute value** of x, denoted $|x|$, is the distance x s from 0.

Solving Absolute Value Equations
If C is positive, then $|x| = C$ if and only if $x = \pm C$.

Special Cases for $|x| = C$:

Case 1: If C is negative then the equation $|x| = C$ has no solution since absolute value cannot be negative.

Case 2: The solution of the equation $|x| = 0$ is $x = 0$.

Solve the following:

a. $|2x - 3| = 7$

b. $|6 - 2x| + 6 = 14$

c. $2|-3(2x - 8)| + 4 = 30$

d. $-4\left|\frac{1}{2}x+1\right|+3=-11$

$4 + |x + 8| = 12$

$|2x + 4| = 3$

$|3x – 2| + 1 = 4$

$|x + 3| = -4$

Next, we'll look at inequalities. The approach to these problems will depend on whether the problem is a "less than" problem or a "greater than" problem. If C is zero, then $x = 0$.

Solving Absolute Value Inequalities

If C is positive, then

a. $|x| < C$ if and only if $-C < x < C$.

b. $|x| \leq C$ if and only if $-C \leq x \leq C$.

c. $|x| > C$ if and only if $x > C$ or $x < -C$.

d. $|x| \geq C$ if and only if $x \geq C$ or $x \leq -C$

Example 2: Solve the inequality. Graph the solution on the real number line. Write the solution using interval notation:

a. $|x + 3| \leq 8$

b. $|4 - 2x| < 12$

c. $3|2x - 6| \leq 6$

d. $|-3x + 1| < 4$

e. $2|1-4x|+1>7$

f. $-\frac{2}{3}|x-4| \leq -\frac{4}{3}$

Special Cases:

Case 1:

If C is negative, then:

a) The inequalities $|x| < C$ and $|x| \leq C$ have no solution.

b) Every real number satisfies the inequalities $|x| > C$ and $|x| \geq C$

Case 2:

a) The inequality $|x| < 0$ has no solution.

b) The solution of the inequality $|x| \leq 0$ is $x = 0$.

c) Every real number satisfies the inequality $|x| \geq C$