MATH 1314

Section 2.8

Absolute Value Equations

In this lesson, you'll learn to solve absolute value equations and inequalities.

Definition: The **absolute value** of *x*, denoted $|x|$, is the distance *x* s from 0.

Solving Absolute Value Equations
If C is positive, then $|x| = C$ if and only if $x = \pm C$.

$|7| = 7 \rightarrow$ Distance from zero

$|-7| = 7 \rightarrow$ Distance from zero

$|x| = 3$

which numbers are a distance of 3 from zero.

$x = 3$

$x = -3$

$|x+5| = 3$

$x + \cancel{5} = 3$
$\cancel{-5} \quad -5$
$x = -2$

$x + \cancel{5} = -3$
$\cancel{-5} \quad -5$
$x = -8$

Special Cases for |x| = C:

Case 1: If C is negative then the equation $|x|$ = C has no solution since absolute value cannot be negative.

Case 2: The solution of the equation $|x| = 0$ is $x = 0$.

Case 1: $|x| =$ Negative

$|x+3| = -2$ No Solution.

Case 2: $|x| = 0$ → single answer

$|x+3| = 0$

$x + \cancel{3} = 0$

$\quad \cancel{-3} \quad -3$ $\quad x = -3$

Solve the following:

① Isolate the Absolute Value
② Do Not Distribute over an absolute value.

a. $|2x - 3| = 7$

$2x - \cancel{3} = 7$
$\quad + \cancel{3} \quad +3$
$\frac{\cancel{2}x}{\cancel{2}} = \frac{10}{2}$
$x = 5$

$2x - \cancel{3} = -7$
$\quad + \cancel{3} \quad +3$
$\frac{\cancel{2}x}{\cancel{2}} = \frac{-4}{2}$
$x = -2$

$\boxed{\{-2, 5\}}$

b. $|6 - 2x| + \cancel{6} = 14$
$\quad \cancel{-6} \quad -6$

$|6 - 2x| = 8$

$\boxed{\{-1, 7\}}$

$\cancel{6} - 2x = 8$
$\cancel{-6} \quad\quad -6$
$\frac{-\cancel{2}x}{\cancel{-2}} = \frac{2}{-2}$
$x = -1$

$\cancel{6} - 2x = -8$
$\cancel{-6} \quad\quad -6$
$\frac{-\cancel{2}x}{\cancel{-2}} = \frac{-14}{-2}$
$x = 7$

c. $2|-3(2x - 8)| + 4 = 30$

$-4 \quad -4$

$2|-3(2x-8)| = 26$

$\frac{2|-3(2x-8)|}{2} = \frac{26}{2}$

$|-3(2x-8)| = 13$

$-3(2x-8) = 13$

$-6x + 24 = 13$

$-24 \quad -24$

$\frac{-6x}{-6} = \frac{-11}{-6}$

$x = 11/6$

$-3(2x-8) = -13$

$-6x + 24 = -13$

$-24 \quad -24$

$\frac{-6x}{-6} = \frac{-37}{-6}$

$x = 37/6$

$\boxed{\{11/6, 37/6\}}$

d. $-4\left|\frac{1}{2}x+1\right|+3=-11$

$$-3 \qquad -3$$

$$\frac{-4\left|\frac{1}{2}x+1\right|}{-4} = \frac{-14}{-4}$$

$$\left|\frac{1}{2}x+1\right| = \frac{14 \div 2}{4 \div 2} = \frac{7}{2}$$

$$2\left(\frac{1}{2}x+1 = \frac{7}{2}\right) \qquad 2\left(\frac{1}{2}x+1 = -\frac{7}{2}\right)$$

$$x+2=7 \qquad x+2=-7$$

$$-2 \quad -2 \qquad -2 \quad -2$$

$$x=5 \qquad x=-9$$

$$\boxed{\{-9, 5\}}$$

e. $|2x - 1| = |x + 7|$

Keep one side the same
change one side
check answers (only if x is on both sides)

$$\begin{aligned} 2x - 1 &= \cancel{x} + 7 \\ -x \quad & \quad \cancel{-x} \\ x \cancel{-1} &= 7 \\ +1 \quad & \quad +1 \\ x &= 8 \end{aligned}$$

check

$$|2(8) - 1| = |8 + 7|$$
$$|16 - 1| = |15|$$
$$15 = 15 \checkmark$$

$$\begin{aligned} 2x - 1 &= -\cancel{x} - 7 \\ +x \quad & \quad \cancel{+x} \\ 3x \cancel{+1} &= -7 \\ +1 \quad & \quad +1 \\ \frac{\cancel{3}x}{\cancel{3}} &= \frac{-6}{3} \\ x &= -2 \end{aligned}$$

Check

$$|2(-2) - 1| = |-2 + 7|$$
$$|-4 - 1| = |5|$$
$$|-5| = |5|$$
$$5 = 5 \checkmark$$

$$\boxed{\{-2, 8\}}$$

Popper 8:

$|x+8| = 8$

$x + 8 = 8$
$-8 \quad -8$
$x = 0$

$x + 8 = -8$
$-8 \quad -8$
$x = -16$

1. $4 + |x + 8| = 12$
   $-4 \qquad -4$

   a. {-8, 8} b. {0, 16} c. {-16, 0} d. No Answer

2. $|2x + 4| = 3$

   a. {-0.5} b. {-3.5} c. {-3.5, -0.5} d. No Answer

$2x + 4 = 3$
$-4 \quad -4$
$\frac{2x}{2} = \frac{-1}{2}$
$x = -1/2 = -0.5$

$2x + 4 = -3$
$-4 \quad -4$
$\frac{2x}{2} = \frac{-7}{2}$
$x = -7/2 = -3.5$

Popper 8…continued

3. $|3x-2|+1=4$ $\rightarrow |3x-2|=3$

   a. {-1/3, 5/3}   b. {1/3, 5/3}   c. {5/3}   d. No Answer

$3x-\cancel{2}=3$
$\underline{+\cancel{2}\quad +2}$
$\frac{\cancel{3}x}{\cancel{3}}=\frac{5}{3}$
$x=5/3$

$3x-\cancel{2}=-3$
$\underline{+\cancel{2}\quad +2}$
$\frac{\cancel{3}x}{\cancel{3}}=\frac{-1}{3}$
$x=-1/3$

4. $|x+3|=-4$

   a. {-7, 7}   b. {-7}   c. {-7, -1}   d. No Answer

|whatever| = Neg → No Answer

Try this one:

$$\cancel{8}\times\frac{3|x^2+2x-1|-5}{\cancel{8}}=2\times 8$$

$$3|x^2+2x-1|-5=16$$
$$+5 \quad +5$$
$$\frac{3|x^2+2x-1|}{3}=\frac{21}{3}$$
$$|x^2+2x-1|=7$$

$$x^2+2x-1=7$$
$$-7 \quad -7$$
$$x^2+2x-8=0$$
$$(x+4)(x-2)=0$$
$$x+4=0 \qquad x-2=0$$
$$x=-4 \qquad x=2$$

$$x^2+2x-1=-7$$
$$+7 \quad +7$$
$$x^2+2x+6=0$$
$$x=\frac{-2\pm\sqrt{2^2-4\cdot1\cdot6}}{2\cdot1}$$
$$x=\frac{-2\pm\sqrt{4-24}}{2}$$
$$x=\frac{-2\pm\sqrt{-20}}{2}=\frac{-2\pm2i\sqrt{5}}{2}$$
$$x=-1\pm i\sqrt{5}$$

$$\boxed{\{-4, 2, -1\pm i\sqrt{5}\}}$$

Next, we'll look at inequalities. The approach to these problems will depend on whether the problem is a "less than" problem or a "greater than" problem. If C is zero, then $x = 0$.

Solving Absolute Value Inequalities

If C is positive, then

a. $|x| < C$ if and only if $-C < x < C$.

b. $|x| \leq C$ if and only if $-C \leq x \leq C$.

c. $|x| > C$ if and only if $x > C$ or $x < -C$.

d. $|x| \geq C$ if and only if $x \geq C$ or $x \leq -C$

"Less Than" → 1 interval Answer

|whatever| < Number

becomes

−Number < inside < +Number

$|x+2| \leq 5 \rightarrow -5 \leq x+2 \leq 5$

$-2 \qquad -2 \qquad -2$

$-7 \leq x \leq 3$

"Greater Than" → 2 interval Answer

|whatever| > Number

becomes

inside < −Number or inside > +Number

$|x+2| \geq 5 \rightarrow x+2 \leq -5$ or $x+2 \geq 5$

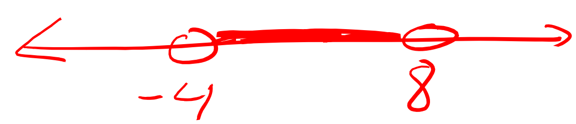
Example 2: Solve the inequality. Graph the solution on the real number line. Write the solution using interval notation:

a. $|x + 3| \leq 8$ (Less Than)

$$-8 \leq x + \cancel{3} \leq 8$$
$$-3 \qquad -\cancel{3} \qquad -3$$
$$-11 \leq x \leq 5 \rightarrow \boxed{[-11, 5]}$$

b. $|4 - 2x| < 12$ (Less Than)

$$-12 < \cancel{4} - 2x < 12$$
$$-4 \qquad -\cancel{4} \qquad -4$$
$$\frac{-16}{-2} < \frac{-\cancel{2}x}{-\cancel{2}} < \frac{8}{-2}$$
$$8 > x > -4 \rightarrow -4 < x < 8$$
$$\boxed{(-4, 8)}$$



c. $\cancel{3}|2x - 6| \leq 6$

$\frac{\cancel{3}|2x-6|}{\cancel{3}} \leq \frac{6}{3}$

$|2x-6| \leq 2$ (Less Than)

$-2 \leq 2x\cancel{-6} \leq 2$

$+6 \qquad \cancel{+6} \qquad +6$

$\frac{4}{2} \leq \frac{\cancel{2}x}{\cancel{2}} \leq \frac{8}{2}$

$2 \leq x \leq 4 \longrightarrow [2, 4]$

d. $|-3x + 1| < 4$ (Less Than)

$-4 < -3x\cancel{+1} < 4$

$-1 \qquad \cancel{-1} \qquad -1$

$\frac{-5}{-3} < \frac{\cancel{-3}x}{\cancel{-3}} < \frac{3}{-3}$

$5/3 > x > -1 \longrightarrow -1 < x < 5/3 \longrightarrow (-1, 5/3)$

e. $2|1-4x|+1>7$

$\quad -1 \qquad -1$

$\cancel{2}\,|1-4x| > 6$

$\quad \cancel{2} \qquad 2$

$|1-4x| > 3$ (Greater Than)

$1-4x < -3$ or $1-4x > 3$

$-1 \qquad -1 \qquad\qquad -1 \qquad -1$

$-4x < -4 \qquad -4x > 2$

$-4 \qquad -4 \qquad\qquad -4 \qquad -4$

$x > 1$ or $x < -1/2$

higher values $\qquad$ Lower values

$(-\infty, -1/2) \cup (1, \infty)$

smallest ⟶ largest

f. $-\frac{2}{3}|x-4| \le -\frac{4}{3}$ $\cdot -\frac{3}{2}$ $\quad \frac{-3}{2} = \frac{12}{6}$

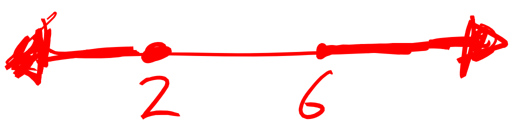
$|x-4| \ge 2$ (Greater Than)

$x-4 \le -2$ or $x-4 \ge 2$

$+4 \qquad +4 \qquad\qquad +4 \qquad +4$

$x \le 2$ or $x \ge 6$

$(-\infty, 2] \cup [6, \infty)$



Special Cases: Absolute Values Must be Positive Numbers!

$|x+2| < -3$
$|x+2| \leq -3$ No Solution

Case 1:

If C is negative, then:
a) The inequalities | *x* | < *C* and | *x* | ≤ *C* have no solution.
b) Every real number satisfies the inequalities | *x* | > *C* and | *x* | ≥ *C*

$|x+2| > -3$
$|x+2| \geq -3$ All Real Numbers $(-\infty, \infty)$

Case 2:

a) The inequality | *x* | < 0 has no solution.
b) The solution of the inequality | *x* | ≤ 0 is *x* = 0.
c) Every real number satisfies the inequality | *x* | ≥ 0

$|x+2| < 0$: No Solution

$|x+2| \leq 0$: $|x+2| < 0 \rightarrow$ No Solution
or
$|x+2| = 0 \rightarrow x+2=0$
$x=-2$

d) $|x| > 0$

$|x+2| \geq 0 \rightarrow (-\infty, \infty)$

$|x+2| > 0$ All Real Numbers except for the solution to $x+2=0$

$(-\infty, -2) \cup (-2, \infty)$
(Everything exc. $x=-2$)

Popper 8, continued:

5. $|2x + 6| \geq 8$ $\rightarrow$ $2x+\cancel{6} \leq -8$ or $2x+\cancel{6} \geq 8$

$-\cancel{6} \quad -6 \qquad -6 \quad -6$

a. [-7, 1] b. [-7, 7] c. (-∞, -1] U [7, ∞) d. (-∞, -7] U [1, ∞)

$\frac{\cancel{2}x}{\cancel{2}} \leq \frac{-14}{2}$ $\qquad$ $\frac{\cancel{2}x}{\cancel{2}} \geq \frac{2}{2}$

$x \leq -7$ $\qquad$ $x \geq 1$

6. $-4|x - 3| + \cancel{5} > -7$ $\rightarrow$ $\frac{-\cancel{4}|x-3|}{-\cancel{4}} > \frac{-12}{-4}$ $\rightarrow$ $|x-3| < 3$

$-\cancel{5} \quad -5$

a. (-∞, 0) U (6, ∞) b. (0, 6) c. (-6, 6) d. No Solution

$-3 < x - \cancel{3} < 3$

$+3 \quad +\cancel{3} \quad +3$

$0 < x < 6$

7. $|5x + 5| + \cancel{3} < 28$ $\rightarrow$ $|5x+5| < 25$

$-\cancel{3} \quad -3$

a. (-30, 20) b. (-6, 4) c. (-∞, 4) U (6, ∞) d. (-∞, -30) U (20, ∞)

$-25 < 5x + \cancel{5} < 25$

$-5 \quad -\cancel{5} \quad -5$

$-30 < 5x < 20$ $\rightarrow$ $\frac{-30}{5} < \frac{\cancel{5}x}{\cancel{5}} < \frac{20}{5}$

$-6 < x < 4$

Popper 8…continued

8. $5|x-12|+8 \le 8$ → $\frac{5|x-12|}{5} \le \frac{0}{5}$ → $|x-12| \le 0$

$\cancel{+8}$ -8

$|x-12|<0$
N.A.

$|x-12|=0$

$x-12=0$
$+12 \quad +12$

$x=12$

a. {12}

b. {0}

c. (-∞, ∞)

d. No Solution

9. $|2x+7|+9 \ge 4$

$-9 \quad -9$

a. (-∞, -6] U [-1, ∞)

b. [-6, -1]

c. (-∞, ∞)

d. No Solution

$|2x+7| \ge -5$