MATH 1314

Section 2.8

Absolute Value Equations

In this lesson, you'll learn to solve absolute value equations and inequalities.

Definition: The **absolute value** of *x*, denoted $|x|$, is the distance *x* s from 0.

Solving Absolute Value Equations
If C is positive, then $|x| = C$ if and only if $x = \pm C$.

$|7| = 7 \rightarrow$ Distance from zero

$|-7| = 7 \rightarrow$ Distance from zero

$|x| = 3$

which numbers are a distance of 3 from zero.

$x = 3$

$x = -3$

$|x+5| = 3$

$$\begin{aligned} x + \cancel{5} &= 3 \\ -\cancel{5} &\quad -5 \\ \hline x &= -2 \end{aligned} \qquad \begin{aligned} x + \cancel{5} &= -3 \\ -\cancel{5} &\quad -5 \\ \hline x &= -8 \end{aligned}$$

Special Cases for |x| = C:

Case 1: If C is negative then the equation $|x| = C$ has no solution since absolute value cannot be negative.

Case 2: The solution of the equation $|x| = 0$ is $x = 0$.

Case 1: $|x| =$ Negative

$|x+3| = -2$ No Solution.

Case 2: $|x| = 0$ → single answer

$|x+3| = 0$

$x + \cancel{3} = 0$

$\quad\ \cancel{-3} \quad -3$ $\qquad x = -3$

Solve the following:

① Isolate the Absolute Value
② Do Not Distribute over an absolute value.

a. $|2x - 3| = 7$

$$2x - \cancel{3} = 7$$
$$\quad +\cancel{3} \quad +3$$
$$\frac{\cancel{2}x}{\cancel{2}} = \frac{10}{2}$$
$$x = 5$$

$$2x - \cancel{3} = -7$$
$$\quad +\cancel{3} \quad +3$$
$$\frac{\cancel{2}x}{\cancel{2}} = \frac{-4}{2}$$
$$x = -2$$

$$\boxed{\{-2, 5\}}$$

b. $|6 - 2x| + \cancel{6} = 14$

$$\quad -\cancel{6} \quad -6$$
$$|6 - 2x| = 8$$

$$\cancel{6} - 2x = 8$$
$$-\cancel{6} \qquad -6$$
$$\frac{-\cancel{2}x}{-\cancel{2}} = \frac{2}{-2}$$
$$x = -1$$

$$\cancel{6} - 2x = -8$$
$$-\cancel{6} \qquad -6$$
$$\frac{-\cancel{2}x}{-\cancel{2}} = \frac{-14}{-2}$$
$$x = 7$$

$$\boxed{\{-1, 7\}}$$

c. $2|-3(2x-8)|+4=30$

$$\begin{aligned} -4 \quad & -4 \\ 2|-3(2x-8)| &= 26 \\ \div 2 \quad & \div 2 \\ |-3(2x-8)| &= 13 \end{aligned}$$

$-3(2x-8)=13$

$-6x+24=13$

$-24 \quad -24$

$-6x=-11$

$\div(-6) \quad \div(-6)$

$x=11/6$

$-3(2x-8)=-13$

$-6x+24=-13$

$-24 \quad -24$

$-6x=-37$

$\div(-6) \quad \div(-6)$

$x=37/6$

$$\boxed{\{11/6,\ 37/6\}}$$

d. $-4\left|\frac{1}{2}x+1\right|+3=-11$

$$\begin{aligned} -4\left|\tfrac{1}{2}x+1\right| + 3 &= -11 \\ -3 \quad & \quad -3 \\ \hline -4\left|\tfrac{1}{2}x+1\right| &= -14 \\ \overline{-4} \quad & \quad \overline{-4} \\ \left|\tfrac{1}{2}x+1\right| &= \frac{14 \div 2}{4 \div 2} = \frac{7}{2} \end{aligned}$$

$$2\left(\tfrac{1}{2}x+1 = \tfrac{7}{2}\right) \qquad 2\left(\tfrac{1}{2}x+1 = -\tfrac{7}{2}\right)$$

$$\begin{aligned} x+2 &= 7 \\ -2 \quad & \quad -2 \\ \hline x &= 5 \end{aligned} \qquad \begin{aligned} x+2 &= -7 \\ -2 \quad & \quad -2 \\ \hline x &= -9 \end{aligned}$$

$$\boxed{\{-9, 5\}}$$

e. $|2x - 1| = |x + 7|$

Keep one side the same
change one side
check answers (only if x is on both sides)

$$\begin{aligned} 2x - 1 &= x + 7 \\ -x \quad & \quad -x \\ \hline x - 1 &= 7 \\ +1 \quad & \quad +1 \\ \hline x &= 8 \end{aligned}$$

check

$$|2(8) - 1| = |8 + 7|$$
$$|16 - 1| = |15|$$
$$15 = 15 \checkmark$$

$$\begin{aligned} 2x - 1 &= -x - 7 \\ +x \quad & \quad +x \\ \hline 3x - 1 &= -7 \\ +1 \quad & \quad +1 \\ \hline 3x &= -6 \\ \frac{3x}{3} &= \frac{-6}{3} \\ x &= -2 \end{aligned}$$

$$\boxed{\{-2, 8\}}$$

Check

$$|2(-2) - 1| = |-2 + 7|$$
$$|-4 - 1| = |5|$$
$$|-5| = |5|$$
$$5 = 5 \checkmark$$

Popper 5:

1. $4 + |x + 8| = 12$

$-4 \qquad -4$

$|x+8| = 8$

$x + 8 = 8$
$-8 \quad -8$
$x = 0$

$x + 8 = -8$
$-8 \quad -8$
$x = -16$

a. {-8, 8} b. {0, 16} c. {-16, 0} d. No Answer

2. $|2x + 4| = 3$

a. {-0.5} b. {-3.5} c. {-3.5, -0.5} d. No Answer

$2x + 4 = 3$
$-4 \quad -4$
$\frac{2x}{2} = \frac{-1}{2}$
$x = -1/2 = -0.5$

$2x + 4 = -3$
$-4 \quad -4$
$\frac{2x}{2} = \frac{-7}{2}$
$x = -7/2 = -3.5$

Popper 5…continued

3. $|3x - 2| + 1 = 4$ $\rightarrow |3x-2| = 3$

   a. {-1/3, 5/3} b. {1/3, 5/3} c. {5/3} d. No Answer

$3x - \cancel{2} = 3$
$+\cancel{2} \quad +2$
$\frac{\cancel{3}x}{\cancel{3}} = \frac{5}{3}$
$x = 5/3$

$3x - \cancel{2} = -3$
$+\cancel{2} \quad +2$
$\frac{\cancel{3}x}{\cancel{3}} = \frac{-1}{3}$
$x = -1/3$

4. $|x + 3| = -4$

   a. {-7, 7} b. {-7} c. {-7, -1} d. No Answer

|whatever| = Neg $\longrightarrow$ No Answer

Try this one:

$$8 \times \frac{3|x^2+2x-1|-5}{\cancel{8}} = 2 \times 8$$

$$3|x^2+2x-1| - 5 = 16$$
$$\quad +5 \quad +5$$

$$\frac{3|x^2+2x-1|}{3} = \frac{21}{3}$$

$$|x^2+2x-1| = 7$$

$$x^2+2x-1 = 7$$
$$-7 \quad -7$$
$$x^2+2x-8=0$$
$$(x+4)(x-2)=0$$
$$x+4=0 \qquad x-2=0$$
$$x=-4 \qquad x=2$$

$$x^2+2x-1 = -7$$
$$+7 \quad +7$$
$$x^2+2x+6=0$$
$$x = \frac{-2 \pm \sqrt{2^2-4\cdot 1\cdot 6}}{2\cdot 1}$$
$$x = \frac{-2 \pm \sqrt{4-24}}{2}$$
$$x = \frac{-2 \pm \sqrt{-20}}{2} = \frac{-2 \pm 2i\sqrt{5}}{2}$$
$$x = -1 \pm i\sqrt{5}$$

$$\boxed{\{-4, 2, -1 \pm i\sqrt{5}\}}$$

Next, we'll look at inequalities. The approach to these problems will depend on whether the problem is a "less than" problem or a "greater than" problem. If C is zero, then $x = 0$.

Solving Absolute Value Inequalities

If C is positive, then

a. $|x| < C$ if and only if $-C < x < C$.

b. $|x| \le C$ if and only if $-C \le x \le C$.

c. $|x| > C$ if and only if $x > C$ or $x < -C$.

d. $|x| \ge C$ if and only if $x \ge C$ or $x \le -C$

"Less Than" → 1 interval Answer

|whatever| < Number

becomes

−Number < inside < +Number

$|x+2| \le 5 \rightarrow -5 \le x+2 \le 5$

$-2 \quad -2 \quad -2$

$-7 \le x \le 3$

"Greater Than" → 2 interval Answer

|whatever| > Number

becomes

inside < −Number or inside > +Number

$|x+2| \ge 5 \rightarrow x+2 \le -5$ or $x+2 \ge 5$

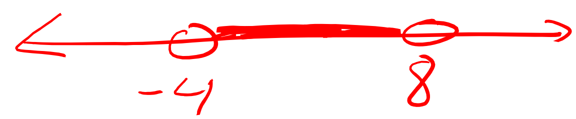
Example 2: Solve the inequality. Graph the solution on the real number line. Write the solution using interval notation:

a. $|x + 3| \le 8$ (Less Than)

$$-8 \le x + \cancel{3} \le 8$$
$$-3 \qquad -\cancel{3} \qquad -3$$
$$-11 \le x \le 5 \rightarrow \boxed{[-11, 5]}$$

b. $|4 - 2x| < 12$ (Less Than)

$$-12 < \cancel{4} - 2x < 12$$
$$-4 \qquad -\cancel{4} \qquad -4$$
$$-16 < -\cancel{2}x < 8$$
$$\frac{-16}{-2} \qquad \frac{-\cancel{2}x}{-\cancel{2}} \qquad \frac{8}{-2}$$
$$8 > x > -4 \rightarrow -4 < x < 8$$
$$\boxed{(-4, 8)}$$



c. $3|2x - 6| \le 6$

$$\frac{\cancel{3}|2x-6|}{\cancel{3}} \le \frac{6}{3}$$

$$|2x-6| \le 2 \quad \text{(Less Than)}$$

$$-2 \le 2x - \cancel{6} \le 2$$
$$+6 \qquad +6 \qquad +6$$

$$\frac{4}{2} \le \frac{\cancel{2}x}{\cancel{2}} \le \frac{8}{2}$$

$$2 \le x \le 4 \longrightarrow [2, 4]$$

d. $|-3x + 1| < 4$ (Less Than)

$$-4 < -3x \cancel{+1} < 4$$
$$-1 \qquad -1 \qquad -1$$

$$\frac{-5}{-3} < \frac{-3x}{-3} < \frac{3}{-3}$$

$$5/3 > x > -1 \longrightarrow -1 < x < 5/3 \longrightarrow (-1, 5/3)$$

e. $2|1-4x|+1>7$

$-1 \quad -1$

$\cancel{2}|1-4x| > 6$

$\cancel{2} \quad 2$

$|1-4x| > 3$ (Greater Than)

$1-4x < -3$ or $1-4x > 3$

$-1 \quad -1$ $\qquad$ $-1 \quad -1$

$-4x < -4$ $\qquad$ $-4x > 2$

$-4 \quad -4$ $\qquad$ $-4 \quad -4$

$x > 1$ or $x < -1/2$

higher values $\qquad$ Lower values

$(-\infty, -1/2) \cup (1, \infty)$

smallest $\longrightarrow$ largest

f. $-\frac{2}{3}|x-4| \le -\frac{4}{3}$ $\cdot -\frac{3}{2}$ $\quad -\frac{3}{2} = \frac{12}{6}$

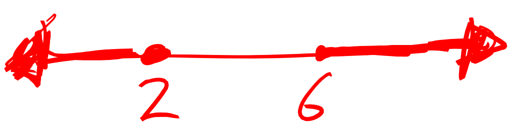
$|x-4| \ge 2$ (Greater Than)

$x-4 \le -2$ or $x-4 \ge 2$

$+4 \quad +4$ $\qquad$ $+4 \quad +4$

$x \le 2$ or $x \ge 6$

$(-\infty, 2] \cup [6, \infty)$



Special Cases: Absolute Values Must be Positive Numbers!

$|x+2| < -3$
$|x+2| \leq -3$ } No Solution

Case 1:

If C is negative, then:

a) The inequalities $|x| < C$ and $|x| \leq C$ have no solution.

b) Every real number satisfies the inequalities $|x| > C$ and $|x| \geq C$

$|x+2| > -3$
$|x+2| \geq -3$ } All Real Numbers $(-\infty, \infty)$

Case 2:

a) The inequality $|x| < 0$ has no solution.

b) The solution of the inequality $|x| \leq 0$ is $x = 0$.

c) Every real number satisfies the inequality $|x| \geq 0$

d) $|x| > 0$

$|x+2| < 0$: No Solution

$|x+2| \leq 0$ < $|x+2| < 0 \rightarrow$ No Solution, or $|x+2| = 0 \rightarrow x+2=0$, $x=-2$

$|x+2| \geq 0 \rightarrow (-\infty, \infty)$

$|x+2| > 0$ All Real Numbers except for the solution to $x+2=0$

$(-\infty,-2)\cup(-2,\infty)$ (only thing exc. $x=-2$)

Popper 6, continued:

1. $|2x + 6| \geq 8$ $\rightarrow$ $2x+\cancel{6} \leq -8$ or $2x+\cancel{6} \geq 8$

$-6 \quad -6 \qquad\qquad -6 \quad -6$

a. [-7, 1] b. [-7, 7] c. (-∞, -1] U [7, ∞) d. (-∞, -7] U [1, ∞)

$\frac{\cancel{2}x}{\cancel{2}} \leq \frac{-14}{2} \qquad \frac{\cancel{2}x}{\cancel{2}} \geq \frac{2}{2}$

$x \leq -7 \qquad x \geq 1$

2. $-4|x - 3| + \cancel{5} > -7$ $\rightarrow$ $\frac{\cancel{-4}|x-3|}{\cancel{-4}} > \frac{-12}{-4}$ $\rightarrow$ $|x-3| < 3$

$-5 \quad -5$

a. (-∞, 0) U (6, ∞) b. (0, 6) c. (-6, 6) d. No Solution

$-3 < x-\cancel{3} < 3$

$+3 \quad +3 \quad +3$

$0 < x < 6$

3. $|5x + 5| + \cancel{3} < 28$ $\rightarrow$ $|5x+5| < 25$

$-3 \quad -3$

a. (-30, 20) b. (-6, 4) c. (-∞, 4) U (6, ∞) d. (-∞, -30) U (20, ∞)

$-25 < 5x+\cancel{5} < 25$

$-5 \qquad -5 \qquad -5$

$-30 < 5x < 20$ $\rightarrow$ $\frac{-30}{5} < \frac{\cancel{5}x}{\cancel{5}} < \frac{20}{5}$

$-6 < x < 4$

Popper 6...continued

4. $5|x-12| + 8 \leq 8$

$-8 \quad -8$

$\rightarrow \frac{5|x-12|}{5} \leq \frac{0}{5} \rightarrow |x-12| \leq 0$

$|x-12| < 0$ N.A.

$|x-12| = 0$

$x-12=0$

$+12 \quad +12$

$x = 12$

a. {12} b. {0} c. (-∞, ∞) d. No Solution

5. $|2x + 7| + 9 \geq 4$

$-9 \quad -9$

a. (-∞, -6] U [-1, ∞) b. [-6, -1] c. (-∞, ∞) d. No Solution

$|2x+7| \geq -5$