MATH 1310

Session 4

MATH 1310

Functions: Basic Ideas

The rest of this course deals with **functions.**

Definition: A **function**, f, is a rule that assigns to each element x in a set A exactly one elements, called $f(x)$, in a set B.
Functions are so important that we use a special notation when working with them. We'll write $f(x)$ to denote the value of function f at x. We read this as "f of x." We can use letters other than f to denote a function, so you may see a function such as $g(x)$, $h(x)$ or $P(x)$.

$f(x)$: function
$g(x)$ "
$h(x)$ "

Definition: The set A is called the **domain** and is the set of all valid inputs for the function.

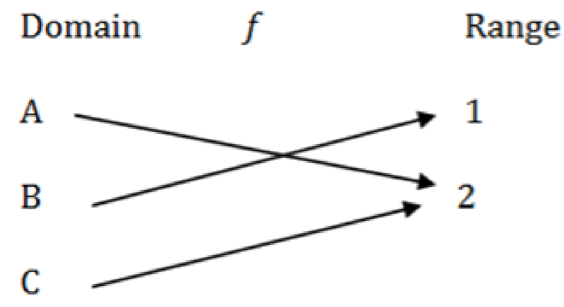
Definition: The set B is called the **range** and is the set of all possible values of $f(x)$ as x varies throughout the domain.
Sets A and B will consist of real numbers.

x-values: domain

y-values: range

Example 1:

a. Given:



Every x-value must have a unique answer in y.

Is f a function?

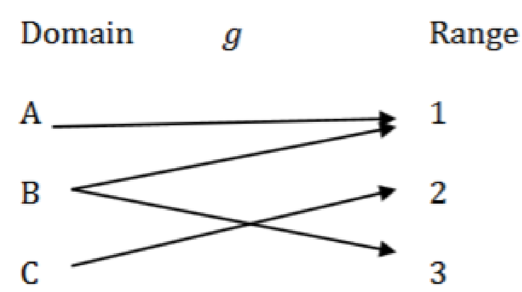
Yes

$f(A) = 2$

$f(B) = 1$

$f(C) = 2$

b. Given:



Is g a function? Not a function

$g(B) = 1$

$g(B) = 3$

Since there is an element of the domain with two images in the range. It is not a function.

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Next we'll consider some things you'll need to be able to do when working with functions. First, you'll need to be able to evaluate all types of functions when given a specific value for the variable.

Example 2: Let $f(x) = x^2 - 4x$ Calculate

a. $f(-3) = (-3)^2 - 4(-3)$

$f(-3) = 9 + 12 = 21$

$f(-3) = 21$

$(-3, 21)$

b. $-2f(x)$

$-2(x^2 - 4x)$

$-2x^2 + 8x$

c. $f(3x) = (3x)^2 - 4(3x)$

$3^2x^2 - 12x$

$9x^2 - 12x$

d. $f(x+2) = (x+2)^2 - 4(x+2)$

$(x+2)(x+2) - 4(x+2)$

$x^2 + \cancel{2x + 2x} + 4 \cancel{-4x} - 8$

$x^2 - 4$

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Example 3: Suppose $g(x) = \begin{cases} 2x - 6, & x < -2 \\ x^2 + 2x + 3, & x \geq -2 \end{cases}$. Calculate the following

a. $g(-5)$ $x = -5$ (First category, since $-5 < -2$)

$g(-5) = 2(-5) - 6 = -10 - 6 = -16$

$(-5, -16)$

b. $g(-2)$ $x = -2$ (category 2)

$g(-2) = (-2)^2 + 2(-2) + 3$

$4 - 4 + 3 = 3$

$g(-2) = 3$ $(-2, 3)$

c. $g(3)$ $x = 3$ (category 2)

$(3)^2 + 2(3) + 3$

$9 + 6 + 3 = 18$

$(3, 18)$

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Finding the Domain of a Function

Recall: The domain is the set of all real numbers for which the expression is defined as a real number. Exclude from a function's domain real numbers that cause division by zero or real numbers that result in an even root of a negative number.

We express the set of real numbers as $(-\infty, \infty)$.
The domain of any polynomial function is $(-\infty, \infty)$.

Exclude zeroes in denominators of fractions

or

Negatives inside even roots

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Example 4: Find the domain of each function below and express your answer in interval notation.

a. $f(x) = -17$

D: $(-\infty, \infty)$

b. $f(x) = 3x - 4$

$(-\infty, \infty)$

c. $h(x) = \frac{5x}{2x-8}$

$2x - 8 = 0$

$2x = 8$

$x = 4$

D: $(-\infty, 4) \cup (4, \infty)$

d. $f(x) = \frac{x-1}{2x-6}$

$2x - 6 = 0$

$2x = 6$

$x = 3$

D: $(-\infty, 3) \cup (3, \infty)$

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e. $p(x) = \frac{x^2-16}{x^2-4x-12}$

$x^2 - 4x - 12 = 0$

$(x-6)(x+2) = 0$

$x-6=0$	$x+2=0$
$x=6$	$x=-2$

$D: (-\infty, -2) \cup (-2, 6) \cup (6, \infty)$

f. $h(x) = \sqrt{x}$

$x \geq 0$

(cannot have negatives inside even roots)

$D: [0, \infty)$

Popper 4

1. $q(x) = \sqrt{x-4}$

$x - 4 \geq 0$

2. $f(x) = \sqrt[3]{2x+4}$

Defined everywhere

3. $g(x) = \sqrt[10]{42-2x}$

$42 - 2x \geq 0$

a. (-∞, 21]

b. [21, ∞)

c. (-∞, ∞)

d. [4, ∞)

Popper 4

Functions and Graphs

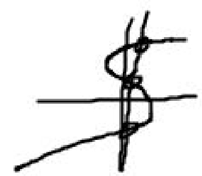
You can answer many questions given a graph.

Definition: The graph of a function $f(x)$ is the set of points (x, y) whose x coordinates are in the domain of f and whose y coordinates are given by $y = f(x)$.

First, does the graph represent a function? To answer this, you will need to use the **vertical line test (VLT)**.

The Vertical Line Test:
If you can draw a vertical line that crosses the graph more than once, it is NOT the graph of a function.

function

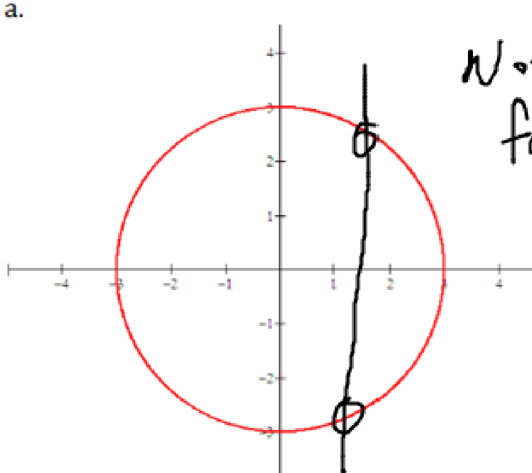


Not a function

Does the graph represent a function

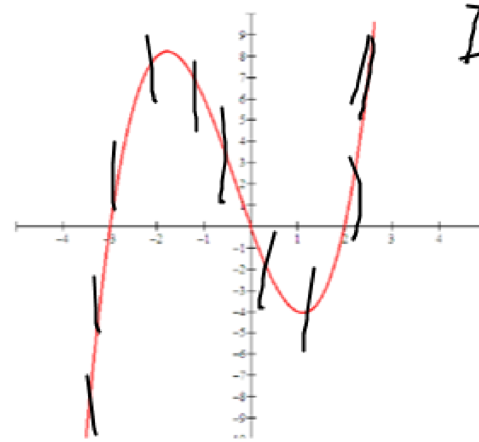
No

a.



Not a function

Yes



Is a function

Does the graph represent a function

Definition: An **equation defines *y* as a function of *x*** if when one value for x is substituted in the equation, **exactly one value for *y* is returned.**

Example 2: Does the following equation define y as a function of x?

$y - x^2 = 4$

1. Solve for y.
2. For each value x, do we get exactly one value for y back?

$$\begin{aligned} y - \cancel{x^2} &= 4 \\ + \cancel{x^2} \quad & \quad + x^2 \\ \hline y &= x^2 + 4 \end{aligned}$$

If we plug in several (or any) x value(s) we get one answer back.

This is a function.

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b. $x^2 + y^2 = 9$

1. Solve for y.
2. For each value x, do we get exactly one value for y back?

$$\cancel{x^2} + y^2 = 9$$
$$-\cancel{x^2} \qquad -x^2$$
$$\sqrt{y^2} = \sqrt{-x^2 + 9}$$
$$y = \pm\sqrt{-x^2 + 9}$$

Evaluate at $x = 0$

$$y = \pm\sqrt{-0^2 + 9}$$
$$y = \pm\sqrt{9} = \pm 3$$

Not a function

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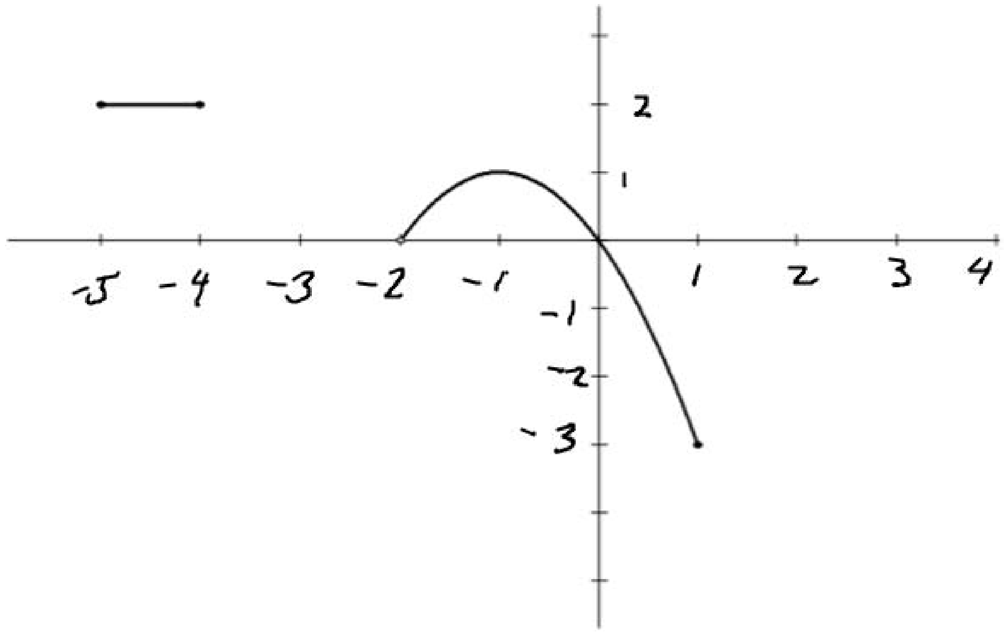
Example 3: Find the domain and range of the function whose graph is shown.

(x-val) Domain: $[-5, -4] \cup (-2, 1]$

(y-val) Range: $[-3, 1] \cup [2, 2]$



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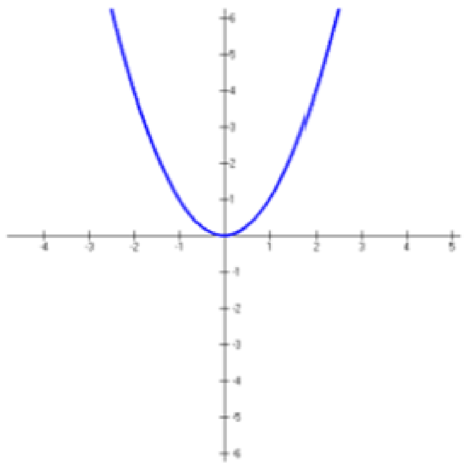
You'll also need to be able to graph functions. For now, you can do so by plotting points. But...

YOU MUST KNOW THESE FUNCTIONS AND GRAPHS

https://youtu.be/sOK4q4OcEQc

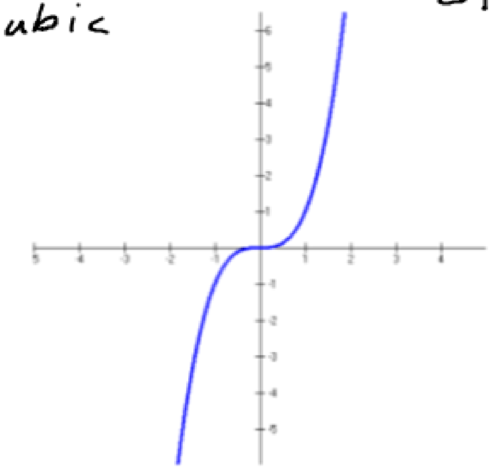
Parent Functions
or
Library of
functions.

$f(x) = x^2$ Quadratic



$f(x) = x^3$

cubic



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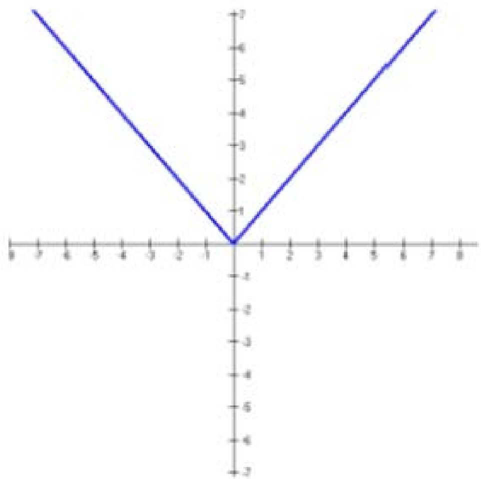
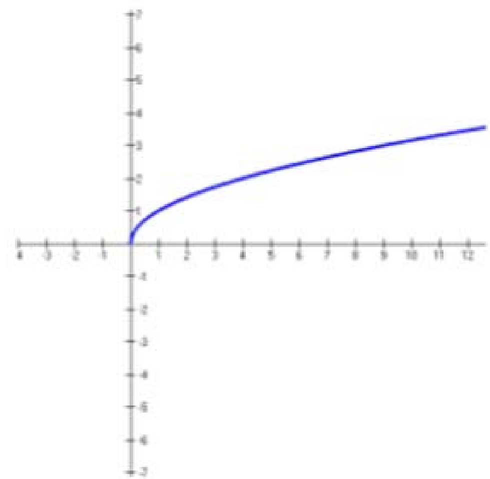
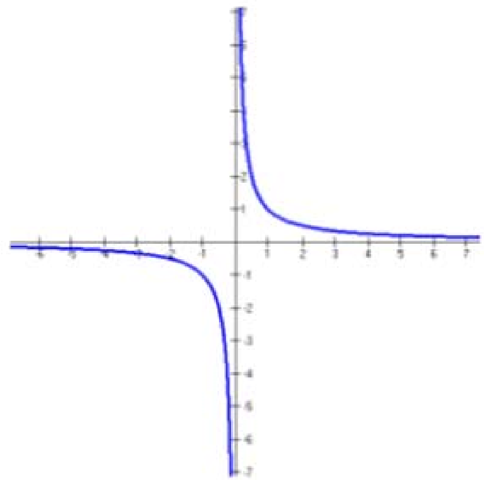
$f(x) = |x|$ absolute value



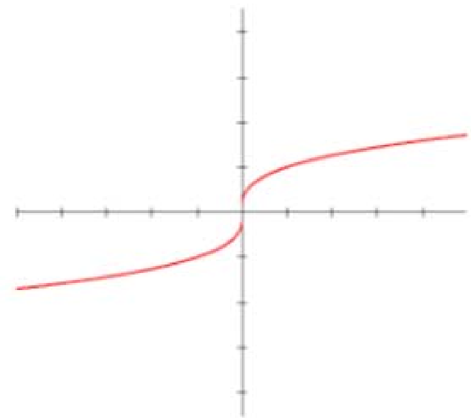
$f(x) = \sqrt{x}$ radical



$f(x) = \frac{1}{x}$ Rational Function



$f(x) = \sqrt[3]{x}$ cube root function



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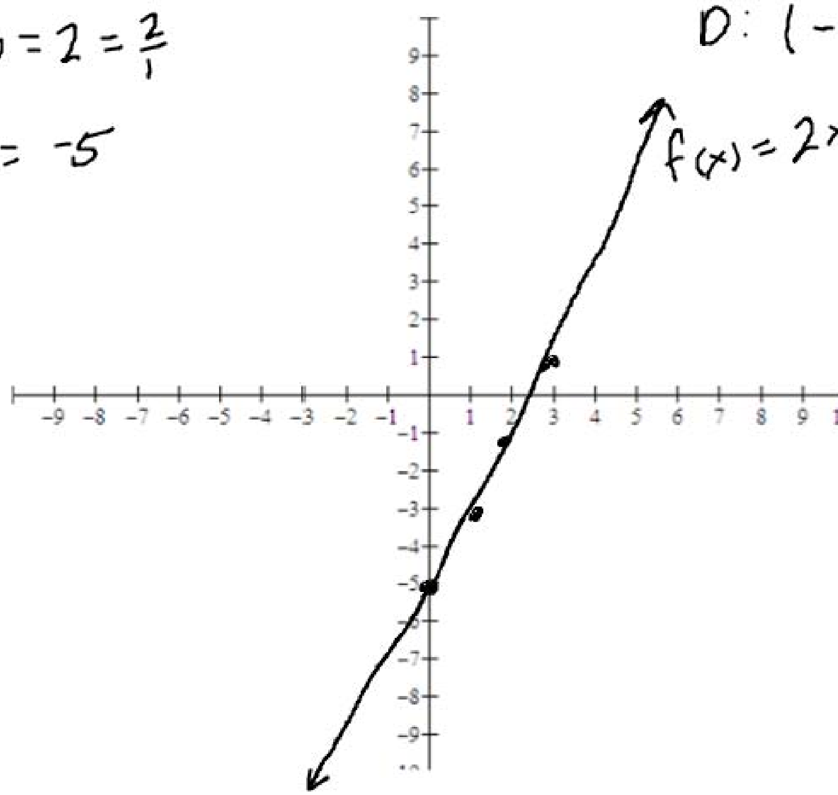
Example 4: Suppose $f(x) = 2x - 5$. State the domain of the function and graph it.

$m = 2 = \frac{2}{1}$

$b = -5$

$D: (-\infty, \infty)$



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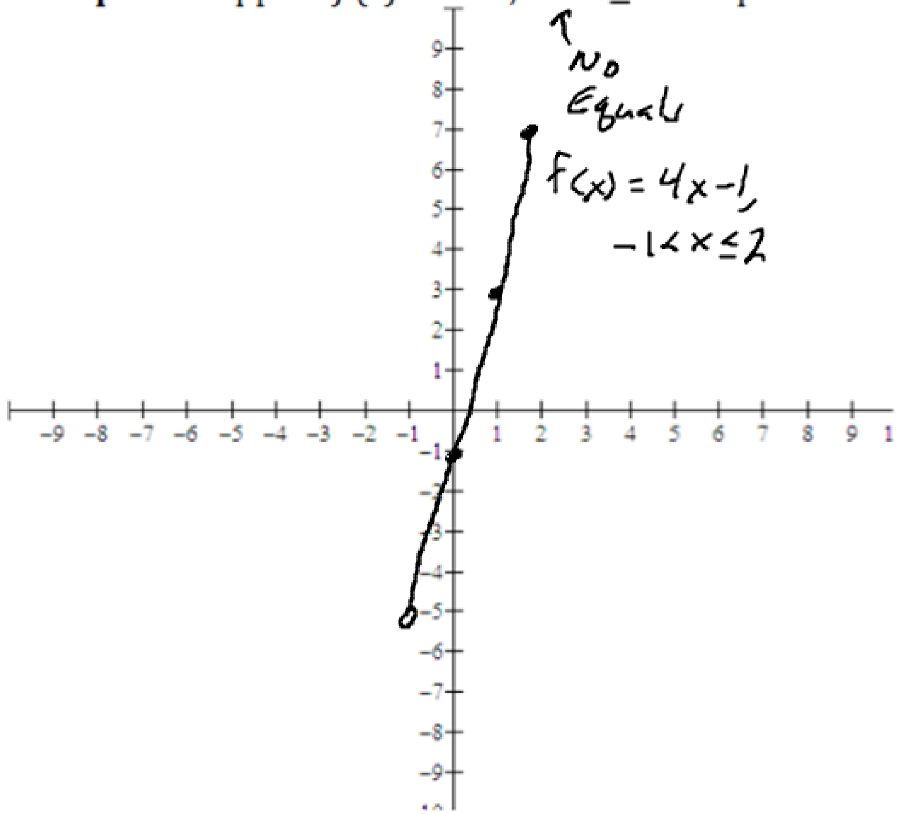
Example 5: Suppose $f(x) = 4x - 1,\ -1 < x \le 2$. Graph the function.

Equals

$m = \frac{4}{1}$



Restricted domain

Find $f(-1) =$

$4(-1) - 1$

$-4 - 1 = -5$

o : $(-1, -5)$

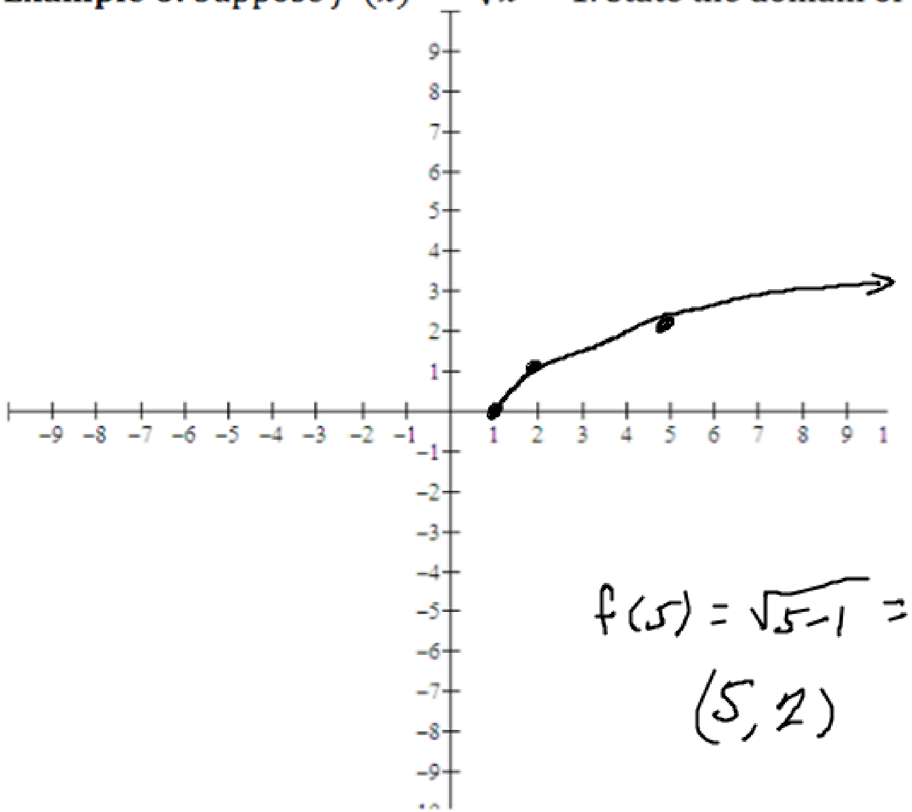
Find $f(2) =$

$4(2) - 1$

$8 - 1 = 7$

• : $(2, 7)$

Example 6: Suppose $f(x) = \sqrt{x-1}$. State the domain of the function and graph it.



$$x - 1 \geq 0$$
$$\underline{+1 \quad +1}$$
$$x \geq 1$$

$D: [1, \infty)$

$f(1) = \sqrt{1-1} = \sqrt{0}$
$= 0$

$(1, 0)$

$f(5) = \sqrt{5-1} = \sqrt{4} = 2$

$(5, 2)$

$f(2) = \sqrt{2-1} = \sqrt{1} = 1$

$(2, 1)$

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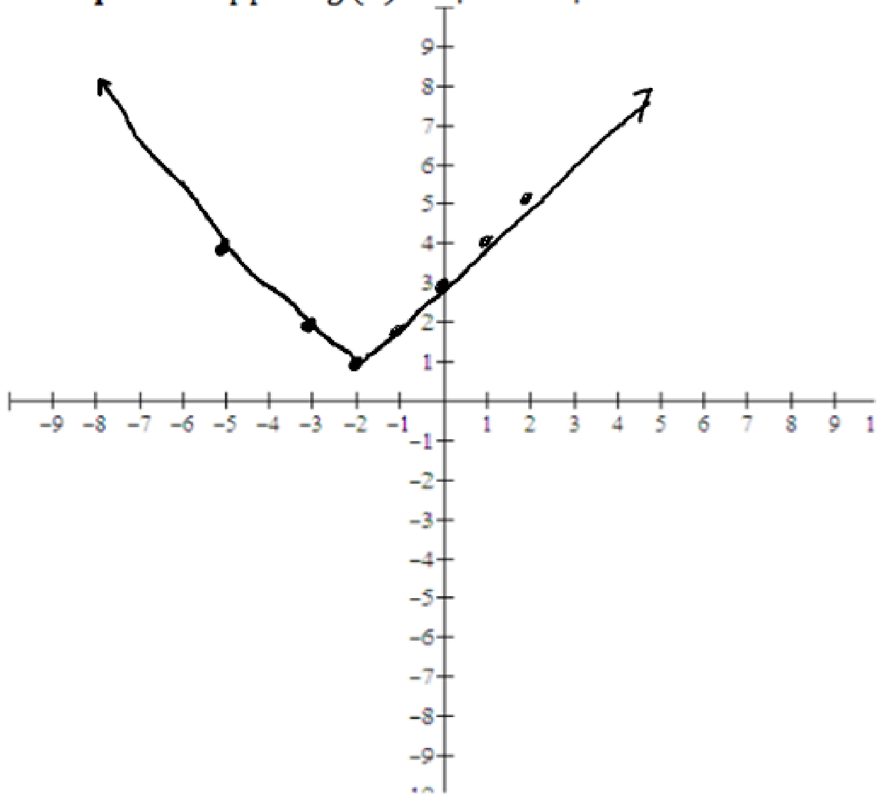
Example 7: Suppose $g(x) = |x + 2| + 1$. State the domain of the function and graph it.



$D: (-\infty, \infty)$

x		Y	
-5	$\|-5+2\|+1$	4	(-5,4)
-3	$\|-3+2\|+1$	2	(-3,2)
-1	$\|-1+2\|+1$	2	(-1,2)
0	$\|0+2\|+1$	3	(0,3)
1	$\|1+2\|+1$	4	(1,4)
2	$\|2+2\|+1$	5	(2,5)

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Example 8: Let $P(x) = \begin{cases} -3, & x < 2 \\ x^2, & x > 2 \\ 2, & x = 2 \end{cases}$ State the domain of the function and graph it.

— No equal sign

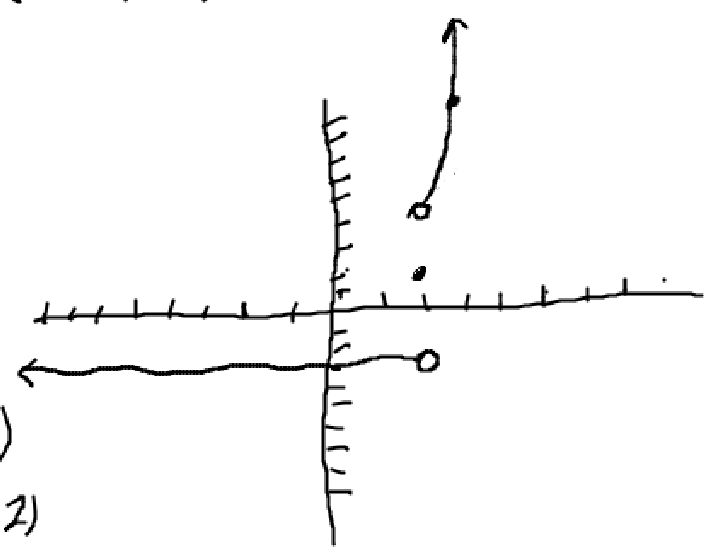
D: $(-\infty, \infty)$

Find $p(-2)$, $p(2)$ and $p(3)$.

$x<2$ $\quad x=2$ $\quad x>2$ $\quad p(3) = 3^2 = 9$

$p(-2) = -3$ $\quad p(2) = 2$

$(-2,-3)$ $\quad (2,2)$ $\quad (3,9)$



$x^2 \to (2)^2 = 4$: O

$Y = -3$ (Left of $x=2$)

Coo-dinate: $(2,2)$

$Y = x^2$ (Right of 2)

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Odd and Even Functions:

Odd Functions have only odd exponents, such as $f(x) = 2x^3 + 8x^1$.

They satisfy the formula: $f(-x) = -f(x)$

They are symmetric about the origin.

If they contain the point (a, b) they also contain (-a, -b).

Even Functions only have even exponents, such as $g(x) = 3x^4 + 2x^2 + 5x^0$

They satisfy the formula: $g(-x) = g(x)$

They are symmetric about the y-axis.

If they contain the point (a, b), they also contain (-a, b).

Odd and Even Functions:

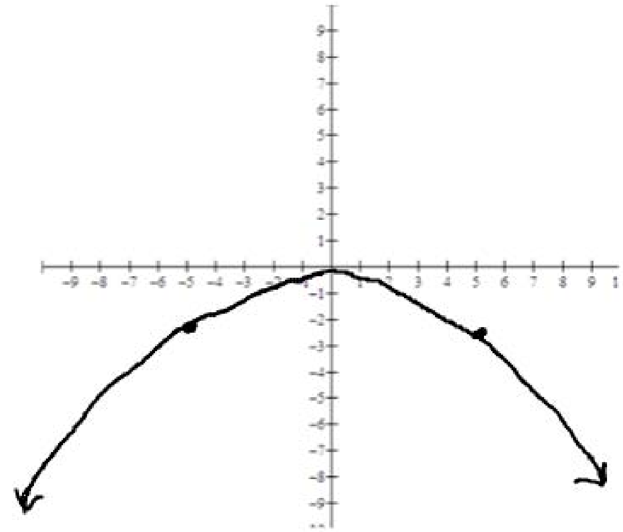
An even function contains the point (-5, -2).

What point must it also contain?

(5, -2)

What is a possible graph of the function?

Must have
y-axis symmetry.



An even function contains the point (-5, -2).

The following function passes through the point (8, -11).

Is the function even or odd?

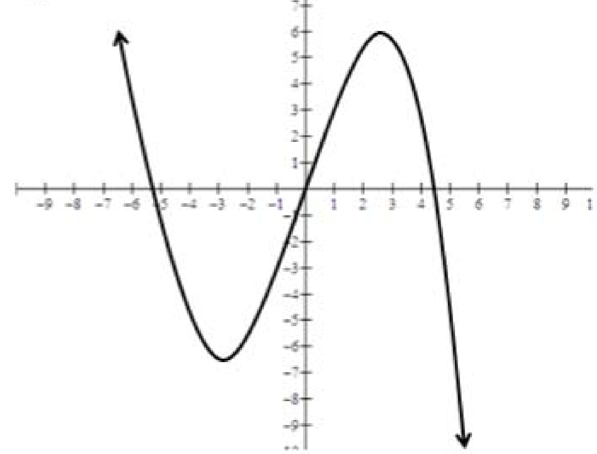
y-axis symmetry? No

origin symmetry? Yes

What other point must it contain?

(-8, 11)

odd function



What is a possible equation?

(assume all letters represent constants)

$\cancel{f(x)} = ax^3 + bx^{2} + cx + dx^{0}$ $h(x) = ax^3 + bx^1$

$\cancel{g(x)} = ax^{2} + bx^{0}$ $\cancel{j(x)} = ax^{2} + bx$

The following function passes through the point (8, -11)

Transforming Functions

In future courses, you will need to be able to sketch the graph of a function quickly and accurately. You can use transformations to do this. There are two types of transformations:

- Translation
- Reflections

We'll start with **translations**. To **translate** a graph means to shift it horizontally, vertically or both.

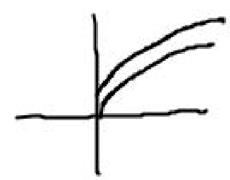
Vertical shifting:

To graph $y = f(x) + c$, $c > 0$, start with the graph of $f(x)$ and shift it upward c units.
To graph $y = f(x) - c$, $c > 0$, start with the graph of $f(x)$ and shift it downward c units.

$f(x) = \sqrt{x} + 1$

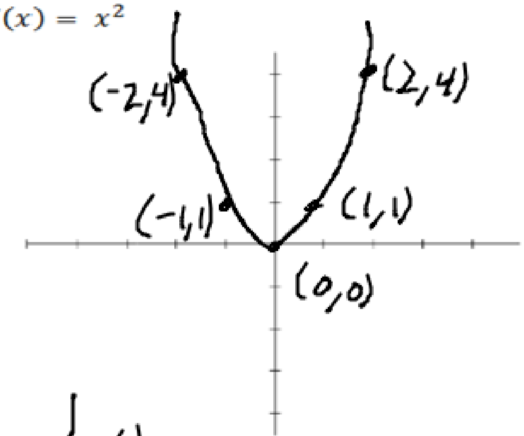
$y = \sqrt{x}$

$y = \sqrt{x} + 1$

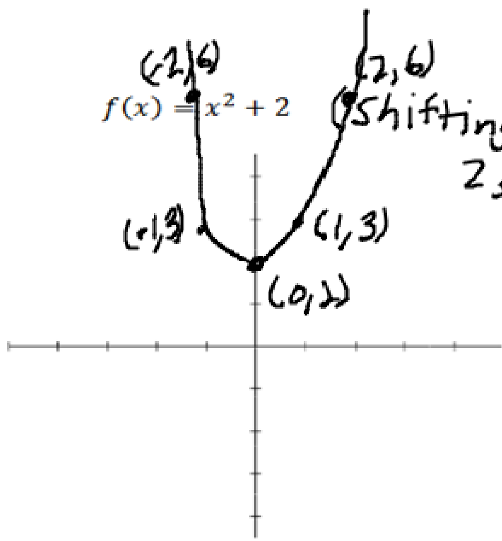


Transforming Functions

Example 1: Sketch $f(x) = x^2$

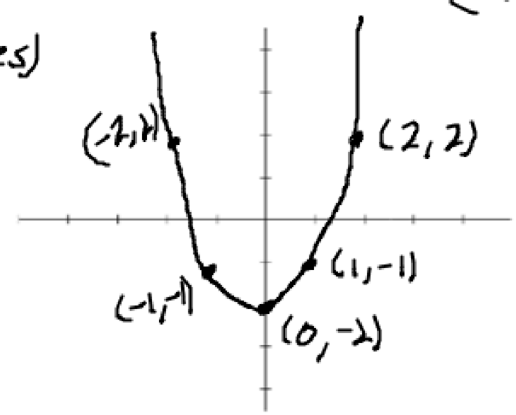


$f(x) = x^2 + 2$



(shifting up 2 spaces)

$f(x) = x^2 - 2$



(shift down 2 spaces)

Horizontal shifting:

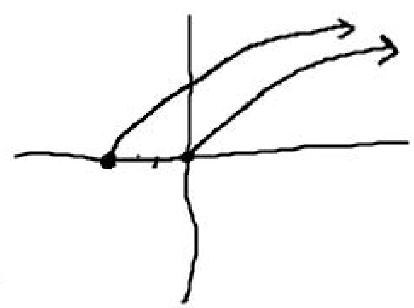
To graph $y = f(x + c)$, $c > 0$, start with the graph of $f(x)$ and shift it left c units.
To graph $y = f(x - c)$, $c > 0$, start with the graph of $f(x)$ and shift it right c units.

$f(x) = \sqrt{x+3}$

$y = \sqrt{x}$

$y = \sqrt{x+3}$

shift left
3 spaces



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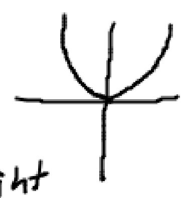
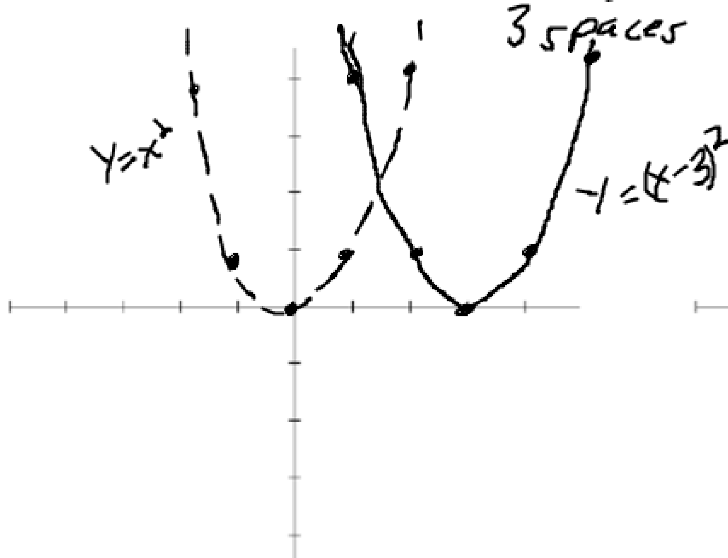
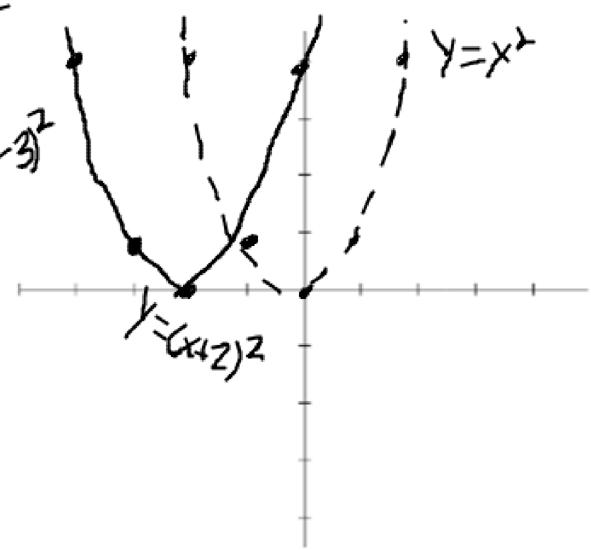
Example 2: Sketch the following graphs



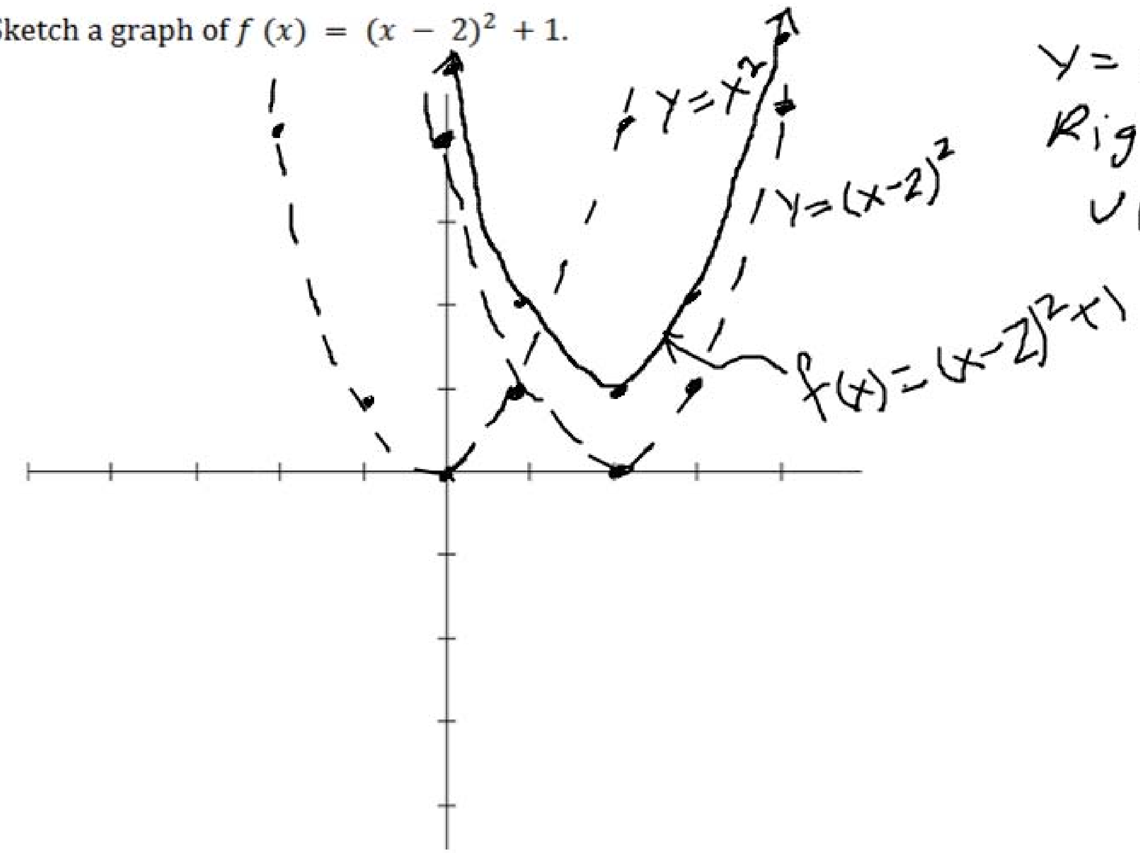
$f(x) = (x - 3)^2$ shift right 3 spaces



$f(x) = (x + 2)^2$ shift left 2 spaces



Example 3: Sketch a graph of $f(x) = (x - 2)^2 + 1$.



$y = x^2$
Right 2
Up 1

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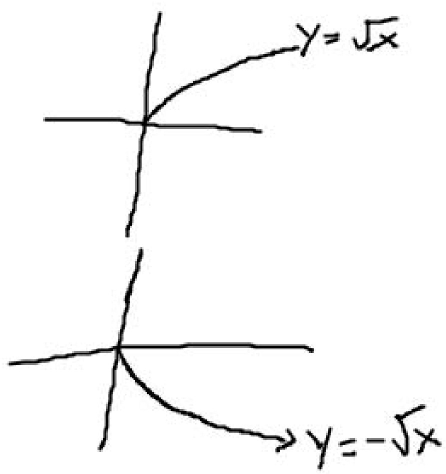
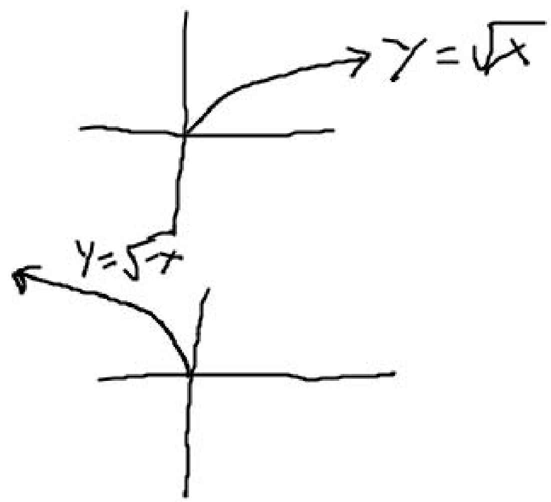
We can also reflect a function. A reflection of a function is its mirror image about the x axis or the y axis.

To graph $-f(x)$, reflect the graph of $f(x)$ about the x axis.
To graph $f(-x)$, reflect the graph of $f(x)$ about the y axis.

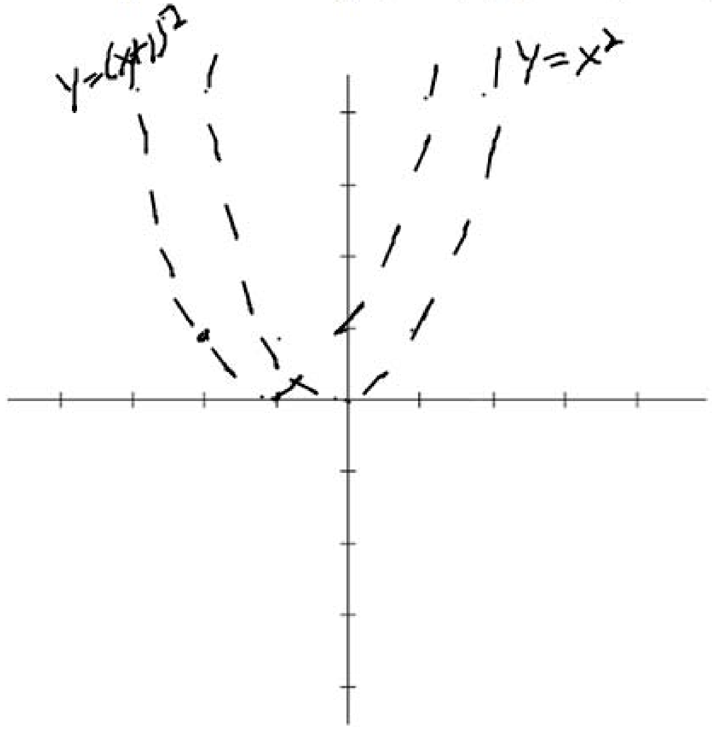
$y = -\sqrt{x}$

$y = \sqrt{-x}$





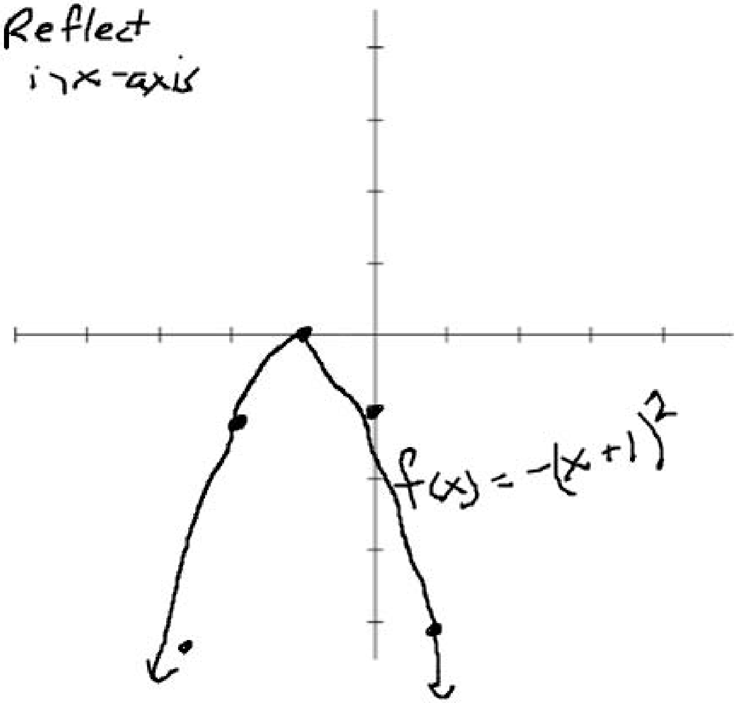
Example 4: Sketch a graph of $f(x) = -(x+1)^2$.



Left 1

Reflect

in x-axis



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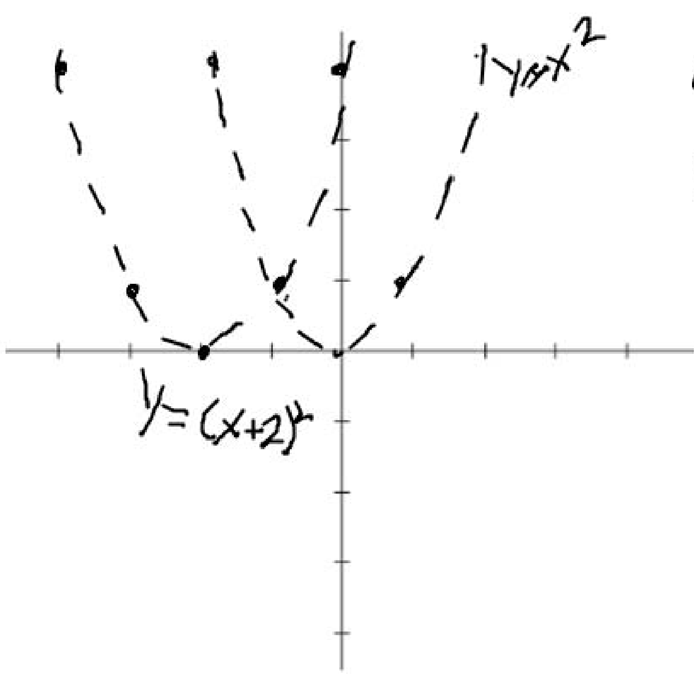
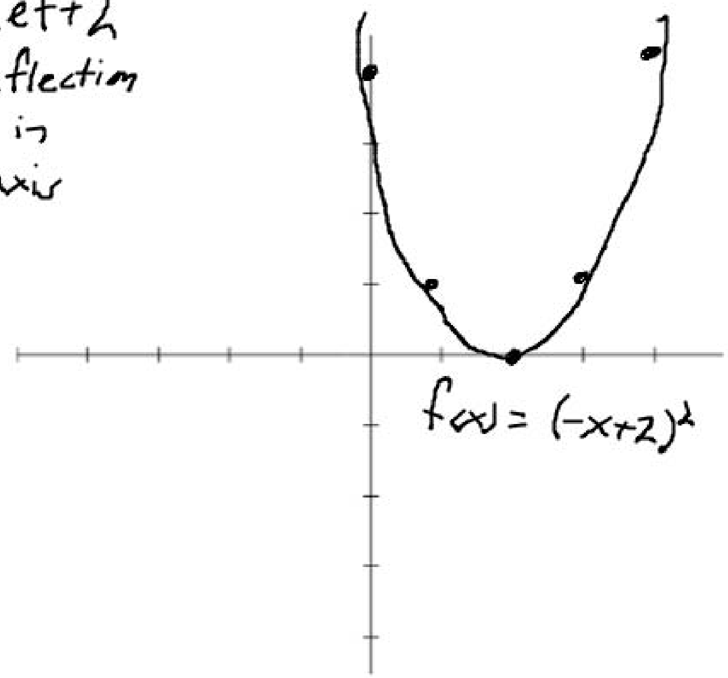
Example 5: Sketch a graph of $f(x) = (-x + 2)^2$.



Left 2
Reflection
in
Y axis



Popper 4, continued: $f(x) = -\sqrt{x+2} - 5$

4. Identify the Parent Function:
   a. Linear   b. Quadratic   c. Rational   d. Radical

5. Is there a vertical shift?
   a. up 5   b. up 2   c. down 5   d. down 2

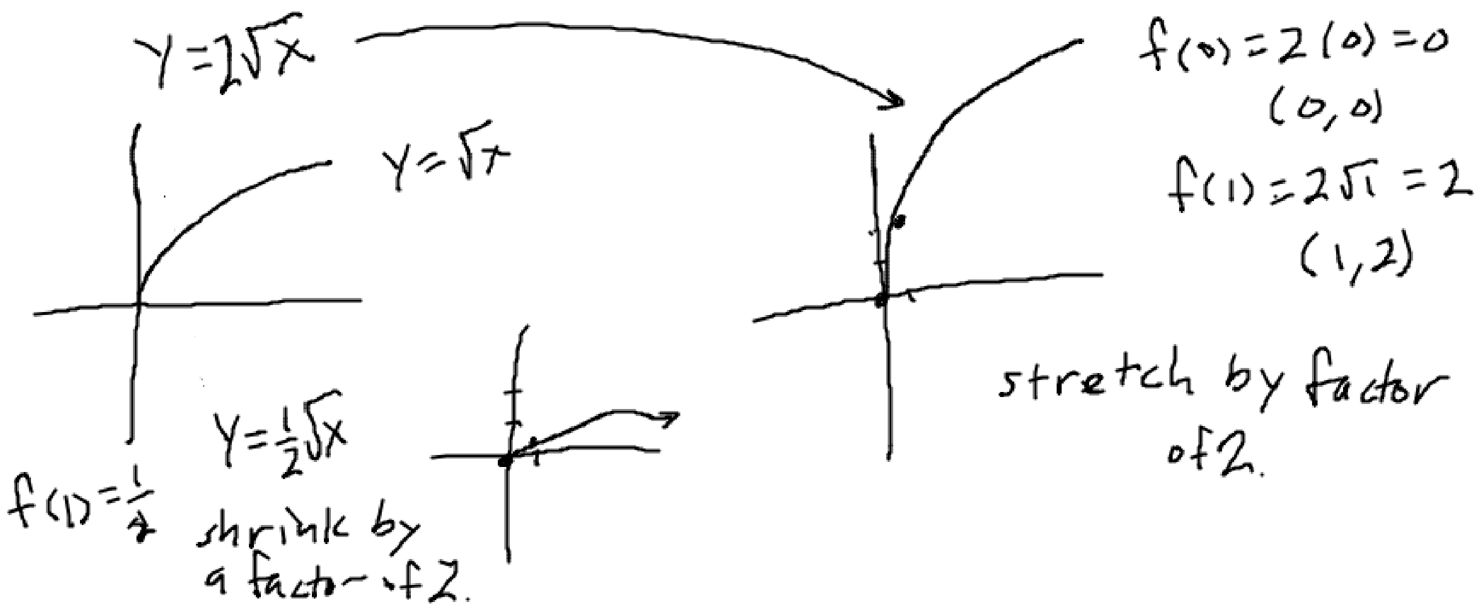
6. Is there a horizontal (y-axis) reflection?
   a. Yes   b. No

7. Is there a vertical (x-axis) reflection?
   a. Yes   b. No

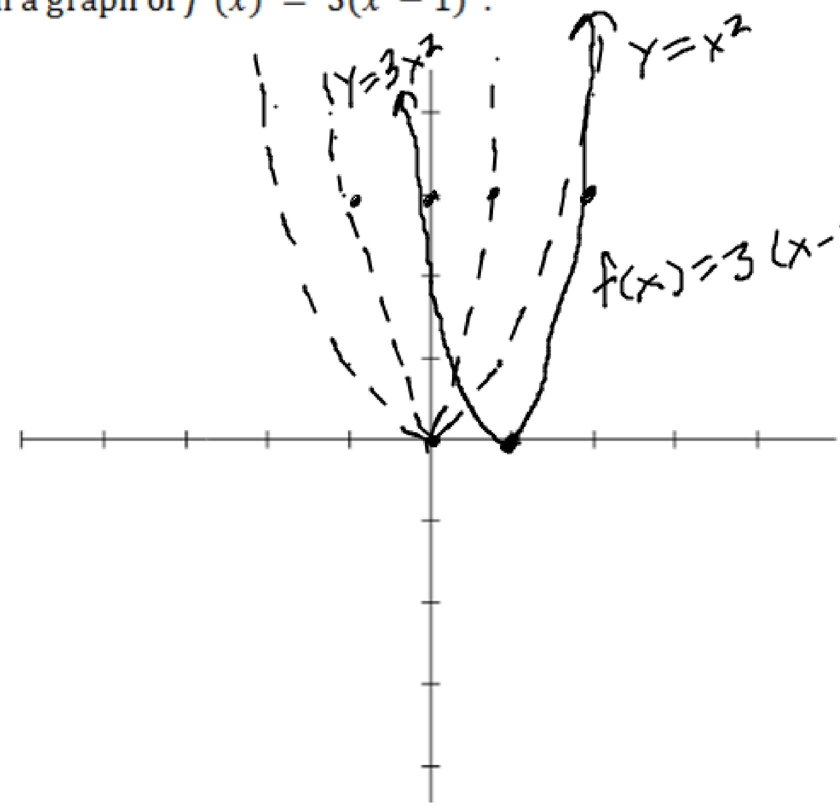
Popper 4, continued: ?? ?? =- ??+2 -5

Finally, you can stretch or shrink your graph vertically. A vertical stretch will move your graph closer to the y axis, while a vertical shrink will move it closer to the x axis. It may be helpful to graph one or two points when your problem has a stretch or shrink.

To graph $y = af(x)$, $a > 1$, stretch the graph of $f(x)$ by a factor of a.
To graph $y = af(x)$, $0 < a < 1$, shrink the graph of $f(x)$ by a factor of $\frac{1}{a}$



Example 6: Sketch a graph of $f(x) = 3(x-1)^2$.

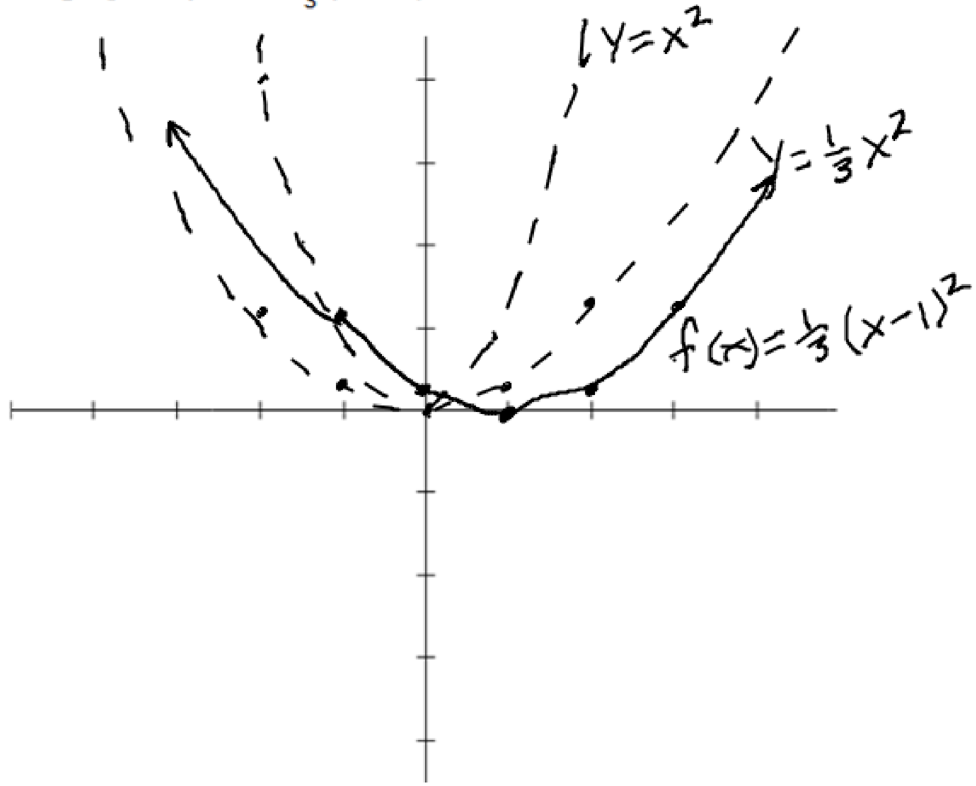


$Y=x^2$

stretch by factor 3

Right 1

Example 7: Sketch a graph of $f(x) = \frac{1}{3}(x-1)^2$.



$y = x^2$

shrink by factor 3

shift Right 1

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Recommended order for transforming functions:

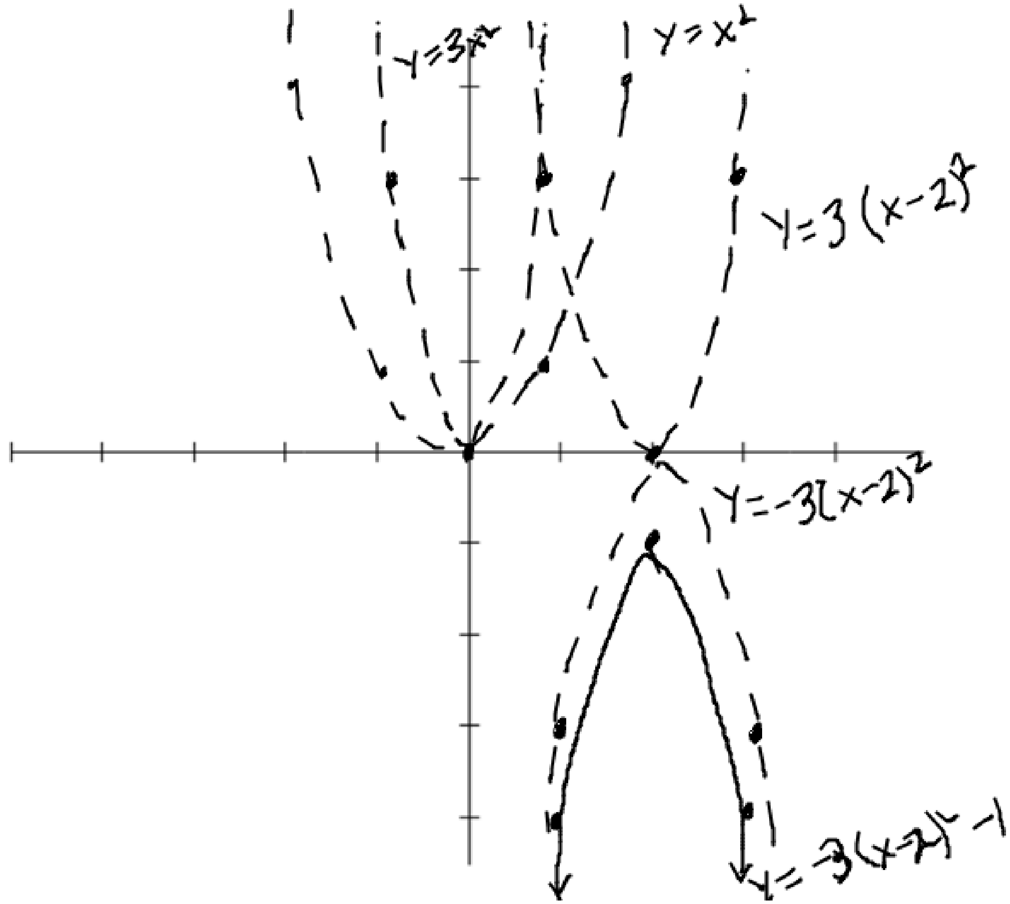
1. Vertically stretch or shrink the function.
2. Reflect the function about the *x* axis.
3. Translate the function vertically and/or horizontally.
4. Reflect the function about the *y* axis.

**Note, not all of these transformations will be presented in each problem. This is not the only order that works, but this order will get the job done with the fewest mistakes. Memorize this order!!!

Heart Rates Vary

(Do stretch/shrink first) Horizontal shifts Reflections Vertical shifts

Example 8: Sketch the graph of $f(x) = -3(x - 2)^2 - 1$.



$Y = x^2$

(3) — stretch by factor 3

(-2) — Right 2 spaces

(-) — x axis reflection

(-1) — Down 1

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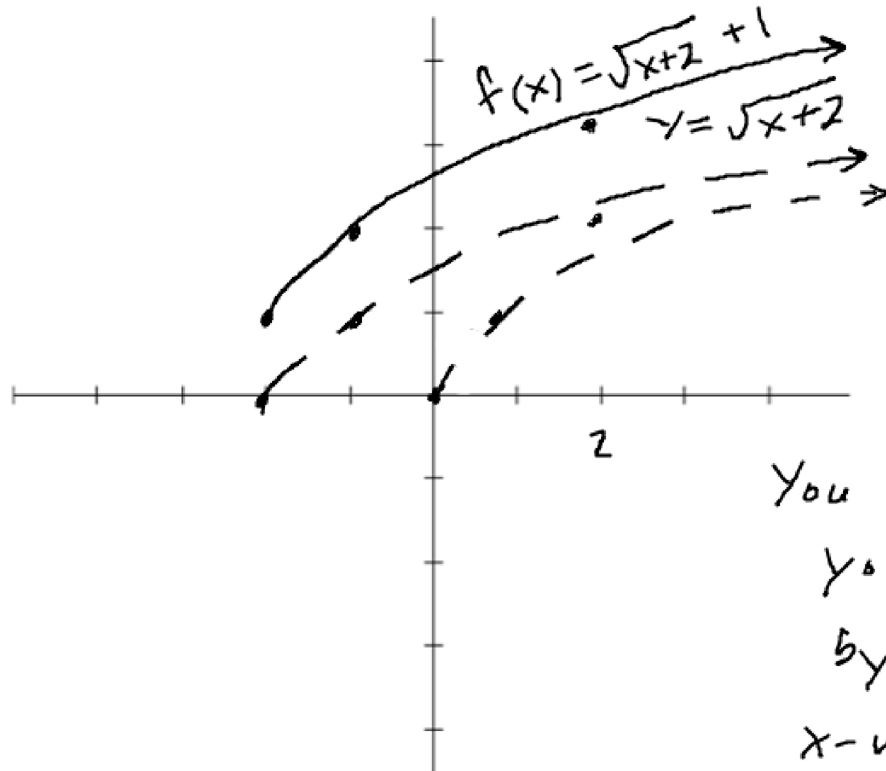
Example 9: Sketch the graph of $f(x) = \sqrt{x+2} + 1$



$y = \sqrt{x}$

left 2

up 1

You can test your final answer by plugging x-values into f(x) to get (x, y) points

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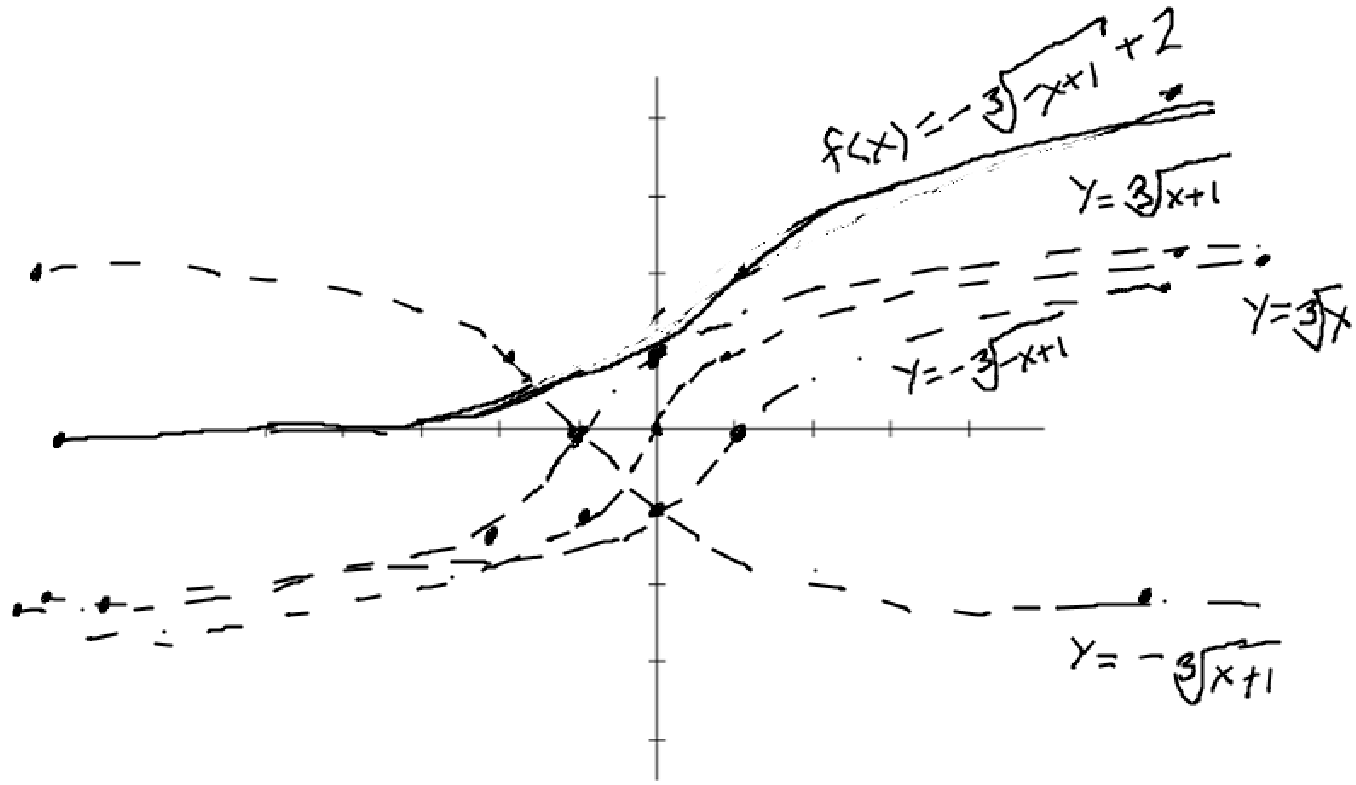
Example 10: Sketch the graph of $f(x) = -\sqrt[3]{-x+1} + 2$



$y = \sqrt[3]{x}$

Left 1

Reflection

(x-axis)

(y-axis)

Up 2

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Example 11: Describe how the graph of g is obtained from the graph of f.

1. $f(x) = \sqrt{x}$ $\qquad g(x) = \sqrt{x} - 3$

g(x) is obtained by shifting f(x) down 3 spaces.

$f(x) = x^3$ $\qquad g(x) = \overset{xR}{-}(x + \overset{HS}{2})^3 + \overset{VS}{1}$

g(x) is obtained by shifting f(x) 2 spaces to the left, reflecting in the x-axis, and shifting 1 space up.