MATH 1314

Section 2.5

Other Techniques for Solving Equations

Solving by Factoring:

Factoring can be used to solve many types of equations. Always begin by Factoring Completely. Then, set each factor equal to zero.

Find all solutions to

$$x^3 + 3x^2 + 2x + 6 = 0$$

$$\underbrace{x^2(x+3)} + 2\underbrace{(x+3)} = 0$$

$$(x+3)(x^2+2) = 0$$

$$x+3=0 \qquad x^2+2=0$$

$$-3 \quad -3 \qquad\qquad -2 \quad -2$$

$$x = -3 \qquad \sqrt{x^2} = \sqrt{-2}$$

$$x = \pm\sqrt{2}\,i$$

$$\{-3, -\sqrt{2}\,i, +\sqrt{2}\,i\}$$

Find all the solutions of $x^3 = x$

$$x^3 = x$$
$$-x \quad -x$$
$$x^3 - x = 0$$
$$x(x^2 - 1) = 0$$
$$x(x+1)(x-1) = 0$$

$$x = 0 \qquad x + 1 = 0 \qquad x - 1 = 0$$
$$-1 \quad -1 \qquad\qquad +1 \quad +1$$
$$x = -1 \qquad\qquad x = 1$$

$$\boxed{\{-1, 0, 1\}}$$

Common mistake:

$$\frac{x^3}{x} = \frac{x}{x}$$
$$\sqrt{x^2} = \sqrt{1}$$
$$x = \pm 1$$

We lost the $x = 0$ answer!

Reason: You cannot divide (or cancel) anything that may equal zero.

Equations Involving Fractions:

Option 1: Rewrite all fractions so that they have the same denominator, then drop all denominators from the equation.

or

Option 2: Multiply the entire equation by the LCD to clear the fractions.

Then: Solve normally.

Be advised: If your answer makes any of the original fractions undefined, it must be rejected!!

$$\frac{4}{x-1}+\frac{3}{x}=\frac{3}{1}$$

LCD: $x(x-1)$

cross cancel:

cancel any numerator with any denominator when multiplying.

$$\frac{4}{\cancel{x-1}}\cdot\frac{x\cancel{(x-1)}}{1}+\frac{3}{\cancel{x}}\cdot\frac{\cancel{x}(x-1)}{1}=\frac{3}{1}\cdot\frac{x(x-1)}{1}$$

$$4x+3x-3=3x^2-3x$$

$$\cancel{7x}\;\cancel{-3}=3x^2-3x$$

$$\cancel{-7x}\;\cancel{+3}\qquad -7x+3$$

$$0=3x^2-10x+3$$

$$3x^2-10x+3=0$$

$3x^2 \underbrace{-10x}+3=0$

$(3)(3)=9$

$-9, -1$

$(3x^2-9x)(-x+3)=0$

$3x(x-3)-1(x-3)=0$

$(x-3)(3x-1)=0$

$x-\cancel{3}=0$

$\cancel{+3} \quad +3$

$x=3$

$3x\cancel{-1}=0$

$\cancel{+1} \quad +1$

$\frac{\cancel{3}x}{\cancel{3}}=\frac{1}{3}$

$x=1/3$

$\{1/3, 3\}$

Equations involving radicals:

If an equation involves a square root (also called a radical), you must isolate the radical, square both sides, and solve the remaining equation. Be certain to check your answers!

Find all solutions to $\sqrt{x+8} - 2 = x$

$\qquad\qquad +2 \qquad +2$

$\left(\sqrt{x+8}\right)^2 = (x+2)^2$

$x+8 = (x+2)(x+2)$

$x+8 = x^2+2x+2x+4$

$\cancel{x+8} = x^2+4x+4$

$-x-8 \qquad -x-8$

$0 = x^2+3x-4$

$0 = (x+4)(x-1)$

$x+4=0 \qquad x-1=0$

reject $x=-4 \qquad \boxed{x=1}$

Check $x=-4$

$\sqrt{-4+8}-2=-4$

$\sqrt{4}-2=-4$

$2-2=-4$

$0 \neq -4$

Check $x=1$

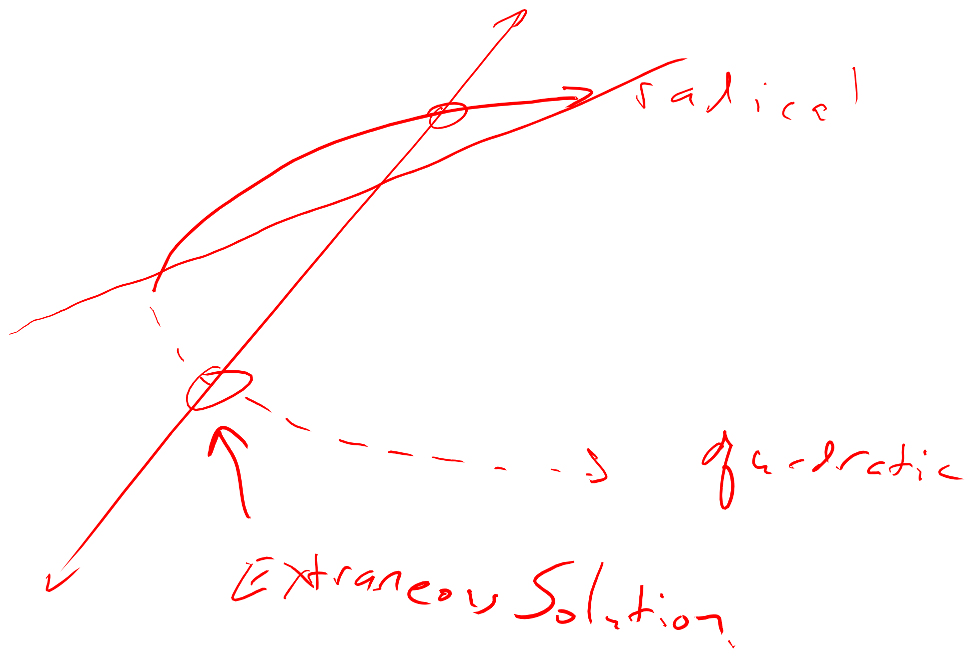
$\sqrt{1+8}-2=1$

$\sqrt{9}-2=1$

$3-2=1$

$1=1$ ✓

Extraneous Solutions: In a radical solution, you may "create" additional answers that are not correct. These must be rejected!



Find all solutions to $\sqrt{3x+1} - 1 = x$

$+1 \qquad +1$

$\left(\sqrt{3x+1}\right)^2 = (x+1)^2$

$3x+1 = (x+1)(x+1)$

$3x+1 = x^2+x+x+1$

$3x+1 = x^2+2x+1$

$-3x-1 \qquad -3x \quad -1$

$0 = x^2 - x$

$0 = x(x-1)$

$x = 0 \qquad x-1=0$

$x = 1$

Check $x=0$

$\sqrt{3(0)+1} - 1 = 0$

$\sqrt{1} - 1 = 0$

$1-1=0$

$0=0$ ✓

Check $x=1$

$\sqrt{3(1)+1} - 1 = 1$

$\sqrt{4} - 1 = 1$

$2 - 1 = 1$

$1 = 1$ ✓

$\{0, 1\}$

Popper # 1

1. Solve the following: $x_3 = 9x$

   a. 0, 3   b. 0, 9, -9   c. 0,-3, 3   d. -3, 3

$x^3 - 9x = 0$

$x(x^2 - 9) = 0$

$\boxed{x = 0}$ $\quad x^2 - 9 = 0 \rightarrow x^2 = 9$

$\boxed{x = \pm 3}$

2. Solve the following: $x + \sqrt{x+1} = 5$

   a. 8, 3   b. 3   c. 8   d. No Answer

$\rightarrow (\sqrt{x+1})^2 = (-x+5)^2$

$x+1 = (-x+5)(-x+5)$

$x+1 = x^2 - 5x - 5x + 25$

$x+1 = x^2 - 10x + 25$

$-x \quad -1 \qquad -x \quad -1$

$0 = x^2 - 11x + 24$

$0 = (x-3)(x-8)$

$x - 3 = 0 \qquad x - 8 = 0$

$x = 3 \qquad x = 8$ reject

3. Solve the following: $\frac{8}{x+1} + \frac{3}{x} = 3$

   a. 3   b. $^{-1}/_3$ c. -1   d. 3, $^{-1}/_3$

Check $x=3$

$3 + \sqrt{3+1} = 5$

$3 + \sqrt{4} = 5$

$3 + 2 = 5$ ✓

Check $x=8$

$8 + \sqrt{8+1} = 5$

$8 + \sqrt{9} = 5 \rightarrow 8 + 3 \neq 5$

$$\frac{8}{x+1} + \frac{3}{x} = \frac{3}{1} \qquad LCD: x(x+1)$$

$$\frac{8}{\cancel{x+1}} \cdot \frac{x(\cancel{x+1})}{1} + \frac{3}{\cancel{x}} \cdot \frac{\cancel{x}(x+1)}{1} = \frac{3}{1} \cdot \frac{x(x+1)}{1}$$

$$8x + 3x+3 = 3x^2+3x$$

$$\begin{aligned} \cancel{11x} + \cancel{3} &= 3x^2 + 3x \\ -11x \quad -3 & \quad\;\; -11x \quad -3 \\ \hline 0 &= 3x^2 - 8x - 3 \end{aligned}$$

$$0 = (3x^2 - 9x) + (x - 3)$$

$$0 = 3x(x-3) + 1(x-3)$$

$$0 = (x-3)(3x+1)$$

$$x - 3 = 0 \qquad x = 3$$

$$\begin{aligned} 3x + \cancel{1} &= 0 \\ -1 \quad & -1 \\ \hline \frac{\cancel{3}x}{\cancel{3}} &= \frac{-1}{3} \\ x &= -\tfrac{1}{3} \end{aligned}$$

$$\left\{ -\tfrac{1}{3}, 3 \right\}$$

Solving by Substitution:

When a function looks, “almost” quadratic, you may want to solve it by relating it to another function.

÷2   Not

$$x^{10} - x^5 - 6 = 0$$

If the exponents go “full amount → half amount → nothing” then you can rewrite as a quadratic.

Let u = x^5*, then*

$$\boxed{x^{10}} - \boxed{x^5} - 6 = 0 \rightarrow u^2 - u - 6 = 0$$

Middle Term: x^{power} only

(unless inside terms)

$u = x^5$

$u^2 = (x^5)^2 = x^{10}$

$x^{10} - x^5 - 6 = 0$

$u = x^5$, $u^2 = x^{10}$

$u^2 - u - 6 = 0$

$(u-3)(u+2)=0$

$u - \cancel{3} = 0$

$+3 \quad +3$

$u = 3$

$u + \cancel{2} = 0$

$-2 \quad -2$

$u = -2$

Since $u = x^5$

$u = 3$

$x^5 = 3$

$\boxed{x = \sqrt[5]{3}}$

$u = -2$

$x^5 = -2$

$\boxed{x = \sqrt[5]{-2}}$

$$x^{1/2} + 2x^{1/4} - 15 = 0$$

$u = x^{1/4}$, $u^2 = (x^{1/4})^2 = x^{1/2}$

$u^2 + 2u - 15 = 0$

$(u+5)(u-3) = 0$

$u + 5 = 0 \qquad u - 3 = 0$

$u = -5 \qquad u = 3$

alternate check: $x = 81$

$\sqrt{81} + 2\sqrt[4]{81} - 15 = 0$

$9 + 2(3) - 15 = 0$

$9 + 6 - 15 = 0$ ✓

solving a radical

$u = -5$

$(x^{1/4})^4 = (-5)^4$

$x = 625$

check $x = 625$

$\sqrt[4]{625} = -5$

$5 = -5$

reject

$u = 3$

$(x^{1/4})^4 = (3)^4$

$\boxed{x = 81}$

check $x = 81$

$\sqrt[4]{81} = 3$

$3 = 3$ ✓

$$2(x+5)-\sqrt{x+5}-10=0$$

$\boxed{x = 5/4}$

$u = \sqrt{x+5}$ $\quad u^2 = (\sqrt{x+5})^2 = x+5$

$2u^2 - u - 10 = 0$

$(2u^2 + 4u)(-5u - 10) = 0$

$2u(u+2) - 5(u+2) = 0$

$(u+2)(2u-5) = 0$

$u+2=0$ $\quad 2u-5=0$

$u=-2$ $\quad 2u=5$

$u = 5/2$

$u = -2$

$(\sqrt{x+5})^2 = (-2)^2$

$x+5 = 4$

$-5 \quad -5$

$x = -1$

reject

Check $x=-1$

$\sqrt{-1+5} = -2$

$\sqrt{4} = -2$

$2 \cancel{=} -2$

$u = 5/2$

$(\sqrt{x+5})^2 = (5/2)^2$

$x+5 = \frac{25}{4}$

$-5 \quad -5$

$x = \frac{25}{4} - \frac{5 \cdot 4}{1 \cdot 4}$

$x = \frac{25}{4} - \frac{20}{4} = \boxed{5/4}$

Check $x = 5/4$

$\sqrt{\frac{5}{4} + \frac{20}{4}} = 5/2$

$\sqrt{\frac{25}{4}} = \frac{5}{2}$ ✓

Popper 2: $\frac{1}{x^2} + \frac{5}{x} + 6 = 0$

$\frac{5}{x} = 5 \cdot \frac{1}{x}$

u

$u^2 + 5u + 6 = 0$

1. What substitution should be made?

a. u = x   b. u = 5/x   c. u = 1/x   d. u = 5/x^2

$u^2 = \frac{1}{x^2}$

2. How does the equation re-write?

a. $u^2 + 5u + 6 = 0$ b. $0.2u^2 + u + 6 = 0$ c. $u^2 + 5u = 0$ d. $5u^2 + 6 = 0$

3. What is the value(s) of u?

a. {-1, 6}   b. {2, 3}   c. {-3, -2}   d. {0, 5}

4. What is the value(s) of x?

a. {2, 3, 5}   b. {2, 3}   c. {-3, -2}   d. {-1/2, -1/3}

$u^2 + 5u + 6 = 0$

$(u+2)(u+3) = 0$

$u+2=0$ $\quad$ $u+3=0$

$u=-2$ $\quad$ $u=-3$

$\frac{1}{x} = \frac{-2}{1}$ $\quad$ $\frac{1}{x} = \frac{-3}{1}$

$x = \frac{-1}{2}$ $\quad$ $x = \frac{-1}{3}$

$$x + 4\sqrt{x} - 5 = 0$$

$-x \qquad +5 \qquad -x+5$

$$\frac{4\sqrt{x}}{4} = \frac{-x+5}{4}$$

$$\sqrt{x}^2 = \left(\frac{-x+5}{4}\right)^2$$

$$16 \cdot x = \frac{(-x+5)(-x+5)}{16} \cdot 16$$

$$16x = x^2 - 5x - 5x + 25$$

$$16x = x^2 - 10x + 25$$

$-16x \qquad -16x$

$$x^2 - 26x + 25 = 0$$

$$(x-25)(x-1) = 0$$

$x - 25 = 0 \qquad x - 1 = 0$

$x = 25 \qquad \boxed{x = 1}$

reject

Check $x = 25$

$$25 + 4\sqrt{25} - 5 = 0$$

$$25 + 4(5) - 5 = 0$$

$$25 + 20 - 5 = 0$$

$$40 = 0$$

Check $x = 1$

$$1 + 4\sqrt{1} - 5 = 0$$

$$1 + 4 - 5 = 0$$

$$0 = 0 \checkmark$$

$$x + 4\sqrt{x} - 5 = 0$$

$u = \sqrt{x}$ $\quad$ $u^2 = \sqrt{x}^2 = x$

$u^2 + 4u - 5 = 0$

$(u+5)(u-1) = 0$

$u + 5 = 0$ $\quad$ $u - 1 = 0$

$u = -5$ $\quad$ $u = 1$

$\sqrt{x}^2 = (-5)^2$ $\quad$ $\sqrt{x}^2 = 1^2$

$x = 25$ $\quad$ $\boxed{x = 1}$

reject

Check $x = 25$

$\sqrt{25} = -5$

$5 \neq -5$

Check $x = 1$

$\sqrt{1} = 1$

$1 = 1$ ✓