MATH 1314

Section 2.4

Complex Numbers

Definition: A **complex number** is a number that can be written in the form $a + bi$, where a is called the **real part** and bi is called the **imaginary part**. The a and b are real numbers and $i = \sqrt{-1}$

$x^2 + 1 = 0$
$-1 \quad -1$
$x^2 = -1$

$i = \sqrt{-1}$

$5i = 5 \cdot \sqrt{-1}$

$2 + 5i$

a

Real Part

imaginary Part

$\sqrt{-72}$

$\sqrt{36}\sqrt{2}\sqrt{-1}$

$6\sqrt{2}\,i$

$6\sqrt{2}\,i$ Yes

$6i\sqrt{2}$ Yes

$6\sqrt{2i}$ No!

Here are several properties of complex numbers:

Addition of Complex Numbers:

$(a + bi) + (c + di) = (a + c) + (b + d)i$

Add the real parts together and add the imaginary parts together.

$(2+3i)+(5-i) = 2+5+3i-i$

$= 7+2i$

Subtraction of Complex Numbers

$(a + bi) - (c + di) = (a - c) + (b - d)i$

Subtract the real parts and subtract the imaginary parts.

$(2+3i)-(5-i) = 2+3i-5+i$

$-3+4i$

Multiplication of Complex Numbers:

Multiply in the same manner as multiplying binomials and remember that $i^2 = -1$

$(2+3i)(5-i)$

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$10-2i+15i-3i^2$

$10-2i+15i-3(-1)$

$10-2i+15i+3 = 13+13i$

Example 1: Simplify each.

a. $\sqrt{-16}$

$\sqrt{16}\ \sqrt{-1}$

$4\ i$

b. $\sqrt{-40}$

$\sqrt{4}\ \sqrt{10}\ \sqrt{-1}$

$2\sqrt{10}\ i$

$2i\sqrt{10}$

Simplify the following:

$\sqrt{-50}$

$\sqrt{-50}$

$\sqrt{25}\sqrt{2}\sqrt{-1}$

$5\sqrt{2}\,i$

2. $\sqrt{-80} + \sqrt{45}$

$\sqrt{16}\sqrt{5}\sqrt{-1} + \sqrt{9}\sqrt{5}$

$4\sqrt{5}\,i + 3\sqrt{5} = (4i+3)\sqrt{5}$

Example 2: Simplify each of the following and write the answer in form a + b*i*.

a. $(5 + 4i) + (2 - i)$

$5 + 2 + 4i - i$

$\boxed{7 + 3i}$

b. $(-6 - 3i) - (-2 + 2i)$

$-6 - 3i + 2 - 2i$

$\boxed{-4 - 5i}$

c. $-i(-3 + 6i)$

$3i - 6i^2 \rightarrow i^2 = -1$

$3i - 6(-1)$

$3i + 6 \longrightarrow \boxed{6 + 3i}$

d. $(-1 - i)(2 + 5i)$

$-2 - 5i - 2i - 5i^2 \rightarrow -1$

$-2 - 5i - 2i + 5$

$\boxed{3 - 7i}$

$(5 + 7i) + (2 - 3i)$ a. $= 5 + 2 + 7i - 3i = 7 + 4i$

$(5 + 7i)(2 - 3i)$

$10 - 15i + 14i \overbrace{- 21i^2}^{-21(-1)}$

$10 - 15i + 14i + 21$

$31 - i$

Next, you'll need to be able to find various powers of *i*. You'll need to know these 4 powers:

$$i = \sqrt{-1}$$
$$i^2 = -1$$
$$i^3 = i^2 * i = -1 * i = -i$$
$$i^4 = i^2 * i^2 = -1 * -1 = 1$$

For other powers of *i*, divide the exponent by 4 and find the remainder. Your answer will be *i* raised to the remainder power. If the remainder is zero, your answer will be i^4 or 1.

$$i^0 = 1$$
$$i^1 = i$$
$$i^2 = -1$$
$$i^3 = i \cdot i^2 = i(-1) = -i$$
$$i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$$

Any power of i:

1. divide exponent by 4.
2. The remainder is your new exponent.
3. Evaluate from chart.

Example 3: Simplify each.

a. i^{15}

$4 \overline{)15} = 3 \text{ R}3$, $\quad 15 - 12 = 3$

$i^{15} = i^{3} = \boxed{-i}$

b. $i^{72} = i^{0} = \boxed{1}$

$4 \overline{)72} = 18 \text{ R}0$, $\quad 7 - 4 = 3$, $\quad 32 - 32 = 0$

c. $i^{42} = i^{2} = \boxed{-1}$

$4 \overline{)42} = 10 \text{ R}2$, $\quad 4 - 4 = 0$, $\quad 02 - 0 = 2$

d. $i^{313} = i^{1} = \boxed{i}$

$4 \overline{)313} = 78 \text{ R}1$, $\quad 31 - 28 = 3$, $\quad 33 - 32 = 1$

Division of Complex Numbers

The **complex conjugate** of the complex number $a + bi$ is the complex number $a - bi$.

To simplify the quotient $\frac{a+bi}{c+di}$ multiply both the numerator and denominator by the complex conjugate of the denominator.

Change the sign on the imaginary term.

a. $\dfrac{(5+4i)}{(2-3i)} \cdot \dfrac{(2+3i)}{(2+3i)} = \dfrac{10+15i+8i+12i^2}{4+\cancel{6i}-\cancel{6i}-9i^2} = \dfrac{-2+23i}{13} = \boxed{-\dfrac{2}{13}+\dfrac{23}{13}i}$

$12i^2 \to -12$, $\quad -9i^2 \to +9$

b. $\dfrac{-1-i}{i} \cdot \dfrac{-i}{-i} = \dfrac{i+i^2}{-i^2} = \dfrac{i-1}{1} = \boxed{-1+i}$

$i^2 \to -1$, $\quad -i^2 \to +1$

c. $\dfrac{1}{4-i} + \dfrac{4}{4+i}$

$$\frac{1}{4-i}\cdot\frac{4+i}{4+i} = \frac{4+i}{16+\cancel{4i}-\cancel{4i}-i^2} = \frac{4+i}{17} = \frac{4}{17}+\frac{1}{17}i$$

$-i^2 \to +1$

$$\frac{4}{4+i}\cdot\frac{4-i}{4-i} = \frac{16-4i}{16-\cancel{4i}+\cancel{4i}-i^2} = \frac{16-4i}{17} = \frac{16}{17}-\frac{4}{17}i$$

$-i^2 \to +1$

$$\frac{4}{17}+\frac{1}{17}i+\frac{16}{17}-\frac{4}{17}i = \boxed{\frac{20}{17}-\frac{3}{17}i}$$

d. $\dfrac{(5-2i)}{(3+4i)}$

$$\frac{(5-2i)}{(3+4i)}\cdot\frac{(3-4i)}{(3-4i)} = \frac{15-20i-6i+8i^2}{9-\cancel{12i}+\cancel{12i}-16i^2} = \frac{7-26i}{25}$$

$8i^2 \to -8$

$-16i^2 \to +16$

$$\boxed{\frac{7}{25}-\frac{26}{25}i}$$

Complex Roots of Quadratic Equations

Using complex numbers, we can now find all solutions to quadratic equations. We can use any of the techniques from the previous section to solve, but usually, we will just take the square root of both sides of the equation, complete the square or use the quadratic formula.

Example 5: Find all complex solutions of the following equations. Express your answer in form a + b*i*.

a. $x^2 + 100 = 0$

$$-100 \quad -100$$

$$\sqrt{x^2} = \sqrt{-100}$$

$$\boxed{x = \pm 10i}$$

b. $49x^2 + 36 = 0$

$$-36 \quad -36$$

$$\frac{49x^2}{49} = \frac{-36}{49}$$

$$\sqrt{x^2} = \sqrt{\frac{-36}{49}}$$

$$x = \pm \frac{\sqrt{36}}{\sqrt{49}} i = \boxed{\pm \frac{6}{7} i}$$

c. $x^2 - 6x = -13 \rightarrow x^2 - 6x + 9 = -13 + 9$

Complete the square

$b = -6$

$\frac{b}{2} = -3$

$\left(\frac{b}{2}\right)^2 = 9$

$\sqrt{(x-3)^2} = \sqrt{-4}$

$x - \cancel{3} = \pm 2i$

$\quad \cancel{+3} \quad +3$

$\boxed{x = 3 \pm 2i}$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

d. $x^2 + 12x + 75 = 0$

$a=1$
$b=12$
$c=75$

$$x = \frac{-12 \pm \sqrt{12^2-4(1)(75)}}{2(1)}$$

$$x = \frac{-12 \pm \sqrt{144-300}}{2} = \frac{-12 \pm \sqrt{-156}}{2}$$

$$= \frac{-12 \pm \sqrt{4}\sqrt{39}\sqrt{-1}}{2}$$

$$= \frac{-12 \pm 2i\sqrt{39}}{2}$$

$$\boxed{x = -6 \pm \sqrt{39}\,i}$$

$a=4 \quad b=8 \quad c=9$

e. $4x^2 + 8x + 9 = 0$

$$x = \frac{-8 \pm \sqrt{8^2-4(4)(9)}}{2(4)} = \frac{-8 \pm \sqrt{64-144}}{8}$$

$$x = \frac{-8 \pm \sqrt{-80}}{8} = \frac{-8 \pm \sqrt{16}\sqrt{5}\sqrt{-1}}{8} = \frac{-8 \pm 4i\sqrt{5}}{8}$$

$$= \frac{-8}{8} \pm \frac{4\sqrt{5}}{8}i = \boxed{-1 \pm \frac{\sqrt{5}}{2}i}$$