MATH 1310

Session 7

MATH 1310

Polynomial Functions:

$P(x) = 5x^3 + 2x^2 - 8$

A polynomial function is a function of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^n + \ldots + a_1 x^1 + a_n$$

where $a_n \neq 0$, $a_0, a_1, \ldots, a_n$ are real numbers and n is a whole number.

The degree of the polynomial function is n. We call the term $a_n x^n$ the leading term, and a_n is called the leading coefficient.

Degree: largest exponent

$$P(0) = a_0$$

Our objectives in working with polynomial functions will be, first, to gather information about the graph of the function and, second, to use that information to generate a reasonably good graph without plotting a lot of points. In later examples, we'll use information given to us about the graph of a function to write its equation.

Polynomial Functions:

Graph Properties of Polynomial Functions

Let P be any nth degree polynomial function with real coefficients. The graph of P has the following properties.

1. P is continuous for all real numbers, so there are no breaks, holes, jumps in the graph.
2. The graph of P is a smooth curve with rounded corners and no sharp corners.
3. The graph of P has at most n x-intercepts.
4. The graph of P has at most $n - 1$ turning points.

Degree n: At most n - x-intercepts

At most $n-1$ turning points

1 y-intercept

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Example 1: Given the following polynomial functions, state the leading term, the degree of the polynomial and the leading coefficient.

a. $P(x) = 6x^4 - 4x^3 + 7x - 2$

$6x^4$

Degree: 4th

Leading Coefficient: 6

b. $P(x) = (3x + 4)(x + 1)^2(x - 5)^3$

Multiply the "First" terms throughout

$(3x)(x)^2(x)^3 = 3x^6$

Degree: 6th

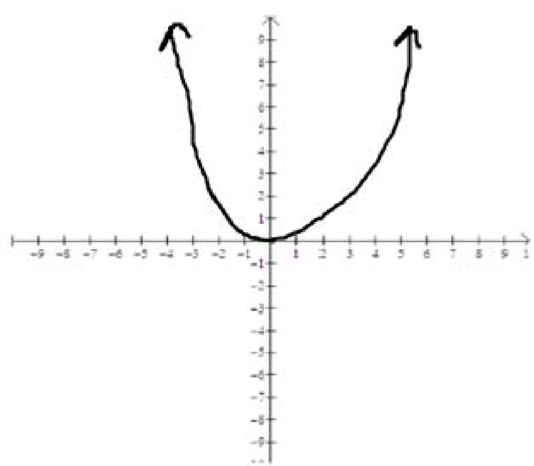
Lead coef: 3

We'll start with the shapes of the graphs of functions of the form $f(x) = x^n, n > 0$.

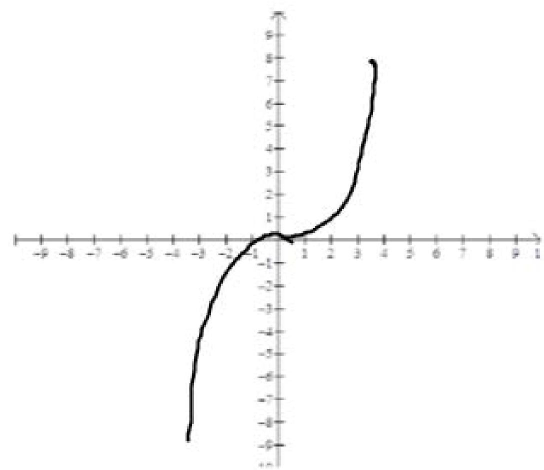
You should be familiar with the graphs of $f(x) = x^2$ and $g(x) = x^3$.

The graph of $f(x) = x^n, n > 0.$, n is even, will resemble the graph of $f(x) = x^2$, and the graph of $f(x) = x^n, n > 0$, n is odd, will resemble the graph of $f(x) = x^3$.

x^n : n is even



x^n : n is odd



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Next, you will need to be able to describe the end behavior of a function.

End Behavior of Polynomial Functions Far left, Far Right

The behavior of a graph of a function to the far left or far right is called its **end behavior.**

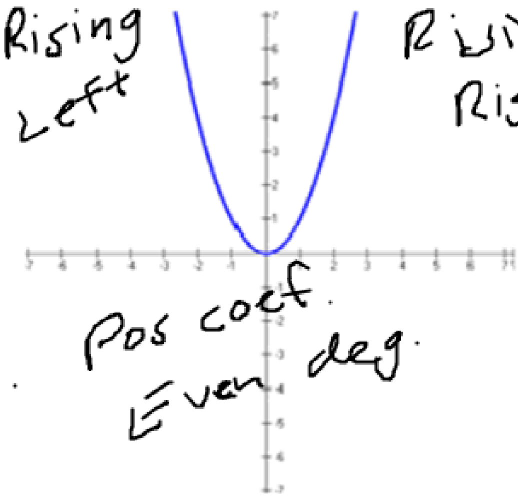
The end behavior of a polynomial function is revealed by the leading term of the polynomial function.

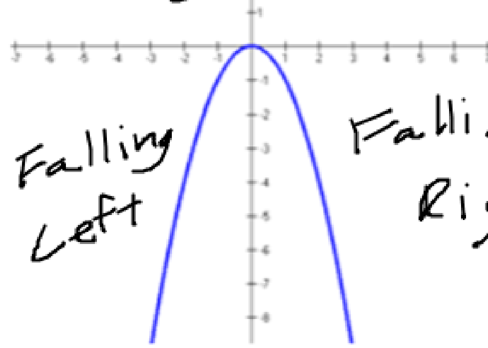
1. Even-degree polynomials look like . $y = \pm x^2$

Leading term:

Degree: even or odd

Coefficient: + or −





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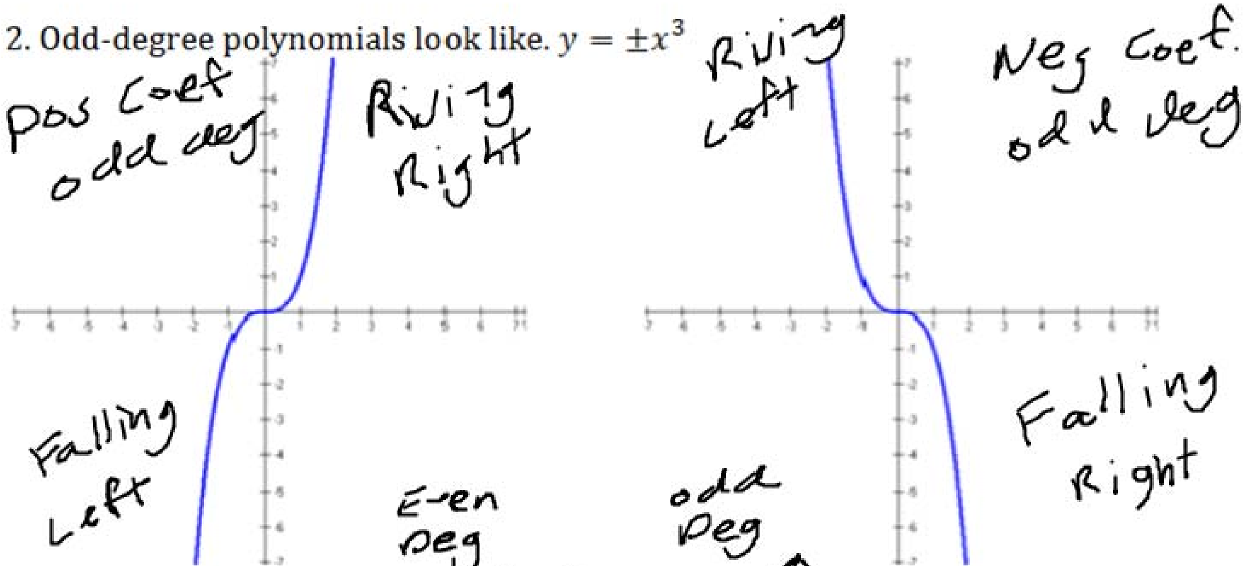
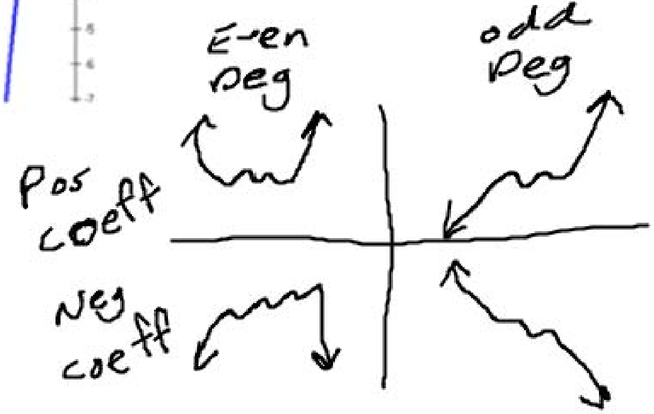
2. Odd-degree polynomials look like. $y = \pm x^3$





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Next, you should be able to find the x intercept(s) and the y intercept of a polynomial function.

Zeros of Polynomial Functions (x-intercepts)

You will need to set the function equal to zero and then use the Zero Product Property to find the x-intercept(s). That means if $ab = 0$, then either $a = 0$ or $b = 0$. To find the y intercept of a function, you will find $f(0)$.

Example 2: Find the zeros of:

a. $f(x) = x^4 - x^2$

$x^2(x^2-1)$

$x^2(x+1)(x-1)$

$x^2=0$ $\quad x+1=0$ $\quad x-1=0$

$x=0$ $\quad x=-1$ $\quad x=1$

$\{-1, 0, 1\}$

b. $f(x) = -3x\left(x + \frac{1}{2}\right)(x - 4)^3$

$-3x=0$ $\quad x+\frac{1}{2}=0$ $\quad x-4=0$

$x=0$ $\quad x=-1/2$ $\quad x=4$

$\{-1/2, 0, 4\}$

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In some problems, one or more of the factors will appear more than once when the function is factored. The power of a factor is called its multiplicity. So given $P(x) = x^3(x-3)^2(x+2)^1$, then the multiplicity of the first factor is 3, the multiplicity of the second factor is and the multiplicity of the third factor is 1.

$P(x) = x^3 \quad (x-3)^2 \quad (x+2)$

$x=0$ Mult: 3 $\quad$ $x=3$ Mult: 2 $\quad$ $x=-2$ Mult: 1

Description of the Behavior at Each *x*-intercept

1. Even Multiplicity: The graph touches the *x*-axis, but does not cross it. It looks like a parabola there.
2. Multiplicity of 1: The graph crosses the *x*-axis. It looks like a line there.
3. Odd Multiplicity greater than or equal 3: The graph crosses the *x*-axis. It looks like a cubic there.

You can use all of this information to generate the graph of a polynomial function.

- degree of the function
- end behavior of the function
- *x* and *y* intercepts (and multiplicities)
- behavior of the function through each of the *x* intercepts (zeros) of the function

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Steps to graphing other polynomials:

1. Determine the **leading term**. Is the degree even or odd? Is the sign of the leading coefficient positive or negative?
2. Determine the **end behavior**. Which one of the 4 cases will it look like on the ends?
3. Factor the polynomial.
4. Make a table listing the factors, x intercepts, multiplicity, and describe the behavior at each x intercept.
5. Find the y- intercept. $(x=0)$
6. Draw the graph, being careful to make a nice smooth curve with no sharp corners.

Note: without calculus or plotting lots of points, we don't have enough information to know how high or how low the turning points are

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Example 3: Find the x and y intercepts. State the degree of the function. Sketch the graph of $f(x) = x^3 + 4x^2 + 4x$

Deg: 3

Lead Coef: +

End Behavior:

Left: ↓ Right: ↑

$\cancel{x}(x^2 + 4x + 4)$

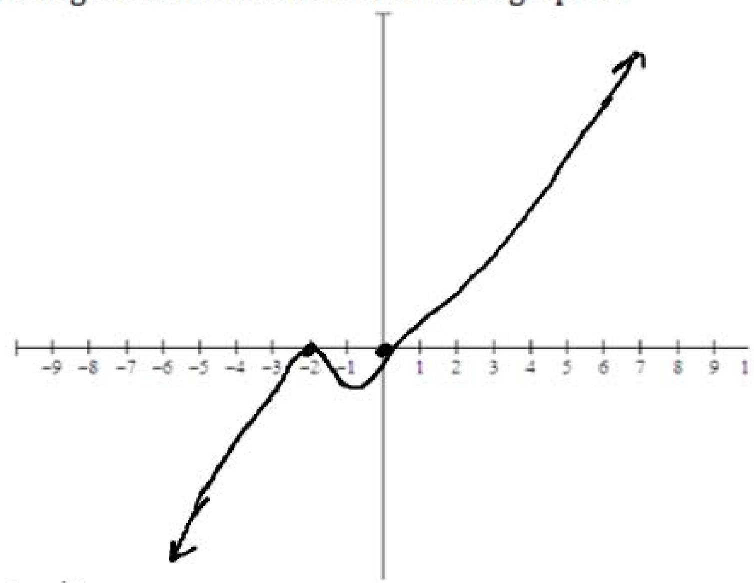
$x(x+2)(x+2)$

$x(x+2)^2$

$x = 0$ Mult 1

$x + 2 = 0$

$x = -2$ Mult 2



Y-int:

$f(0) = 0$

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Example 4: Find the x and y intercepts. State the degree of the function. Sketch the graph of $P(x) = (x-3)^2(x+1)^5(x+2)^3$

Leading Term:

$(x)^2(x)^5(x)^3 = x^{10}$

Even Degree

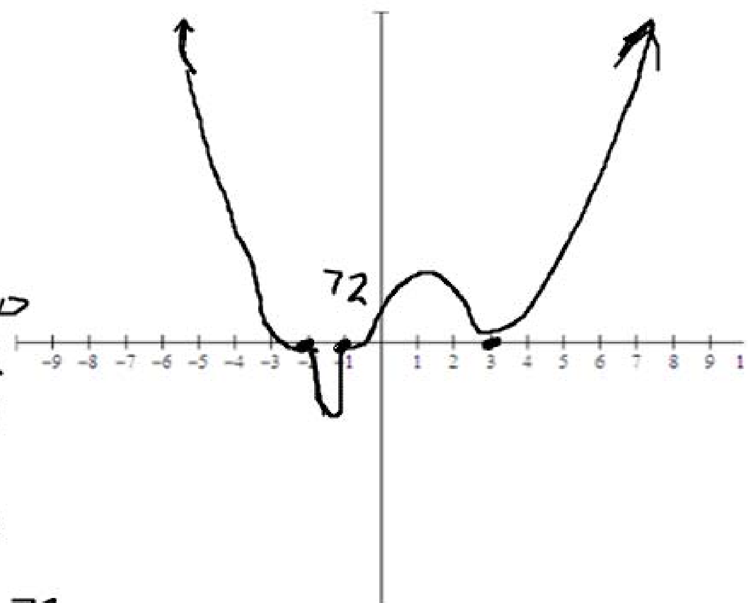
Positive Coeff

End Behavior:

L: ↑ R: ↑

x-int:

$x-3=0$	$x+1=0$	$x+2=0$
$x=3$	$x=-1$	$x=-2$
M: 2	M: 5	M: 3



y-int:

$P(0) = (-3)^2(1)^5(2)^3$

$(9)(1)(8) = 72$

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Example 5: Find the x and y intercepts. State the degree of the function. Sketch the graph of $g(x) = (3 - x)(x + 1)(x + 5)^2$

$(-x)(x)(x)^2 = -x^4$

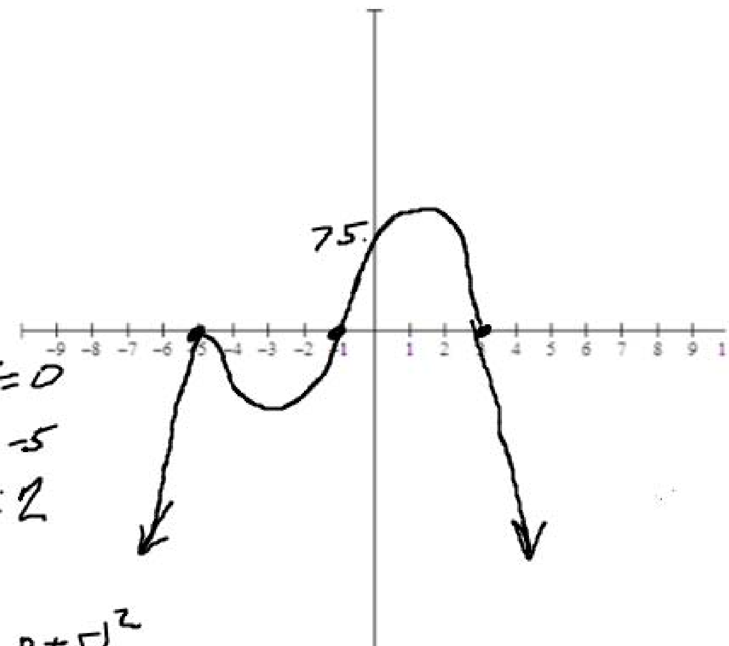
Even deg.

Neg Coeff.

End Beh: L: ↓ R: ↓

X-int:

$3 - x = 0$	$x + 1 = 0$	$x + 5 = 0$
$x = 3$	$x = -1$	$x = -5$
M: 1	M: 1	M: 2



Y-int:

$g(0) = (3-0)(0+1)(0+5)^2$

$(3)(1)(5)^2 = 75$

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Example 6: Find the x and y intercepts. State the degree of the function. Sketch the graph of $f(x) = (x-2)^3(-x+1)^2(x+5)$

$(x)^3(-x)^2(x)^1 = x^6$

Even deg.

pos Coef.

End Behav: L ↑ R ↑

x-int:

$(x-2)=0$

$x=2$

M: 3

$-x+1=0$

$-x=-1$

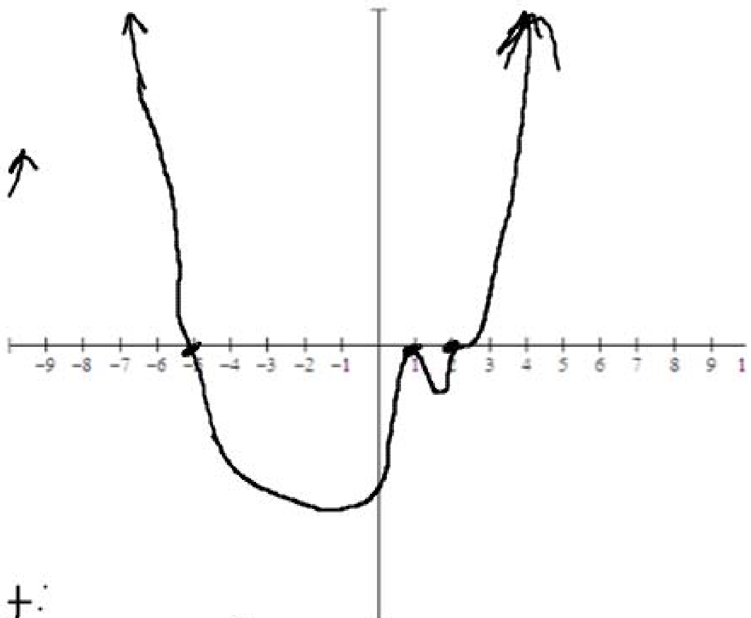
$x=1$

M: 2

$x+5=0$

$x=-5$

M: 1



y-int:

$f(0) = (0-2)^3(-0+1)^2(0+5)$

$-8(5) = -40$

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Deg: 3

Example 7: Write the equation of the cubic polynomial $P(x)$ that satisfies the following conditions: zeros at $x = 3$, $x = -1$, and $x = 4$ and passes through the point (-3, 7)

$x = 3 \qquad x = -1 \qquad x = 4$

$x - 3 = 0 \qquad x + 1 = 0 \qquad x - 4 = 0$

$P(x) = a(x-3)(x+1)(x-4)$

$7 = a(-3-3)(-3+1)(-3-4)$

$$\frac{\cancel{7}}{\cancel{7}} = \frac{a(-6)(-2)(\overset{-1}{\cancel{-7}})}{\cancel{7}}$$

$1 = a(-6)(-2)(-1)$

$1 = -12a \qquad a = \frac{-1}{12}$

$P(x) = \frac{-1}{12}(x-3)(x+1)(x-4)$

Example 8: Write the equation of the quartic function with y intercept 4 which is tangent to the x axis at the points (-1, 0) and (1, 0).

(appea-ing quaxratic) → 4th degree → (0, 4)

$x = -1$ $\quad$ $x = +1$

$x + 1 = 0$ $\quad$ $x - 1 = 0$

$M: 2$ $\quad$ $M: 2$

$P(x) = a(x+1)^2(x-1)^2$

$4 = a(0+1)^2(0-1)^2$

$4 = a$

$P(x) = 4(x+1)^2(x-1)^2$

$P(x) = 4((x+1)(x-1))^2$

$P(x) = 4(x^2-1)^2$

$P(x) = 4(x^2-1)(x^2-1)$

$P(x) = 4(x^4-x^2-x^2+1)$

$P(x) = 4(x^4-2x^2+1)$

$P(x) = 4x^4-8x^2+4$

Dividing Polynomials

In this section, you'll learn two methods for dividing polynomials, long division and synthetic division. You'll also learn two theorems that will allow you to interpret results when you divide.

Suppose $P(x)$ and $D(x)$ are polynomial functions and $D(x) \neq 0$. Then there are unique polynomials $Q(x)$ (called the quotient) and $R(x)$ (called the remainder) such that $P(x) = D(x) * Q(x) + R(x)$.

We call $D(x)$ the divisor. The remainder function, $R(x)$, is either 0 or of degree less than the degree of the divisor.

You can find the quotient and remainder using long division. Recall the steps you learned in elementary school to perform long division:

Dividing Polynomials

Example 1: Divide

$$\frac{x^6 + 4x^4 + 4x^2 + 16}{x^2 + 4} \to \frac{x^6 + 0x^5 + 4x^4 + 0x^3 + 4x^2 + 0x + 16}{x^2 + 4}$$

$$\begin{array}{r|l} & x^4 + 4 \\ \hline x^2 + 4 & x^6 + 0x^5 + 4x^4 + 0x^3 + 4x^2 + 0x + 16 \\ & -(x^6 + 4x^4) \\ \hline & 4x^2 + 0x + 16 \\ & -(4x^2 + 16) \\ \hline & 0 \end{array}$$

$Q(x) = x^4 + 4$

$R(x) = 0$

Example 2: Divide

$$\frac{12x^3 - x^2 - x}{3x - 1} \rightarrow \frac{12x^3 - x^2 - x + 0}{3x - 1}$$

$$\begin{array}{r|l} & 4x^2 + x \\ \hline 3x - 1 & 12x^3 - x^2 - x + 0 \\ & -(12x^3 - 4x^2) \\ \hline & 3x^2 - x + 0 \\ & -(3x^2 - x) \\ \hline & 0 \end{array}$$

$Q(x) = 4x^2 + x$

$R(x) = 0$

Popper 7 Question 1:

Example 3: If $D(x) = 2x - 5, Q(x) = 3x^2 + 5$ and $R(x) = 12$, find $P(x)$.

a. $6x^3 - 15x^2 + 34x - 85$

b. $6x^3 - 15x^2 + 10x - 13$

c. $3x^2 + 2x + 12$

d. $6x^3 - 15x^2 + 10x - 25$

$P(x) = D(x) \cdot Q(x) + R(x)$

Often it will be more convenient to use synthetic division to divide polynomials. This method is easy to use, as long as your divisor is $x \pm c$, for any real number c.

Dividing Polynomials Using Synthetic Division

Example 4: Divide using synthetic division

$$\frac{6x^4 + x^3 - 10x^2 + 9}{x - 3} \rightarrow \frac{6x^4 + x^3 - 10x^2 + 0x + 9}{x - 3}$$

→ Add the column

→ multiply by outside number

opposite

$$\begin{array}{c|ccccc} & 6 & 1 & -10 & 0 & 9 \\ 3 & \downarrow & 18 & 57 & 141 & 423 \\ \hline & 6 & 19 & 47 & 141 & 432 \end{array}$$

→ Remainder

$$\begin{array}{r} 141 \\ 3 \\ \hline 423 \end{array}$$

$Q(x) = 6x^3 + 19x^2 + 47x + 141$

$R(x) = 432$

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Example 5: Divide using synthetic division

$$\frac{x^3 + 5x^2 - 7x + 2}{x - 1}$$

$$\begin{array}{r|rrrr} 1 & 1 & 5 & -7 & 2 \\ & \downarrow & 1 & 6 & -1 \\ \hline & 1 & 6 & -1 & 1 \end{array}$$

$Q(x) = x^2 + 6x - 1$

$R(x) = 1$

Popper 7 Question 2:

Example 6: Divide using synthetic division

$$\frac{x^3 + 8}{x + 2}$$

a. $x^2 - 2x + 4$

b. $x^2 + 2x - 4$

c. $x^2 - 2x + 4$, R 6

d. $x^2 + 2x + 4$, R 16

Here are two theorems that can be helpful when working with polynomials:

The Remainder Theorem: If $P(x)$ is divided by $x - c$, then the remainder is $P(c)$.

The Factor Theorem: c is a zero of $P(x)$ if and only if $x - c$ is a factor of $P(x)$, that is if the remainder when dividing by $x - c$ is zero.

You can use synthetic division and the remainder theorem to evaluate a function at a given value.

Remainder Theorem: shortcut of evaluating $P(c)$

Factor Theorem: If $R(x) = 0$, $x-c$ is a factor of $P(x)$.

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Example 7: Use synthetic division and the remainder theorem to find P(3) for
$P(x) = 2x^3 - 5x^2 + 4x + 3$

$$\begin{array}{r|rrrr} 3 & 2 & -5 & 4 & 3 \\ & \downarrow & 6 & 3 & 21 \\ \hline & 2 & 1 & 7 & 24 \end{array}$$

$R(x) = 24$

Meaning $P(3) = 24$

$P(3) = 2(3)^3 - 5(3)^2 + 4(3) + 3 = 24$

Example 8: Determine if $x + 2$ is a factor of $P(x) = x^3 + 6x^2 + 3x - 10$. Yes

$$\begin{array}{r|rrrr} & 1 & 6 & 3 & -10 \\ -2 & \downarrow & -2 & -8 & 10 \\ \hline & 1 & 4 & -5 & 0 \end{array} \rightarrow R(x) = 0$$

$P(x) = (x+2)(x^2+4x-5)$

$P(x) = (x+2)(x+5)(x-1)$

$x = -2 \quad x = -5 \quad x = 1$: x-intercepts

And you may also need to work backwards.

Example 9: Find a polynomial with a degree of 4 with zeros at -3, 0, 2, 5.

$x = -3 \qquad x = 0 \qquad x = 2 \qquad x = 5$

$x + 3 = 0 \qquad x = 0 \qquad x - 2 = 0 \qquad x - 5 = 0$

$$P(x) = (x+3)(x)(x-2)(x-5)$$

$$= (x^2 + 3x)(x^2 - 7x + 10)$$

$$x^4 - 7x^3 + 10x^2 + 3x^3 - 21x^2 + 30x$$

$$P(x) = x^4 - 4x^3 - 11x^2 + 30x$$

Popper 7 Question 3:

Example 10: Find a polynomial of degree 3 with zeros at 0, 2 and -3.

a. $p(x) = x\,(x + 2)\,(x - 3)$

b. $p(x) = x\,(2x)(-3x)$

c. $p(x) = x\,(x + 2)^3(x - 3)^3$

d. $p(x) = x\,(x - 2)(x + 3)$

Roots of Polynomials (Zeroes, x - intercepts)

You'll need to be able to find all of the zeros of a polynomial. You'll now be expected to find both real and complex zeros of a function.

A polynomial of degree n has exactly n zeros, counting all multiplicities.

To find all zeros, you'll factor completely. From the factored form of your polynomial, you'll be able to read off all the zeros of the function.

If c is a zero of a polynomial P, then $x = c$ is a **root** of the equation $P(x) = 0$.

If your polynomial has real coefficients, then the polynomial may have complex roots. Complex roots occur in pairs, called complex conjugate pairs. This means that if $a + bi$ is a root of P then so is $a - b$.

Note: $a^2 + b^2 = (a + bi)(a - bi)$

$$x^2 + 9 = (x+3i)(x-3i)$$

Roots of Polynomials

Example 1: Find the zeros of the polynomial write the polynomial in factored form and then state the multiplicity of each zero. (Sometimes it may be easier to factor the polynomial first, then find the zeros.)

a. $f(x) = x^2 - 6x + 9$

$(x-3)(x-3)$

$f(x) = (x-3)^2$

$x - 3 = 0$

$x = 3$, Mult: 2

b. $f(x) = x^2 + x - 12$

$(x+4)(x-3)$

$x+4=0$ $\quad$ $x-3=0$

$x=-4$ $\quad$ $x=3$

Mult: 1 $\quad$ Mult: 1

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Popper 7:

Find all the roots of the following:

Question 4:

$f(x) = 9x^2 + 36$

a. 3, 6 b. 3, 6, 3i, 6i c. 2i, -2i d. 2, -2

Question 5:

$f(x) = x^3 - 4x^2 + x - 4$

a. 4 b. 4, i, -i c. -1, 1, 4 d. -4i, 1, 4i

You can also work backwards to writing a polynomial with integer coefficients that meets stated conditions.

Example 2: Find a 3rd degree polynomial with integer coefficients given -5, and *i* are zeros

$x = -5 \qquad x = i \qquad x = -i$

$P(x) = (x+5) \quad (x-i)\,(x+i)$

$P(x) = (x+5)\,(x^2 - i^2)$

$P(x) = (x+5)\,(x^2+1)$

$P(x) = x^3 + 5x^2 + x + 5$

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Example 3: Find a polynomial with integer coefficients given the zeros at 2 and 2 - 5*i*.

$x = 2$ $\quad$ $x = 2 - 5i$ $\quad$ $x = 2 + 5i$

$$P(x) = (x-2)\,(x - (2-5i))(x - (2+5i))$$

$$P(x) = (x-2)\,(x^2 - (2+5i)x - (2-5i)x + (2-5i)(2+5i))$$

$$P(x) = (x-2)\,(x^2 - 2x \cancel{-5ix} - 2x \cancel{+5ix} + 4 \cancel{+10i} \cancel{-10i} - 25i^2)$$

$\hookrightarrow +25$

$$P(x) = (x-2)(x^2 - 4x + 29)$$

$$P(x) = x^3 - 4x^2 + 29x - 2x^2 + 8x - 58$$

$$P(x) = x^3 - 6x^2 + 37x - 58$$

Example 4: Write a polynomial with integer coefficients with degree 4 and zeros at -3 (multiplicity 2) and - $3i$.

$x = -3 \qquad x = -3i \qquad x = 3i$

$P(x) = (x+3)^2 (x+3i)(x-3i)$

$P(x) = (x+3)^2 (x^2+9)$

$P(x) = (x^2+6x+9)(x^2+9)$

$P(x) = x^4 + 6x^3 + 9x^2 + 9x^2 + 54x + 81$

$P(x) = x^4 + 6x^3 + 18x^2 + 54x + 81$

Example 5: Write a polynomial with integer coefficients with degree 3 and zeros at 5 and $4 + i$ with a constant coefficient of 170. $(-85)(-2) = 170$

$x = 5 \qquad x = 4+i \qquad x = 4-i$

$$P(x) = (x-5)(x-(4+i))(x-(4-i))$$

$$P(x) = (x-5)(x^2-(4-i)x-(4+i)x+(4+i)(4-i))$$

$$P(x) = (x-5)(x^2-4x+\cancel{ix}-4x-\cancel{ix}+16+1)$$

$$P(x) = (x-5)(x^2-8x+17)$$

$$P(x) = x^3-8x^2+17x-5x^2+40x-85$$

$$P(x) = -2(x^3-13x^2+57x-85)$$

$$P(x) = -2x^3+26x^2-114x+170$$

Rational Functions

The objective in this section will be to identify the important features of a rational function and then to use them to sketch an accurate graph of the function.

A **rational function** can be expressed as $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial functions and $q(x) \neq 0$.

Example 1: Find the domain of $f(x) = \frac{x-2}{x^2-9}$

Denominator:

$x^2 - 9 = 0$

$x^2 = 9$

$x = \pm 3$

Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

Rational Functions

Vertical Asymptote of Rational Functions

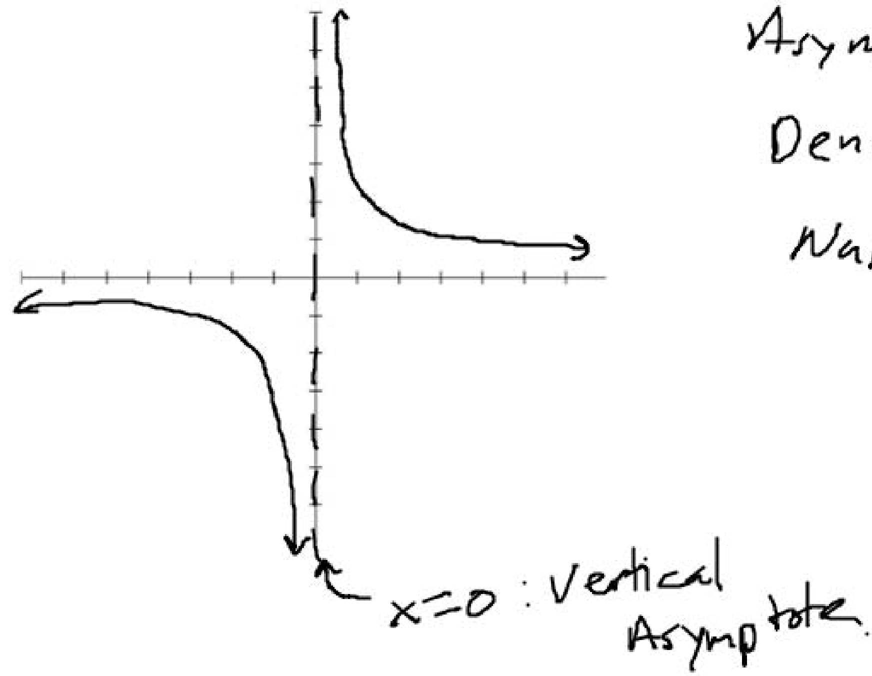
The line $x = a$ is a **vertical asymptote** of the graph of a function f if $f(x)$ increases or decreases without bound as x approaches a.

Basic example is $f(x) = \frac{1}{x}$

Find Vertical Asymptote:

Denominator = 0

Numerator ≠ 0



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Factor the numerator and denominator. Look at each factor in the denominator.

- If a factor cancels with a factor in the numerator, then there is a hole where that factor equals zero.
- If a factor does not cancel, then there is a vertical asymptote where that factor equals zero.

Example 2: Find any vertical asymptote(s) and/or hole(s) of $f(x) = \frac{x^2 - 3x - 10}{x^2 - x - 6} = \frac{(x-5)(x+2)}{(x-3)(x+2)}$

$x^2 - 3x - 10$

$(x-5)(x+2)$

$x^2 - x - 6$

$(x-3)(x+2)$

Hole: Den and Num

$x+2=0$

$x=-2$

Vertical Asymptote: Den Only

$x-3=0$

$x=3$

x-int: Num Only

$x-5=0$

$x=5$

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Example 3: Find any vertical asymptote(s) and/or hole(s) of

$$f(x) = \frac{x^2 + 3x - 4}{x^2 - 9} = \frac{(x+4)(x-1)}{(x+3)(x-3)}$$

No Holes.

Vertical Asymptote:

$x+3=0$ $\qquad$ $x-3=0$

$x=-3$ $\qquad$ $x=3$

Example 4: Find any vertical asymptote(s) and/or hole(s) of

$$f(x) = \frac{x^2 - 16}{x^2 - 2x - 8} = \frac{(x+4)(x-4)}{(x-4)(x+2)}$$

Holes: $x - 4 = 0$

$x = 4$

Vertical Asymptote:

$x + 2 = 0$

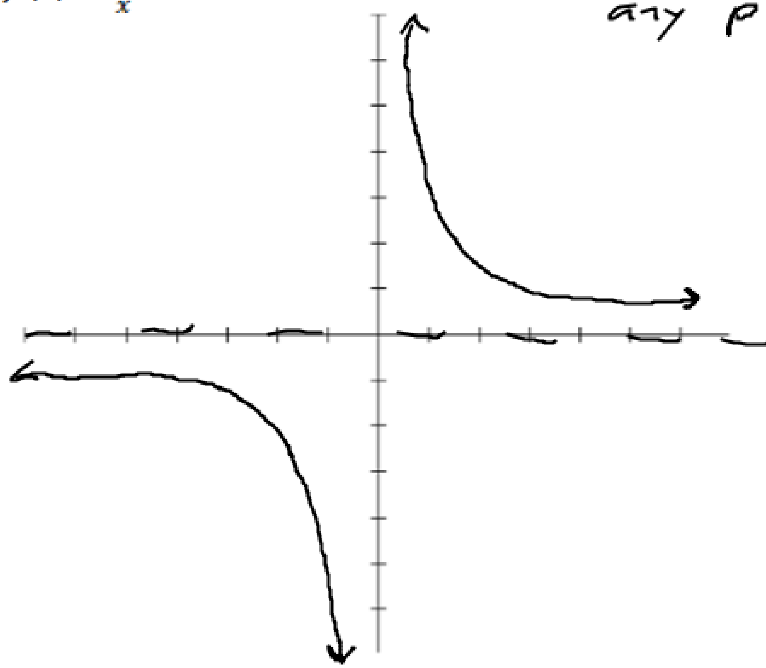
$x = -2$

Horizontal Asymptote of Rational Functions

The line $y = b$ is a **horizontal asymptote** of the graph of a function f if $f(x)$ approaches b as x increases or decreases without bound.

Again we look at $f(x) = \frac{1}{x}$

Does the graph level out to any particular y-value



$y = 0$

Horizontal Asymptote

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Horizontal asymptotes really have to do with what happens to the y-values as x becomes very large or very small. If the y-values approach a particular number at the far left and far right ends of the graph, then the function has a horizontal asymptote.

Note: A rational function may have several vertical asymptotes, but only at most one horizontal asymptote. Also, a graph cannot cross a vertical asymptote, but may cross a horizontal asymptote.

Locating Horizontal Asymptotes

To find the location of any horizontal asymptote, determine the degree of the numerator and the degree of the denominator. Then

- If the degree of the numerator is less than the degree of the denominator, the horizontal asymptote is $y = 0$.
- If the degree of the numerator is the same as the degree of the denominator, then the horizontal asymptote is $y = \frac{a}{b}$, where a and b are the leading coefficients of the numerator and denominator
- If the degree of the numerator is greater than the degree of the denominator, then there is no horizontal asymptote.

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To summarize:

$$\frac{x^3+2x^2-1}{x+5}$$

$$\frac{Larger\ Degree}{Smaller\ Degree}: HA\ does\ not\ exist$$

$$\frac{x+5}{x^2+5x-6}$$

$$\frac{Smaller\ Degree}{Larger\ Degree}: HA\ is\ at\ y = 0$$

$$\frac{Equal\ Degree}{Equal\ Degree}: HA\ exists: y = (quotient\ of\ leading\ coefficients)$$

$$\frac{2x^2+5}{3x^2+2x-1}$$

$$y = \frac{2}{3}$$

To summarize:

Example 5: Find the horizontal, if there is one

$$f(x) = \frac{x+2}{x^2+6x+9}$$

Larger exponent in the denominator.

HA: $y = 0$

Example6: Find the horizontal, if there is one

$$f(x) = \frac{3x^4 + 12x^2 + 12}{1x^4 + 7x^3 + 10}$$

Equal Exponents in Numerator and the Denominator

H.A.

$$Y = \frac{3}{1} = 3$$

Example 7: Find the horizontal, if there is one

$$f(x) = \frac{x^2 + 3x + 2}{3x + 6}$$

Numerator has the

larger degree.

HA does not exist

Steps to Graphing a Rational Function

1. Factor numerator and denominator. If a factor in the numerator cancels with a factor in the denominator then there is a **hole** in the graph when that cancelled factor equal zero.
2. Find x-intercept(s) by setting numerator equal to zero.
3. Find y-intercept (if there is one) by substituting $x = 0$ in the function.
4. Find horizontal asymptote (if there is one).
5. Find vertical asymptote, if any, by setting the denominator equal to zero.
6. Use the x-intercept(s) and vertical asymptote(s) to divide the x-axis into intervals. Choose a test point in each interval to determine if the function is positive or negative there. This will tell you whether the graph approaches the vertical asymptote in an upward or downward direction.
7. Graph! *Except for the breaks at the vertical asymptotes, the graph should be a nice smooth curve with no sharp corners.*

Example 8: Sketch the graph of

$f(x) = \dfrac{1}{x-1}$

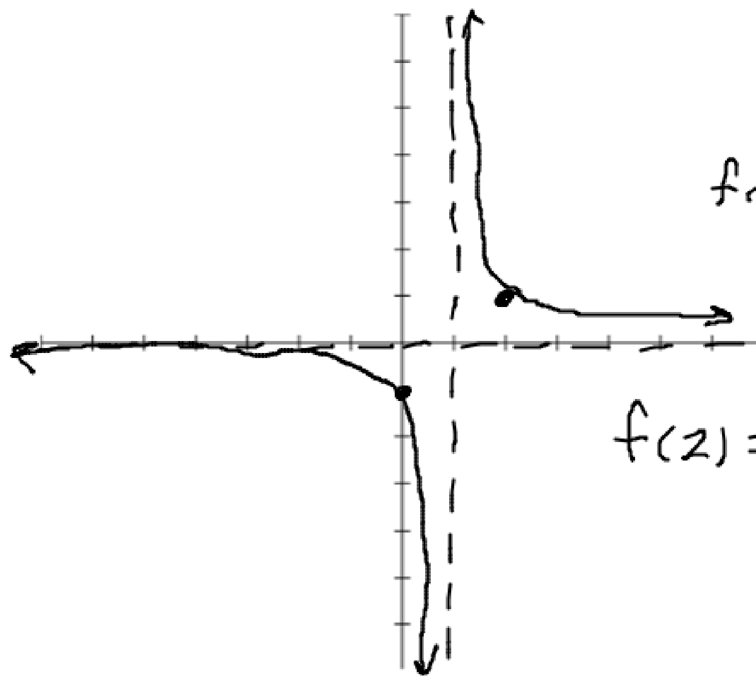
Denominator:

$x - 1 = 0$

VA: $x = 1$

Bigger degree in denominator:

HA: $y = 0$



Numerator $= 0$

$1 \neq 0$

No x-int.

$f(0) = \dfrac{1}{0-1} = -1$

y int: $(0, -1)$

$f(2) = \dfrac{1}{2-1} = \dfrac{1}{1} = 1$

$(2, 1)$

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Example 9: Sketch the graph of

$$f(x) = \frac{x^2 + x - 6}{x^2 + 3x - 10} = \frac{(x+3)(x-2)}{(x+5)(x-2)}$$

N:D:

$x - 2 = 0$

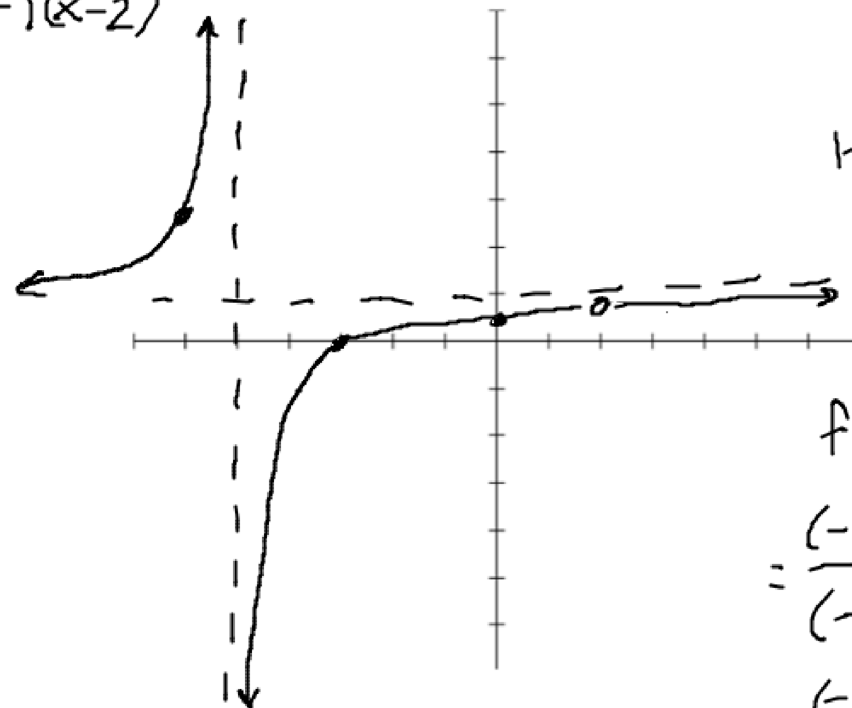
$x = 2$: Hole

D: $x + 5 = 0$

$x = -5$: VA

N: $x + 3 = 0$

$x = -3$: x-int



$f(0) = \frac{-6}{-10} = \frac{3}{5}$

y-int

HA: equal exp.

$y = \frac{1}{1} = 1$

$f(-6) =$

$= \frac{(-6+3)(-6-2)}{(-6+5)(-6-2)}$

$= \frac{(-3)\cancel{(-8)}}{(-1)\cancel{(-8)}} = 3$

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Example 10: Sketch the graph of

$f(x) = \frac{1x^2 + 5}{1x^2 - 4} = \frac{(x^2+5)}{(x+2)(x-2)}$

N:D: No Holes

D: $x+2=0$ $\quad$ $x-2=0$

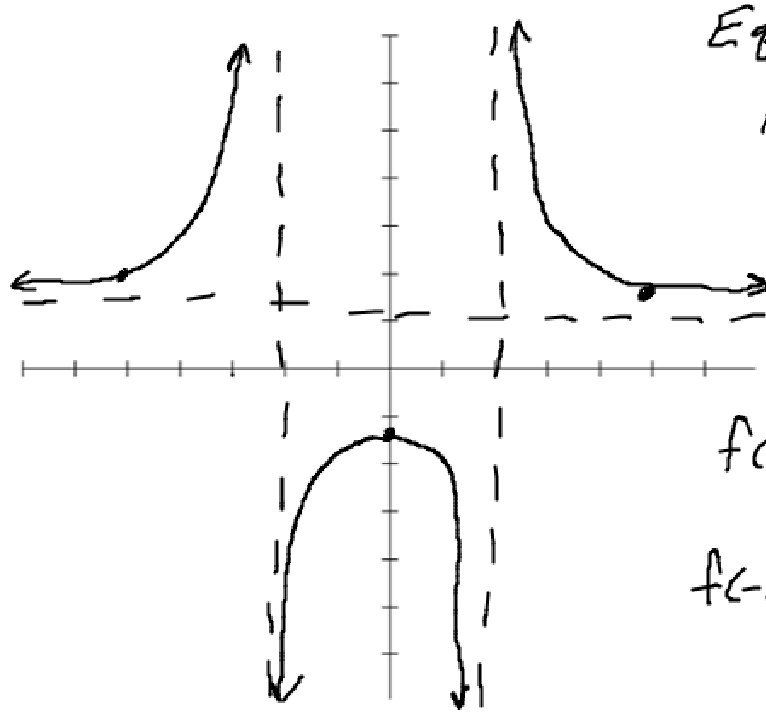
$x=-2$ $\quad$ $x=2$: VA

N: $x^2+5=0$

$x^2=-5$

$x=\pm\sqrt{-5}$

No x-int.



$f(0) = \frac{5}{-4} = -1.25$

Equal Exp:

HA: $y=1$

$f(5) = \frac{25+5}{25-4} = \frac{30}{21}$

$f(-5) = \frac{30}{21}$

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Popper 7: $f(x) = \frac{2x^2-4x-70}{x^2-49}$

6. Find the domain

   a. (-∞, 7) b. (-∞, -7) U (7, ∞)

   c. (-∞, -7) U (-7, 7) U (7, ∞) d. (-∞, ∞)

7. Find any holes

   a. x = -7, 7 b. x = -7 c. x = 7 d. None

8. Find any vertical asymptotes

   a. x = -7, 7 b. x = -7 c. x = 7 d. None

9. Find any horizontal asymptotes

   a. y = 0 b. y = -5 c. y = 2 d. None

10. Find any x-intercepts

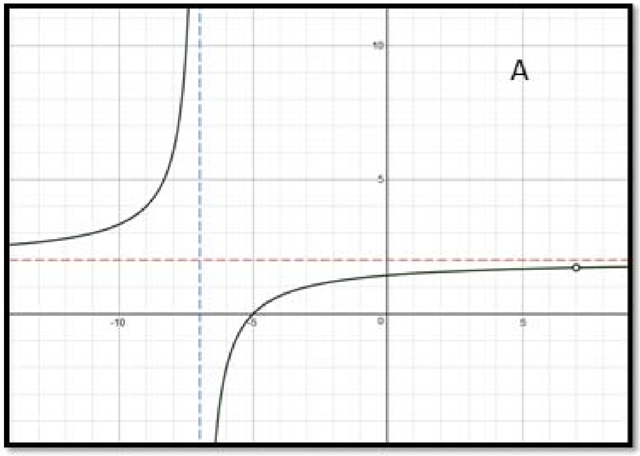
    a. x = -5, 7 b. x = -5 c. x = 7 d. None

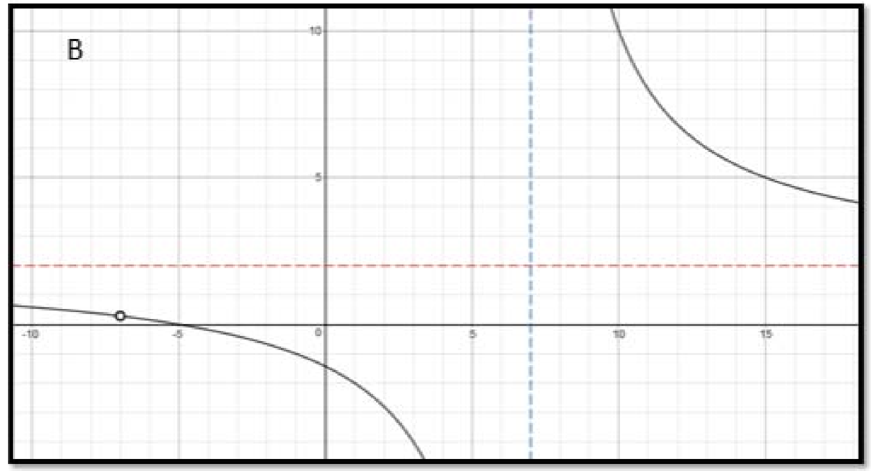
11. Find any y-intercepts

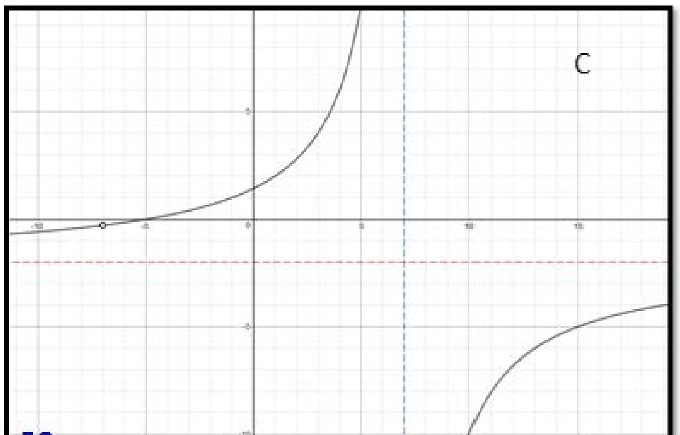
    a. y = -70 b. y = -10/7 c. y = 10/7 d. None

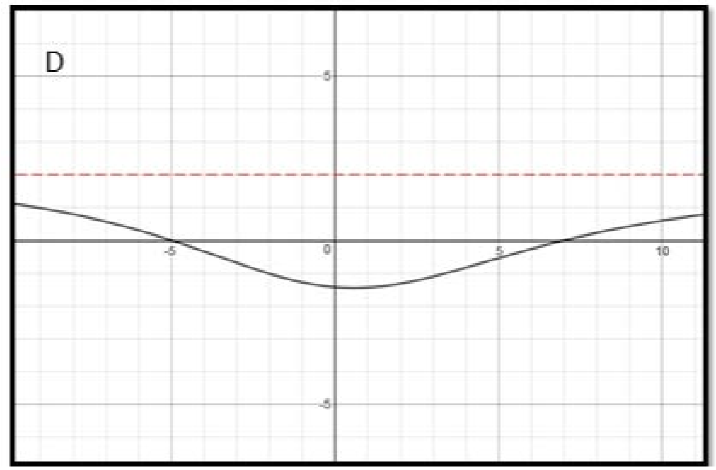
(7. Sketch (choices on next slide)

Popper 7: ?? ?? = 2?? 2 -4??-70 ?? 2 -49









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