MATH 1310

Session 5

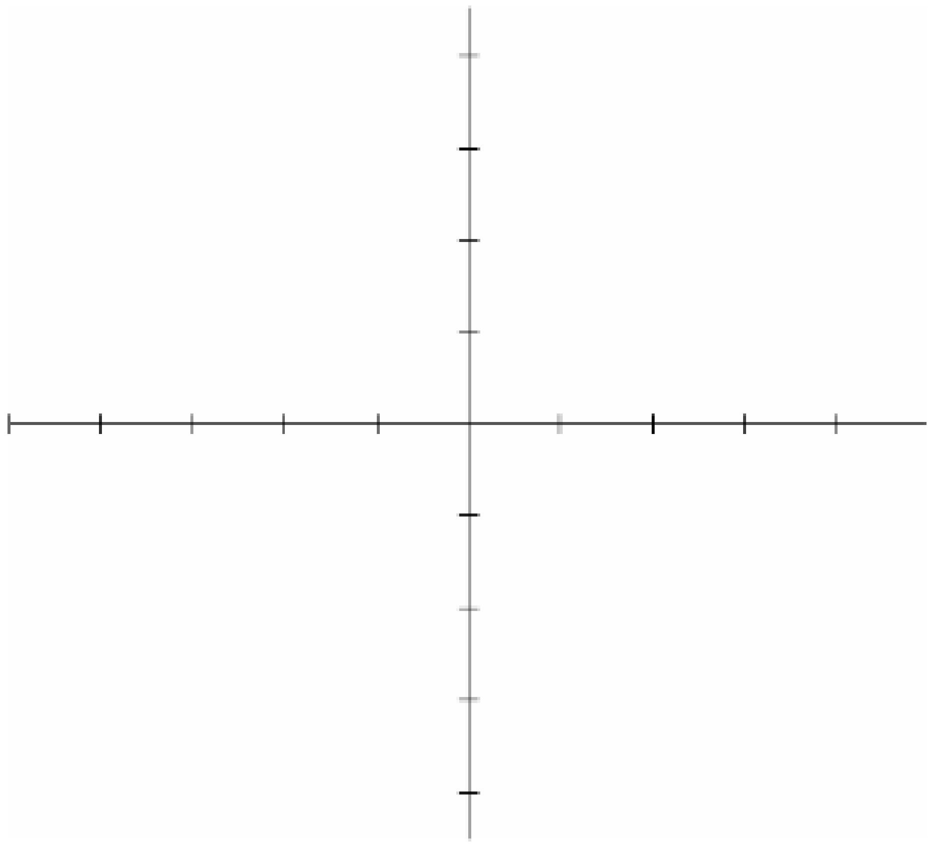
Maximum and Minimum Values

A quadratic equation is of the form $f(x) = ax^2 + bx + c$, where a, b, and c are real and $a \neq 0$

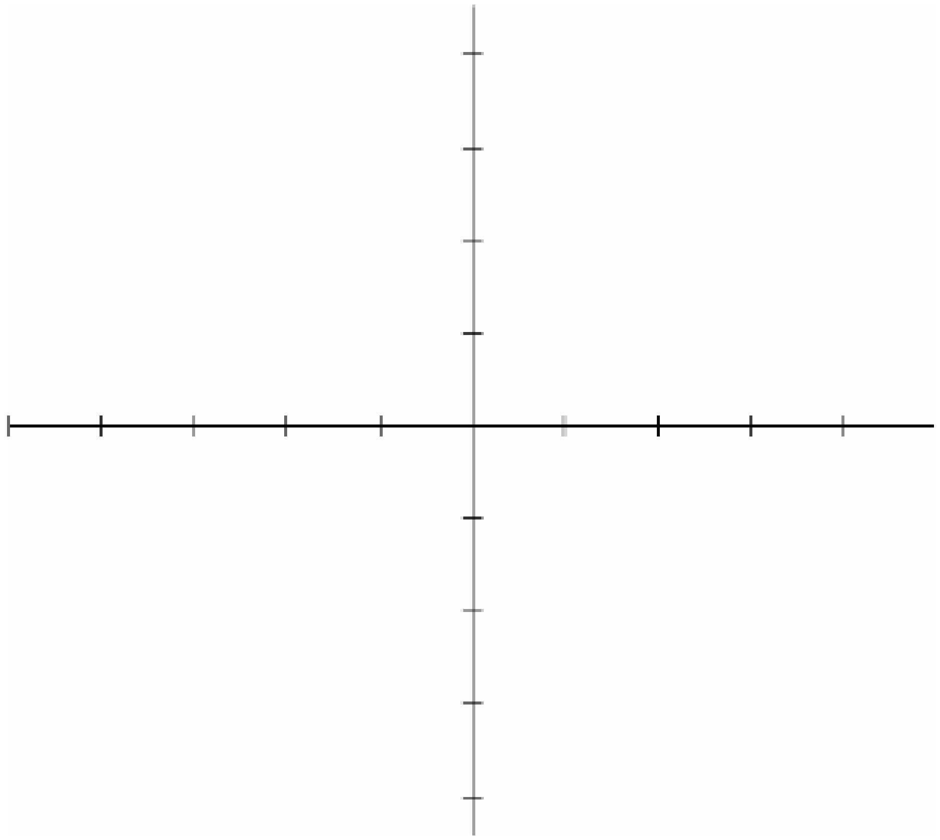
We have seen the graphs of **parabolas**.

Opening Up or Opening Down

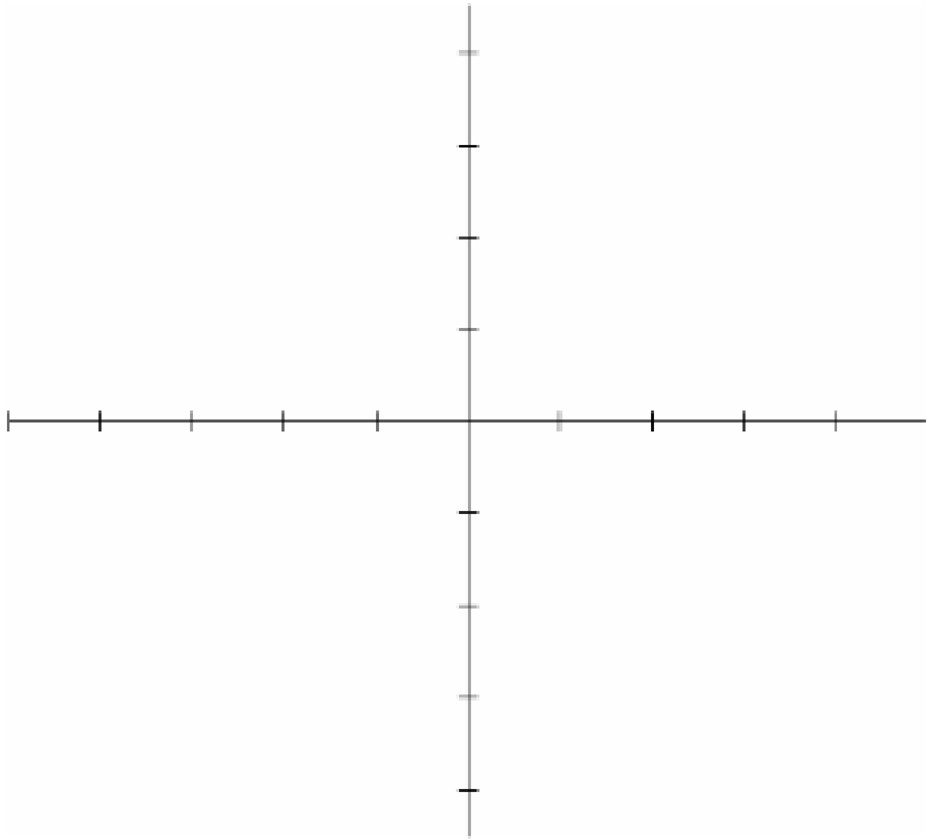
If $a > 0$ then the parabola will open upwards.

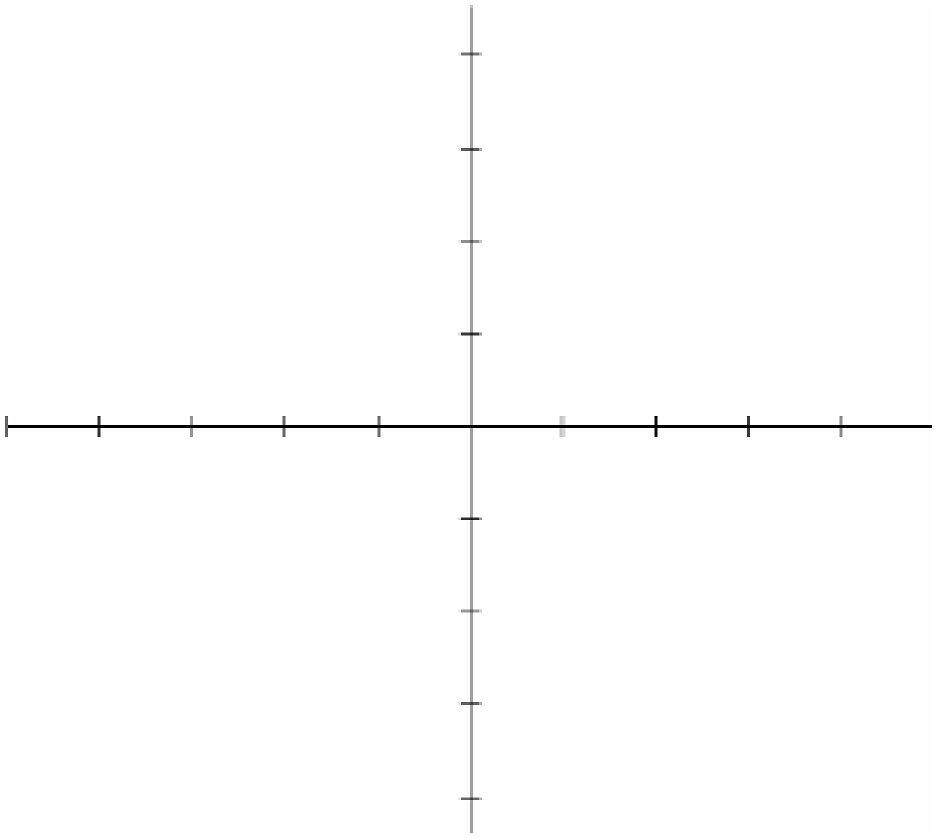


If $a < 0$ then the parabola will open downwards.



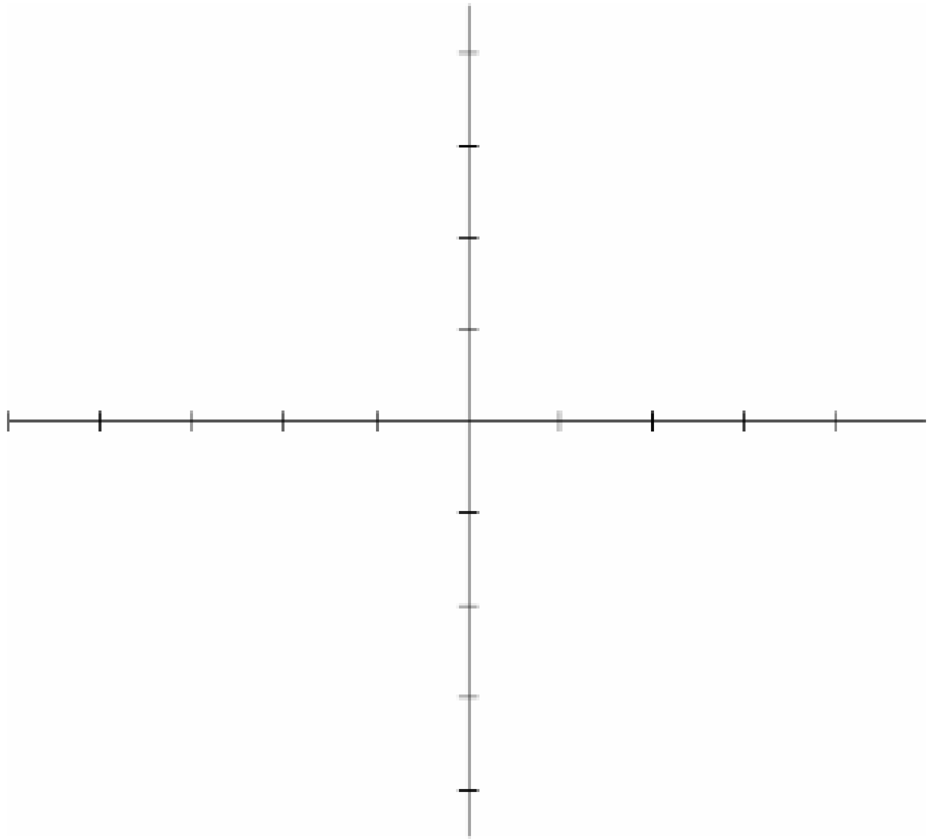
Note: The larger $|a|$, the narrower the parabola

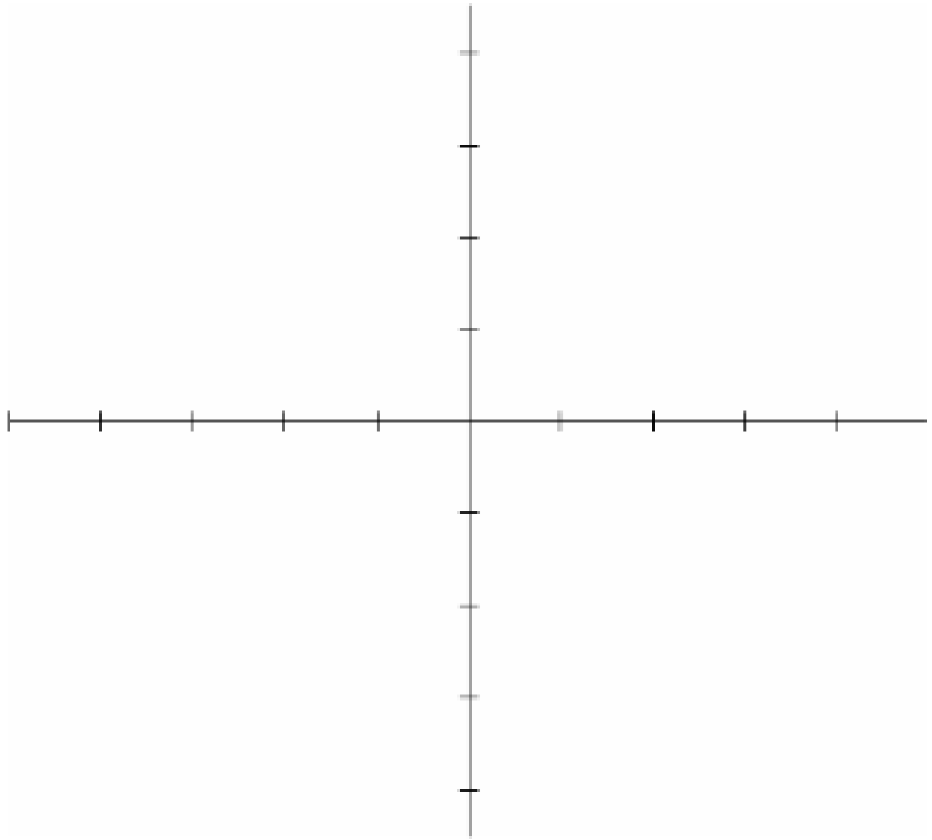




The **vertex** is the turning point of the parabola and is the **minimum point** on the graph when it opens upward and the **maximum point** on the graph when it opens downward. Every parabola has a maximum or minimum, but **NOT** both.

The **axis of symmetry** is a vertical line through the vertex that divides the graph in half.





The Standard form of a Quadratic Function

The quadratic function $f(x) = a(x-h)^2 + k$ is in **standard form**

The vertex is the point $(h,\ k)$ and the axis of symmetry is $x = h$

The domain is $(-\infty, \infty)$.

The range is $[k,\ \infty)$ if $a > 0$ or $(-\infty,\ k]$ if $a < 0$

Our first task will be to change a given quadratic function from the form $f(x) = ax^2 + bx + c$ to standard form. We'll complete the square to do this. Once the function is in standard form, we can sketch a graph using transformations and then read off the maximum or minimum value

Example 1: Write the following quadratic in standard form. Then find the vertex and the axis of symmetry.

a. $f(x) = 3x^2 - 12x - 1$

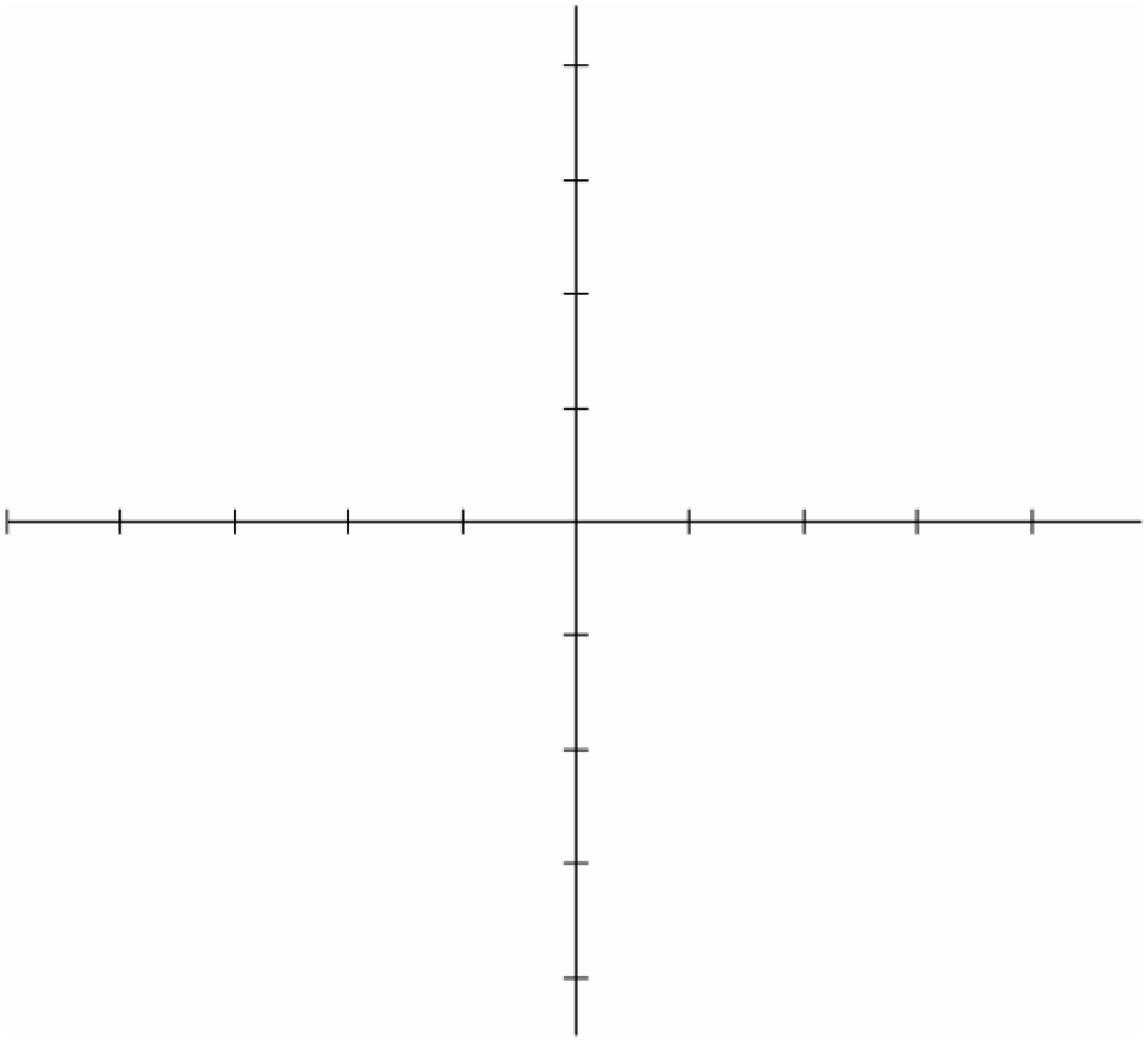
b. $f(x) = -x^2 + 2x + 3$

c. $f(x) = -10x^2 + 60x$

Graphing Quadratic Functions with Equations in Standard Form

1. Determine whether the parabola opens upward or downward.
2. Determine the vertex.
3. Find any x-intercept by replacing $f(x)$ with 0 and then solving for x.
4. Find the y-intercept by replacing x with 0.
5. Plot the intercept(s) and vertex, sketch the graph and draw the axis of symmetry.

Example 2: Sketch the graph of $f(x) = -x^2 + 2x + 3$



Shortcut:

For $f(x) = ax^2 + bx + c$, the vertex is $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$. So the axis of symmetry is $x = -\frac{b}{2a}$.

Example 3: Let $f(x) = 2x^2 + 4x + 7$. Determine, without graphing, whether the given quadratic function has a minimum or maximum value. Then find the coordinates of the minimum or maximum point

Example 4: Suppose $f\ (x) = 5x^2 - 30x + 41$. Write the equation in standard form. State the coordinates of the vertex. Determine, without graphing, whether the given quadratic function has a minimum or maximum value. Then find the coordinates of the minimum or maximum point.

Finally, given the vertex of a quadratic function and one other point that lies on the graph of the quadratic function, you should be able to write the quadratic function.

Example 5: Find a quadratic function with vertex (2, 6) which passes through (-1, 4).

Example 6: Find a quadratic function with vertex (3, -1) which passes through (5, 7).

Popper 5: $f(x) = x^2 + 2x - 3$

1. Determine the equation of the axis of symmetry:

   a. $x = -1$ b. $x = -4$ c. $x = 2$ d. $x = -\frac{1}{2}$

2. Determine the coordinates of the vertex:

   a. (-1, -3)   b. (1, 4)   c. (-1, -4)   d. (2, -3)

3. Determine the coordinates of the y-intercept:

   a. (0,3)   b. (0,2)   c. (0,-3)   d. (0,-4)

Popper 5: $f(x) = x^2 + 2x - 3$

4. Determine the coordinates of the left x-intercept:

a. (3,0) b. (-3,0) c. (2,0) d. (-1,0)

5. Determine the coordinates of the right x-intercept:

a. (3,0) b. (4,0) c. (1,0) d. (5,0)

6. Determine the direction of the parabola:

a. Opening Up b. Opening Down

7. Is the vertex a minimum or a maximum?

a. Minimum b. Maximum

Combining Functions:

Suppose we have two functions $f(x)$ and $g(x)$. The domain of $f(x)$ is the set A. The domain of $g(x)$ is the set B. We can combine these two functions together in five different ways:

Sum of Functions
Difference of Functions
Product of Functions
Quotient of Functions
Composition of Functions

Sum of Functions: $(f + g)(x) = f(x) + g(x)$ with domain $A \cap B$

Difference of Functions: $(f - g)(x) = f(x) - g(x)$ with domain $A \cap B$

Product of Functions: $(fg)(x) = f(x)g(x)$ with domain A ∩ B

Quotient of Functions: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ with domain $\{x \in A \cap B |\ g(x) \neq 0\}$

Example 1: Suppose $f(x) = 2x - 5$ and $g(x) = x^2 - 3x + 5$. Find each of the following and state the domain:

a. $(f + g)(x)$

b. $(f - g)(x)$

Example 1: Suppose $f\ (x) = 2x - 5$ and $g(x) = \ x^2 - 3x + 5$. Find each of the following and state the domain:

c. $(fg)(x)$

d. $\left(\frac{f}{g}\right)(x)$

Example 2: Given $f(x) = 3x^4 + 2x^3 - 8$ and, $g(x) = -x^2$, find each of the following functions and its domain.

a. $(g - f)(x)$

b. $\frac{f(x)}{g(x)}$

Example 3: Let $f(x) = x^2 - 3x - 1$ and $g(x) = -3x - 10$. Find

$(f + g)(1)$

$(gg)(-1)$

Composition of Functions

The composition of the function *f* with g is denoted $f \circ g$ by and is defined by the

$$(f \circ g)(x) = f\big(g(x)\big)$$

The domain of the composition $f \circ g$ is the set of all x such that

1. x is in the domain of g (the "inside" function)
2. $g(x)$ is in the domain of *f* (the "outside" function)

Example 4: Let $f(x) = x^2 + 1$ and $g(x) = -2x + 5$, find $(f \circ g)(x)$.

Example 5: Let $f(x) = \frac{1}{x}$ and $g(x) = \frac{5}{x+4}$, find $(g \circ f)(x)$

Example 6: Let $f(x) = \sqrt{4 - x^2}$ and $g(x) = \sqrt{3 - x}$, find $(f \circ g)(x)$.

Example 7: Suppose $f(x) = 3x - 5$ and $g(x) = x^2 + 4x + 3$. Find each of the following.

a. $(f \circ g)(2)$

c. $(g \circ f)(x)$

b. $(g \circ f)(-1)$

d. $(g \circ g)(0)$

Determine the value of the difference quotient for f(x) = $2x^2$ = 3x – 1

The difference quotient is:

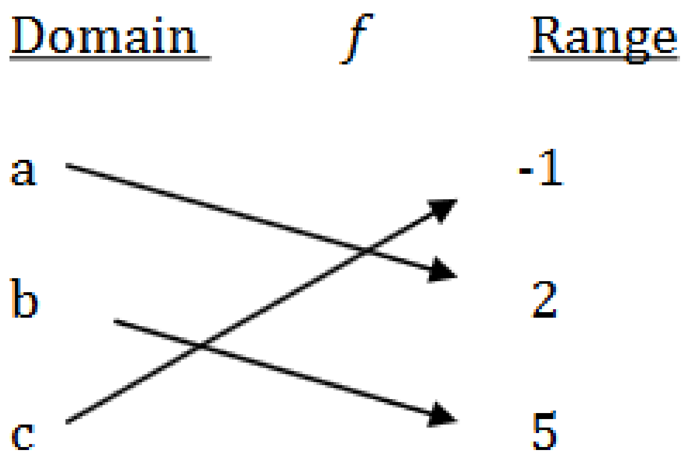
$$\frac{f(x+h)-f(x)}{h}$$

Inverse Functions

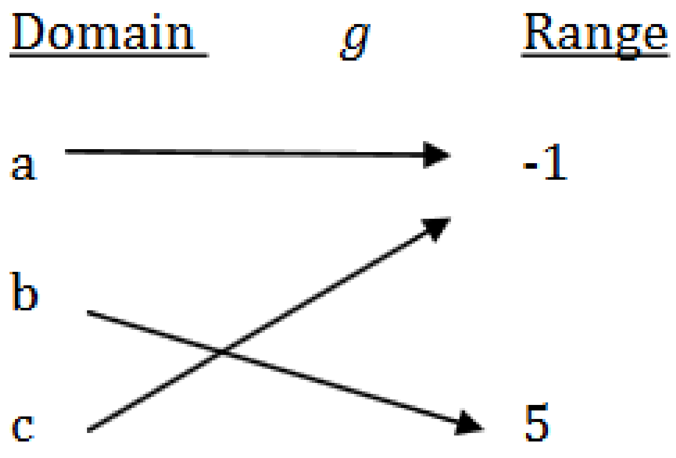
Let f be a function with domain A. f is said to be **one-to-one** if no two elements in A have the same image.

Example 1: Determine if the following function is one-to-one.

a.



b.



A one-to-one function has an inverse. The inverse function reverses whatever the first function did. These two statements mean exactly the same thing:

1. f is one-to-one (1-1)
2. f has an inverse function

The inverse of a function *f* is denoted by f^{-1}, read "*f*-inverse".

Note: $f^{-1}(x) \neq \frac{1}{f(x)}$ like $x^{-3} = \frac{1}{x^3}$

Domain and Range

Suppose f is a one-to-one function with domain A and range B. The inverse function has domain B and range A.

Example 1: Suppose f and g are inverse functions. If $f(3) = -1$ and $f(-1) = 4$, then find $g(-1)$.

Property of Inverse Functions

Let f and g be two functions such that $(f \circ g)(x) = x$ for every x in the domain of g and $(g \circ f)(x) = x$ for every x in the domain of f then **f and g are inverses of each other.**

Example 2: Show that the following functions are inverses of each other.

$f(x) = 3x + 7$ and $g(x) = \frac{x}{3} - \frac{7}{3}$

Example 3: Determine whether the following pair of functions are inverses of each other.

$f(x) = 2x - 1$ and $g(x) = \frac{x}{2} + 1$

How to find the equation of the inverse function of a one-to-one function:

1. Replace $f(x)$ by y.
2. Exchange x and y.
3. Solve for y.
4. Replace y by $f^{-1}(x)$
5. Verify.

Example 4: Write the equation of the inverse function for $f\ (x) = 3x - 3$

Example 5: Write the equation of the inverse for $f(x) = \frac{6}{4-x}$

Popper 5: Question 8:

Example 6: Write the equation of the inverse for $f(x) = (x+1)^3 + 1$

a. $\sqrt[3]{x-2}$ b. $\sqrt[3]{x-1}+1$ c. $\sqrt[3]{x-1}-1$ d. $\sqrt[3]{x-1}$

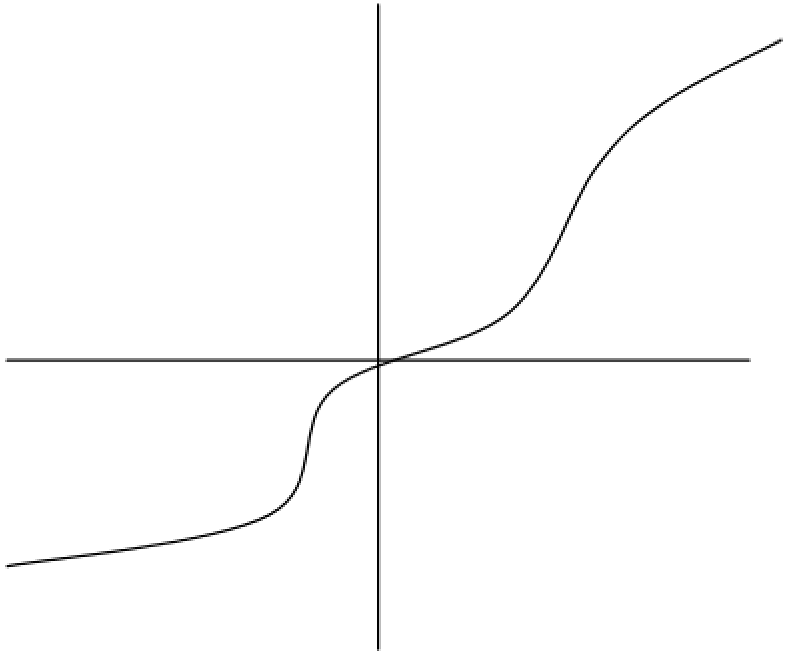
Question 9:

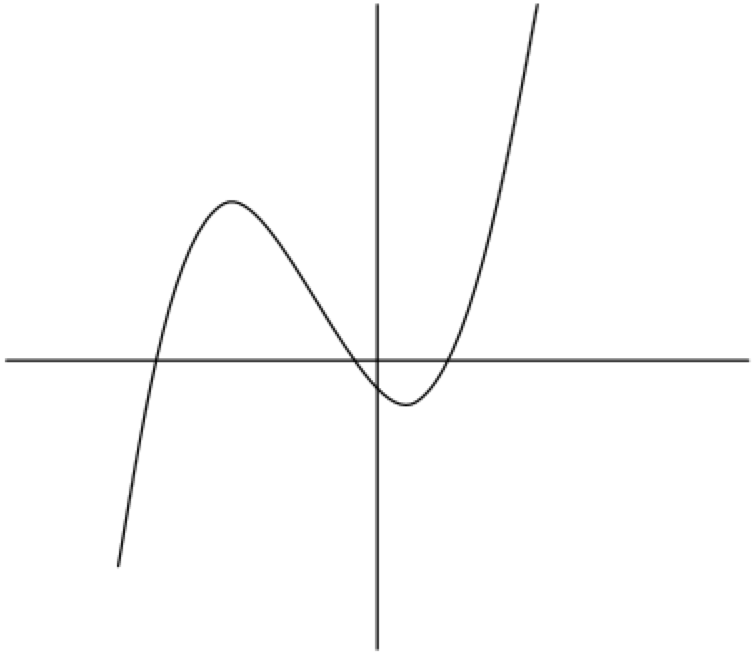
Example 7: Write the equation of the inverse for $f(x) = \sqrt[3]{x+4}$

a. $\sqrt[3]{x-4}$ b. $x^3 - 4$ c. $x^3 + 4$ d. $\sqrt[3]{x}-2$

It is easiest to determine if a function is one-to-one by looking at its graph. We can use the Horizontal Line Test to determine if a function is one-to-one.

Horizontal Line Test: A function is one-to-one if no horizontal line intersects its graph in more than one point.





Graphing the Inverse Function

Given that f is 1-1, the graph of f^{-1} is a reflection of the graph of f about the line $y = x$

Remember:

1. The inverse function reverses whatever the first function did.
2. The Domain of f becomes the Range of and the Range of f becomes the Domain of f^{-1} .