

MATH 1314

Section 2.5

Other Techniques for Solving Equations

Solving by Factoring:

Factoring can be used to solve many types of equations. Always begin by Factoring Completely. Then, set each factor equal to zero.

Find all solutions to

$$(x^3 + 3x^2) + (2x + 6) = 0$$

$$x^2(x+3) + 2(x+3) = 0$$

$$(x+3)(x^2+2) = 0$$

$$\begin{array}{r} x+3=0 \\ -3 \quad -3 \\ \hline x = -3 \end{array}$$

$$\begin{array}{r} x^2+2=0 \\ -2 \quad -2 \\ \hline \sqrt{x^2} = \sqrt{2} \\ x = \pm\sqrt{2}i \end{array}$$

$$\{-3, -\sqrt{2}i, +\sqrt{2}i\}$$

Find all the solutions of $x^3 = x$

$$\begin{array}{r} x^3 = x \\ -x \quad -x \\ \hline x^3 - x = 0 \end{array}$$

$$x(x^2 - 1) = 0$$

$$x(x+1)(x-1) = 0$$

$$x = 0$$

$$\begin{array}{r} x+1 = 0 \\ -1 \quad -1 \\ \hline x = -1 \end{array}$$

$$\begin{array}{r} x-1 = 0 \\ +1 \quad +1 \\ \hline x = 1 \end{array}$$

$$\boxed{\{-1, 0, 1\}}$$

Common mistake:

$$\begin{array}{r} x^3 = x \\ \hline x \quad x \\ \hline \sqrt{x^2} = \sqrt{1} \end{array}$$

$$x = \pm 1$$

We lost the $x=0$ answer!

Reason: You cannot divide (or cancel) anything that may equal zero.

Equations Involving Fractions:

Option 1: Rewrite all fractions so that they have the same denominator, then drop all denominators from the equation.

or

Option 2: Multiply the entire equation by the LCD to clear the fractions.

Then: Solve normally.

Be advised: If your answer makes any of the original fractions undefined, it must be rejected!!

$$\frac{4}{x-1} + \frac{3}{x} = \frac{3}{1}$$

LCD: $x(x-1)$

(cross cancel:

cancel any numerator
with any denominator
when multiplying.

$$\frac{4}{\cancel{x-1}} \cdot \frac{x(\cancel{x-1})}{1} + \frac{3}{\cancel{x}} \cdot \frac{x(x-1)}{1} = \frac{3}{1} \cdot \frac{x(x-1)}{1}$$

$$\underline{4x} + \underline{3x-3} = 3x^2 - 3x$$

$$\begin{array}{r} 7x - 3 \\ - 3x + 3 \\ \hline 0 \end{array} = 3x^2 - 3x$$

$$0 = 3x^2 - 10x + 3$$

$$3x^2 - 10x + 3 = 0$$

$$3x^2 - 10x + 3 = 0$$

$$(3x^2 - 9x)(-x + 3) = 0$$

$$3x(x-3) - 1(x-3) = 0$$

$$(x-3)(3x-1) = 0$$

$$\begin{array}{r} x-3=0 \\ \hline +3 \quad +3 \\ \hline x=3 \end{array}$$

$$\begin{array}{r} 3x-1=0 \\ \hline +1 \quad +1 \\ \hline 3x=1 \\ \hline \frac{1}{3} \\ \hline x = \frac{1}{3} \end{array}$$

$$\begin{array}{r} (3)(3) = 9 \\ \wedge \\ -9, -1 \end{array}$$

$$\left\{ \frac{1}{3}, 3 \right\}$$

Equations involving radicals:

If an equation involves a square root (also called a radical), you must isolate the radical, square both sides, and solve the remaining equation. Be certain to check your answers!

Find all solutions to $\sqrt{x+8} - 2 = x$

$$\begin{array}{l} \cancel{+2} \quad +2 \\ \hline (\sqrt{x+8})^2 = (x+2)^2 \end{array}$$

$$x+8 = (x+2)(x+2)$$

$$x+8 = x^2 + 2x + 2x + 4$$

$$x+8 = x^2 + 4x + 4$$

$$\begin{array}{r} x+8 \\ -x-8 \\ \hline 0 \end{array} \quad \begin{array}{r} x^2 + 4x + 4 \\ -x^2 - 4x - 4 \\ \hline 0 \end{array}$$

$$0 = x^2 + 3x - 4$$

$$0 = (x+4)(x-1)$$

$$x+4=0 \\ \text{reject } x = -4$$

$$x-1=0 \\ \boxed{x=1}$$

Check $x = -4$

$$\sqrt{-4+8} - 2 = -4$$

$$\sqrt{4} - 2 = -4$$

$$2 - 2 = -4$$

$$0 \neq -4$$

Check $x = 1$

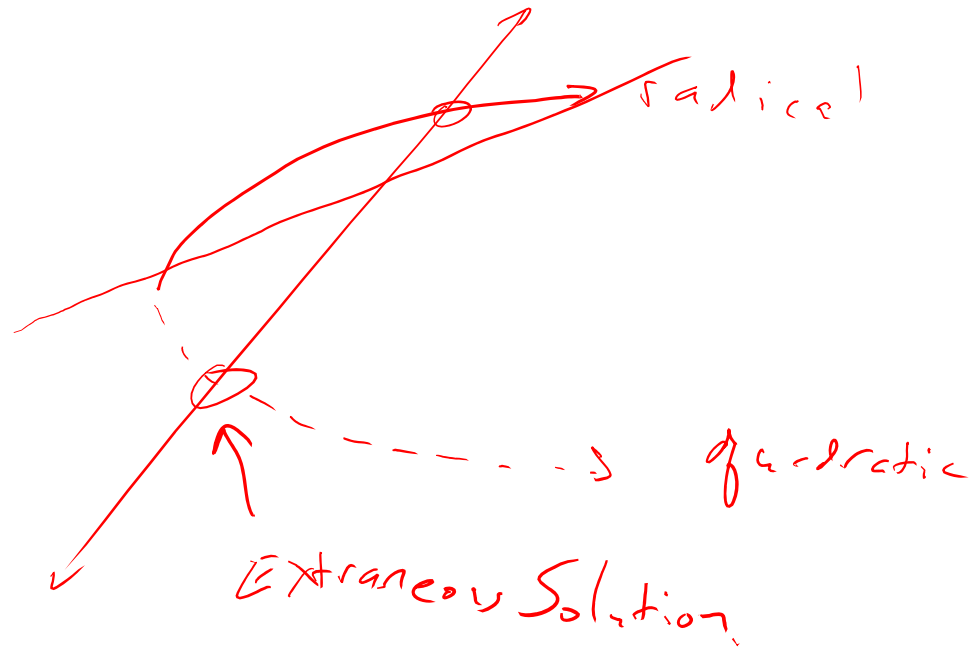
$$\sqrt{1+8} - 2 = 1$$

$$\sqrt{9} - 2 = 1$$

$$3 - 2 = 1$$

$$1 = 1 \checkmark$$

Extraneous Solutions: In a radical solution, you may “create” additional answers that are not correct. These must be rejected!



Find all solutions to $\sqrt{3x+1} - 1 = x$

$$(\sqrt{3x+1})^2 = (x+1)^2$$

$$3x+1 = (x+1)(x+1)$$

$$3x+1 = x^2 + x + x + 1$$

$$3x+1 = x^2 + 2x + 1$$

$$\begin{array}{r} 3x+1 \\ -3x-1 \\ \hline \end{array} = \begin{array}{r} x^2+2x+1 \\ -3x-1 \\ \hline \end{array}$$

$$0 = x^2 - x$$

$$0 = x(x-1)$$

$$x=0$$

$$x-1=0$$

$$x=1$$

Check $x=0$

$$\sqrt{3(0)+1} - 1 = 0$$

$$\sqrt{1} - 1 = 0$$

$$1-1=0$$

$$0=0 \checkmark$$

Check $x=1$

$$\sqrt{3(1)+1} - 1 = 1$$

$$\sqrt{4} - 1 = 1$$

$$2-1=1$$

$$1=1 \checkmark$$

$$\{0, 1\}$$

Popper # 5

1. Solve the following: $x^3 = 9x$

a. 0, 3

b. 0, 9, -9

c. 0, -3, 3

d. -3, 3

$$x^3 - 9x = 0$$

$$x(x^2 - 9) = 0$$

$$x = 0 \quad x^2 - 9 = 0$$

$$x^2 = 9$$

$$x = \pm 3$$

2. Solve the following: $x + \sqrt{x+1} = 5$

a. 8, 3

b. 3

c. 8

d. No Answer

$$\sqrt{x+1} = 5 - x$$

$$x+1 = (-x+5)(-x+5)$$

$$x+1 = x^2 - 5x - 5x + 25$$

$$x+1 = x^2 - 10x + 25$$

3. Solve the following: $\frac{8}{x+1} + \frac{3}{x} = 3$

a. 3

b. $^{-1}/_3$

c. -1

d. 3, $^{-1}/_3$

$$0 = x^2 - 11x + 24$$

$$0 = (x-3)(x-8)$$

$$x-3=0$$

$$x=3$$

$$x-8=0$$

$$x=8$$

reject

check $x=3$

$$3 + \sqrt{3+1} = 5$$

$$3 + \sqrt{4} = 5$$

$$3 + 2 = 5 \checkmark$$

check $x=8$

$$8 + \sqrt{8+1} = 5$$

$$8 + \sqrt{9} = 5 \rightarrow 8 + 3 \neq 5$$

$$\frac{8}{x+1} + \frac{3}{x} = \frac{3}{1} \quad \text{LCD: } x(x+1)$$

$$\frac{8}{\cancel{x+1}} \cdot \frac{x(\cancel{x+1})}{1} + \frac{3}{\cancel{x}} \cdot \frac{x(\cancel{x+1})}{1} = \frac{3}{1} \cdot \frac{x(x+1)}{1}$$

$$8x + 3x + 3 = 3x^2 + 3x$$

$$\begin{array}{r} 11x + 3 = 3x^2 + 3x \\ -11x \quad -3 \\ \hline \end{array}$$

$$0 = 3x^2 - 8x - 3$$

$$0 = (3x^2 - 9x) + (x - 3)$$

$$0 = 3x(x-3) + 1(x-3)$$

$$0 = (x-3)(3x+1)$$

$$x-3=0$$

$$x=3$$

$$\begin{array}{r} 3x+1=0 \\ -x \quad -1 \\ \hline \end{array}$$

$$\frac{3x}{3} = \frac{-1}{3}$$

$$x = -\frac{1}{3}$$

$$\left\{ -\frac{1}{3}, 3 \right\}$$

Solving by Substitution:

When a function looks, “almost” quadratic, you may want to solve it by relating it to another function.

$$x^{10} - x^5 - 6 = 0$$

Handwritten annotations: A red circle around the 10 in x^{10} with an arrow pointing to a $\div 2$ above it. A red arrow points from the x^5 term to the word "Not" written above it.

If the exponents go “full amount \rightarrow half amount \rightarrow nothing” then you can rewrite as a quadratic.

Let $u = x^5$, then

$$\boxed{x^{10}} - \boxed{x^5} - 6 = 0 \rightarrow u^2 - u - 6 = 0$$

Handwritten annotations: A red arrow points from the x^{10} term in the equation above to the definition $u = x^5$. Below that, the equation $u^2 = (x^5)^2 = x^{10}$ is written.

$$u = x^5$$
$$u^2 = (x^5)^2 = x^{10}$$

Handwritten note: Middle Term: x^{power} only
(unless inside term)

$$x^{10} - x^5 - 6 = 0$$

$$u = x^5, \quad u^2 = x^{10}$$

$$u^2 - u - 6 = 0$$

$$(u-3)(u+2) = 0$$

$$\begin{array}{r} u - 3 = 0 \\ +3 \quad +3 \\ \hline u = 3 \end{array}$$

$$\begin{array}{r} u + 2 = 0 \\ -2 \quad -2 \\ \hline u = -2 \end{array}$$

Since $u = x^5$

$$u = 3$$

$$x^5 = 3$$

$$x = \sqrt[5]{3}$$

$$u = -2$$

$$x^5 = -2$$

$$x = \sqrt[5]{-2}$$

$$x^{1/2} + 2x^{1/4} - 15 = 0$$

$$u = x^{1/4}, \quad u^2 = (x^{1/4})^2 = x^{1/2}$$

$$u^2 + 2u - 15 = 0$$

$$(u + 5)(u - 3) = 0$$

$$u + 5 = 0 \quad u - 3 = 0$$

$$u = -5 \quad u = 3$$

alternate check: $x = 81$

$$\sqrt{81} + 2\sqrt[4]{81} - 15 = 0$$

$$9 + 2(3) - 15 = 0$$

$$9 + 6 - 15 = 0 \checkmark$$

solving a radical

$$u = -5$$
$$(x^{1/4})^4 = (-5)^4$$

$$x = 625$$

check $x = 625$

$$\sqrt[4]{625} = -5$$

$$5 = -5$$

reject

$$u = 3$$

$$(x^{1/4})^4 = (3)^4$$

$$x = 81$$

check $x = 81$

$$\sqrt[4]{81} = 3$$

$$3 = 3 \checkmark$$

$$2(x+5) - \sqrt{x+5} - 10 = 0$$

$$u = \sqrt{x+5}, \quad u^2 = (\sqrt{x+5})^2 = \underline{x+5}$$

$$2u^2 - u - 10 = 0$$

$$(2u^2 + 4u)(-5u - 10) = 0$$

$$2u(u+2) - 5(u+2) = 0$$

$$(u+2)(2u-5) = 0$$

$$u+2=0$$

$$u = -2$$

$$2u-5=0$$

$$2u=5$$

$$u = 5/2$$

$$u = -2$$

$$(\sqrt{x+5})^2 = (-2)^2$$

$$x+5 = 4$$

$$\begin{array}{r} -5 \\ -5 \\ \hline \end{array}$$

$$x = -1$$

reject

check $x = -1$

$$\sqrt{-1+5} = -2$$

$$\sqrt{4} = -2$$

$$2 \neq -2$$

$$x = 5/4$$

$$u = 5/2$$

$$(\sqrt{x+5})^2 = (5/2)^2$$

$$x+5 = \frac{25}{4}$$

$$\begin{array}{r} -5 \\ -5 \\ \hline \end{array}$$

$$x = \frac{25}{4} - \frac{5 \cdot 4}{1 \cdot 4}$$

$$x = \frac{25}{4} - \frac{20}{4} = \boxed{5/4}$$

check $x = 5/4$

$$\sqrt{\frac{5}{4} + \frac{20}{4}} = 5/2$$

$$\sqrt{\frac{25}{4}} = \frac{5}{2} \checkmark$$

Popper 5: $\frac{1}{x^2} + \frac{5}{x} + 6 = 0$

$$\frac{5}{x} = 5 \cdot \frac{1}{x}$$

$$u^2 + 5u + 6 = 0$$

4. What substitution should be made?

a. $u = x$

b. $u = 5/x$

c. $u = 1/x$

d. $u = 5/x^2$

5. How does the equation re-write?

a. $u^2 + 5u + 6 = 0$

b. $0.2u^2 + u + 6 = 0$

c. $u^2 + 5u = 0$

d. $5u^2 + 6 = 0$

6. What is the value(s) of u ?

a. $\{-1, 6\}$

b. $\{2, 3\}$

c. $\{-3, -2\}$

d. $\{0, 5\}$

7. What is the value(s) of x ?

a. $\{2, 3, 5\}$

b. $\{2, 3\}$

c. $\{-3, -2\}$

d. $\{-1/2, -1/3\}$

$$u^2 = \frac{1}{x^2}$$

$$u^2 + 5u + 6 = 0$$

$$(u+2)(u+3) = 0$$

$$u+2=0 \quad u+3=0$$

$$u=-2 \quad u=-3$$

$$\frac{1}{x} = \frac{-2}{1} \quad \frac{1}{x} = \frac{-3}{1}$$

$$x = \frac{1}{-2} \quad x = \frac{1}{-3}$$

$$\cancel{x} + 4\sqrt{x} - \cancel{5} = 0$$

$\begin{array}{ccc} +x & & +5 \\ & & -x+5 \end{array}$

$$\frac{4\sqrt{x}}{4} = \frac{-x+5}{4}$$

$$\sqrt{x} = \left(\frac{-x+5}{4}\right)^2$$

$$16x = \frac{(-x+5)(-x+5)}{16}$$

$$16x = x^2 - 5x - 5x + 25$$

$$\begin{array}{r} 16x = x^2 - 10x + 25 \\ -16x \end{array}$$

$$x^2 - 26x + 25 = 0$$

$$(x-25)(x-1) = 0$$

$$\begin{array}{l} x-25=0 \\ x=25 \\ \text{reject} \end{array} \quad \begin{array}{l} x-1=0 \\ x=1 \end{array}$$

check x=25

$$25 + 4\sqrt{25} - 5 = 0$$

$$25 + 4(5) - 5 = 0$$

$$25 + 20 - 5 = 0$$

$$40 = 0$$

check x=1

$$1 + 4\sqrt{1} - 5 = 0$$

$$1 + 4 - 5 = 0$$

$$0 = 0 \checkmark$$

$$x + 4\sqrt{x} - 5 = 0$$

$$u = \sqrt{x} \quad u^2 = \sqrt{x}^2 = x$$

$$u^2 + 4u - 5 = 0$$

$$(u+5)(u-1) = 0$$

$$u+5=0 \quad u-1=0$$

$$u = -5$$

$$u = 1$$

$$\rightarrow \sqrt{x} = (-5)^2$$

$$\sqrt{x} = 1^2 \leftarrow$$

$$x = 25$$

$$\boxed{x = 1}$$

reject

$$\underline{\text{Check } x = 25}$$

$$\sqrt{25} = -5$$

$$5 \neq -5$$

$$\underline{\text{Check } x = 1}$$

$$\sqrt{1} = 1$$

$$1 = 1 \checkmark$$