MATH 1314

Section 2.3

Quadratic Equations

In this section, you'll learn three methods for solving quadratic equations. A quadratic equation is an equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$. Equations of this form can have two solutions, one solution or no solutions. The three methods are:

1. Factoring
2. Completing the Square
3. Quadratic Formula

Factoring

This method makes use of the Zero Product Property, that is, if $ab = 0$, then either $a = 0$ or $b = 0$. The equation must be written in the form $ax^2 + bx + c = 0$, so you may have to rewrite the equation in this form before you get started. Next you'll factor the left hand side of the equation. Once in factored form, you'll use the Zero Product Property to solve the equation.

① All terms to the left ($=0$)

② Factor the left side

- Trinomial Factoring:
  Find 2 numbers that multiply to last number and add to middle number.

$(-5 \cdot +1 = -5, \ -5 + 1 = -4)$

③ Set each factor equal to zero

④ Solve.

Example 1: $x^2 - 4x - 5 = 0$

$(x-5)(x+1)=0$

$$\begin{array}{rr} x-5=0 & x+1=0 \\ +5 \ \ +5 & -1 \ \ -1 \\ \hline x=5 & x=-1 \end{array}$$

$\{-1, 5\}$

Example 2: $x^2 + 5x - 36 = 0$

$(x-4)(x+9)=0$

$x-4=0$

$+4 \quad +4$

$x=4$

$x+9=0$

$-9 \quad -9$

$x=-9$

$\boxed{\{-9, 4\}}$

-36		Sum
-1	+36	35
-2	+18	16
-3	+12	9
-4	+9	5
-6	+6	0

Example 3: $3x^2 - 22x - 16 = 0$

$(3x^2 - 24x) + (2x - 16) = 0$

$3x(x - 8) + 2(x - 8) = 0$

$(x - 8)(3x + 2) = 0$

$x - 8 = 0$

$+8 \quad +8$

$x = 8$

$3x + 2 = 0$

$-2 \quad -2$

$3x = -2$

$\frac{3x}{3} = \frac{-2}{3}$

$x = -2/3$

$\boxed{\{-2/3, 8\}}$

① Multiply $a \cdot c \rightarrow 3 \cdot -16 = -48$

② Find 2 numbers that multiply to -48 and add to -22.

$-24, +2$

③ Rewrite the trinomial as 4 terms using these two numbers.

④ Factor by grouping.

Completing the Square

You'll start with an equation of the form $ax^2 + bx + c = 0$. Move the constant to the right hand side. Next, you'll need to factor so that the coefficient of x^2 is 1. Then you can complete the square.

Example 4: $4x^2 + 16x = 20$

$4(x^2 + 4x) = 20$

$b = 4$

$\frac{b}{2} = \frac{4}{2} = 2$

$\left(\frac{b}{2}\right)^2 = 2^2 = 4$

$\left(\frac{b}{2}\right)^2$

$4(x^2 + 4x + 4) = 20 + \underbrace{4(4)}_{a\left(\frac{b}{2}\right)^2}$

$\frac{\cancel{4}(x+2)^2}{\cancel{4}} = \frac{36}{4}$

$\sqrt{(x+2)^2} = \sqrt{9}$

$x + 2 = \pm 3$

$x + 2 = \pm 3$
$-2 \quad -2$

$x = -2 \pm 3$

$x = -2 + 3 = 1$

$x = -2 - 3 = -5$

$\boxed{\{-5, 1\}}$

① x-terms on left, constant on right.

② Factor the a-value

③ $b =$, $\frac{b}{2} =$, $\left(\frac{b}{2}\right)^2 =$

④ Rewrite equation

⑤ complete the square.

⑥ Solve

Example 5: $x^2 + 8x - \cancel{20} = 0$

$+20 \quad +20$

$x^2 + 8x = 20 \rightarrow x^2 + 8x + 16 = 20 + 16$

$b = 8$

$\frac{b}{2} = 4$

$\left(\frac{b}{2}\right)^2 = 16$

$\sqrt{(x+4)^2} = \sqrt{36}$

$x + 4 = \pm 6$

$-4 \quad -4$

$x = -4 \pm 6$

$x = -4 + 6 = 2$

$x = -4 - 6 = -10$

$\boxed{\{-10, 2\}}$

Example 6: $x^2 + 7x - \cancel{3} = 0$

$+3 \qquad +3$

$x^2 + 7x = 3 \longrightarrow x^2 + 7x + \frac{49}{4} = 3 + \frac{49}{4}$

$b = 7$

$\frac{b}{2} = \frac{7}{2}$

$\left(\frac{b}{2}\right)^2 = \frac{49}{4}$

$3 + \frac{49}{4}$

$4 \cdot \frac{3}{1} + \frac{49}{4}$

$\frac{12}{4} + \frac{49}{4}$

$\frac{61}{4}$

$\sqrt{\left(x + \frac{7}{2}\right)^2} = \sqrt{\frac{61}{4}}$

$x + \frac{7}{2} = \frac{\pm\sqrt{61}}{2}$

$-\frac{7}{2} \qquad -\frac{7}{2}$

$x = -\frac{7}{2} \pm \frac{\sqrt{61}}{2} = \boxed{\frac{-7 \pm \sqrt{61}}{2}}$

The Quadratic Formula

"Quadratic Formula Song"

The third method for solving quadratic equations is the quadratic formula. Here's the formula. You need to memorize it:

For a quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

① All terms left (=0)

② assign a, b, c

③ Plug into the formula.

Example 7: $x^2 - 5x + 3 = 0$

$a = 1 \quad b = -5 \quad c = 3$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(3)}}{2(1)}$$

$$x = \frac{5 \pm \sqrt{25 - 12}}{2}$$

$$\boxed{x = \frac{5 \pm \sqrt{13}}{2}}$$

Example 8: $3x^2 + 2x + 2 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 3 \quad b = 2 \quad c = 2$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(3)(2)}}{2(3)} = \frac{-2 \pm \sqrt{4 - 24}}{6} = \frac{-2 \pm \sqrt{-20}}{6}$$

Cannot have square roots of Negatives.

$\boxed{\text{No Answer}}$ (for Now)

Example 9: $4x^2 + 2x - 4 = 0$

$a=4 \quad b=2 \quad c=-4$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-2 \pm \sqrt{2^2-4(4)(-4)}}{2(4)}$$

$\sqrt{68}$

$\sqrt{4} \cdot \sqrt{17}$

$2\sqrt{17}$

$$x = \frac{-2 \pm \sqrt{4+64}}{8}$$

$$x = \frac{-2 \pm \sqrt{68}}{8} = \frac{-2 \pm 2\sqrt{17}}{8} = \boxed{\frac{-1 \pm \sqrt{17}}{4}}$$

$$\boxed{x = \frac{-1}{4} \pm \frac{\sqrt{17}}{4}}$$

Note: The **discriminant** of the equation $ax^2 + bx + c = 0$ ($a \neq 0$) is given by

$$D = b^2 - 4ac$$

If D > 0, then the equation $ax^2 + bx + c = 0$ has two distinct real solutions.
If D = 0, then the equation $ax^2 + bx + c = 0$ has exactly one real solution.
If D < 0, then the equation $ax^2 + bx + c = 0$ has no real solution (The roots of the equation are complex numbers and appear as complex conjugate pairs.)

$b^2-4ac>0$ (perfect square) : 2 rational solutions

$b^2-4ac>0$ (not perf. sq) : 2 irrational solutions

$b^2-4ac=0$: 1 rational solution

$b^2-4ac<0$: No real solutions

Popper 2:

Solve the following quadratic equations.

1. (Use Factoring): $3x^2 - x = 2(2x+1)$

a. $\{-2, {}^{1}/_{3}\}$ b. $\{-{}^{2}/_{3}, {}^{1}/_{3}\}$ c. $\{-{}^{1}/_{3}, 2\}$ d. $\{-3, ½\}$

$$3x^2 - x = 4x + 2$$
$$-4x - 2 \quad -4x - 2$$
$$3x^2 - 5x - 2 = 0$$
$$(3x^2 - 6x) + (1x - 2) = 0$$
$$3x(x-2) + 1(x-2) = 0$$
$$(x-2)(3x+1) = 0$$

$$(3)(-2) = -6$$
$$-6, +1$$

$$x - 2 = 0$$
$$+2 \quad +2$$
$$x = 2$$

$$3x + 1 = 0$$
$$-1 \quad -1$$
$$\frac{3x}{3} = \frac{-1}{3}$$
$$x = -1/3$$

Popper 2:

2. (Use Completing the Square): $2x^2 - 8x = 9$

a. $x = 2 \pm \sqrt{\frac{13}{2}}$ b. $x = 2 \pm \sqrt{\frac{17}{2}}$ c. $x = \pm\sqrt{\frac{17}{2}}$ d. $x = 2 \pm \sqrt{17}$

$2x^2 - 8x = 9$

$2(x^2 - 4x) = 9 \rightarrow 2(x^2 - 4x + 4) = 9 + 2(4)$

$\left(\frac{b}{2}\right)^2$ $a \cdot \left(\frac{b}{2}\right)^2$

$b = -4$

$\frac{b}{2} = -2$

$\left(\frac{b}{2}\right)^2 = 4$

$\frac{\cancel{2}(x-2)^2}{\cancel{2}} = \frac{17}{2}$

$\sqrt{(x-2)^2} = \sqrt{\frac{17}{2}}$

$x - 2 = \pm\sqrt{17/2} \rightarrow x = 2 \pm \sqrt{17/2}$

Popper 2:

3. (Use Quadratic Formula): $5x^2 - 3x - 7 = 0$

$a = 5$
$b = -3$
$c = -7$

a. $x = \frac{3 \pm \sqrt{149}}{10}$ b. $x = \frac{-3 \pm \sqrt{149}}{5}$ c. $x = \frac{3 \pm \sqrt{131}}{10}$ d. No Answer

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(5)(-7)}}{2(5)} = \frac{3 \pm \sqrt{9 + 140}}{10}$$

$$= \frac{3 \pm \sqrt{149}}{10}$$