MATH 1314

Section 3.5

Maximum and Minimum Values

A quadratic equation is of the form $f(x) = ax^2 + bx + c$, where a, b, and c are real and $a \neq 0$

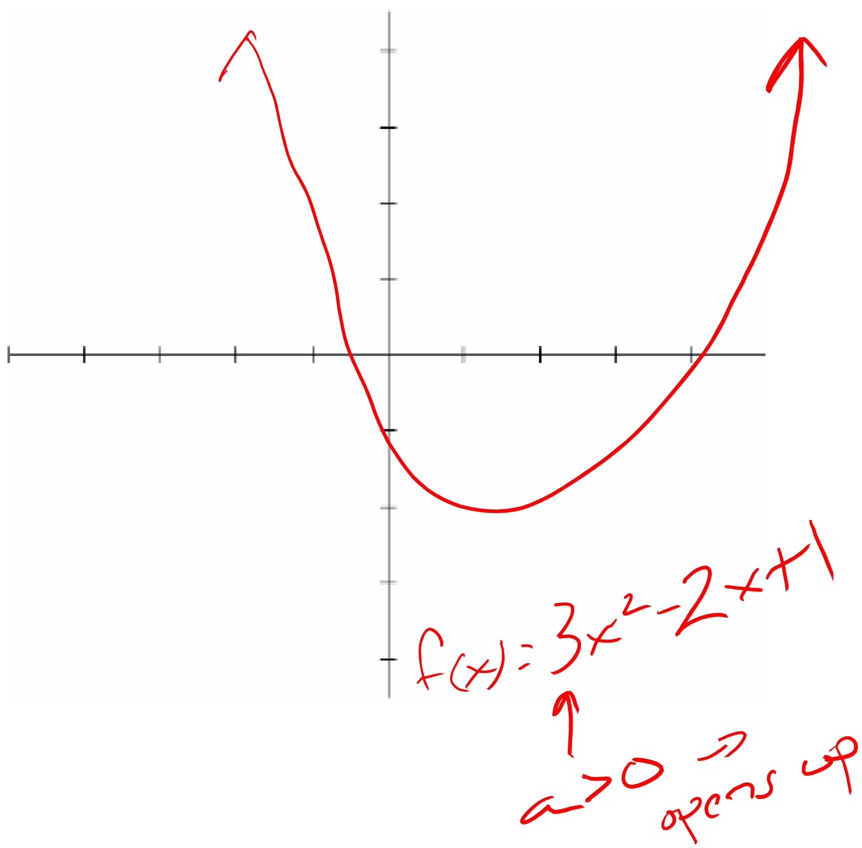
We have seen the graphs of **parabolas**.

$f(x) = ax^2 + bx + c$ Generic Form

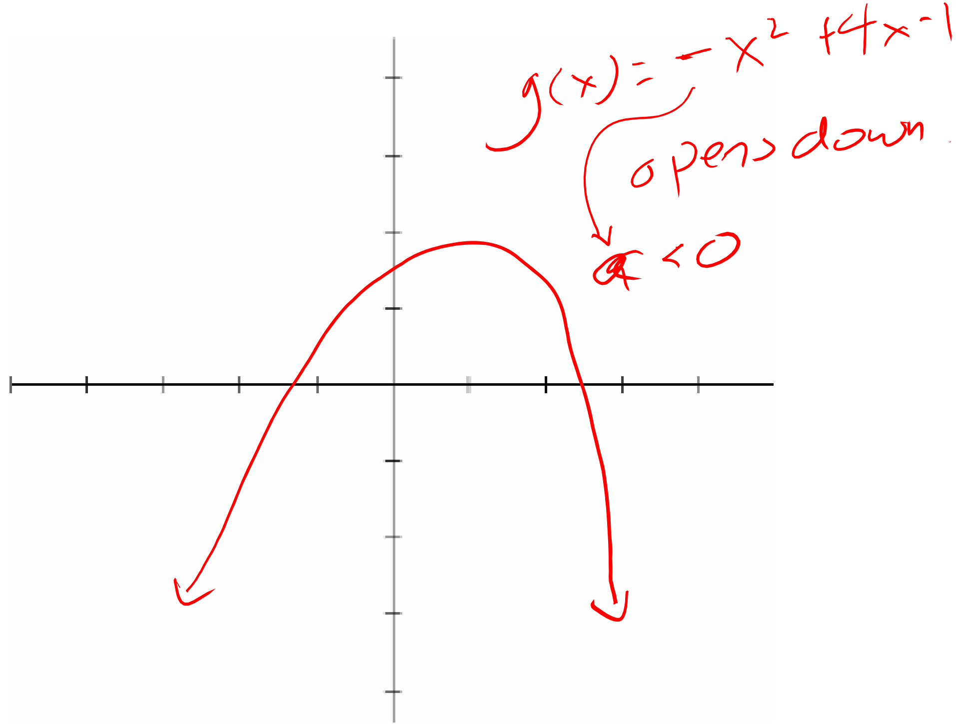
$f(x) = -2x^2 - 3x + 1$ Example

Opening Up or Opening Down

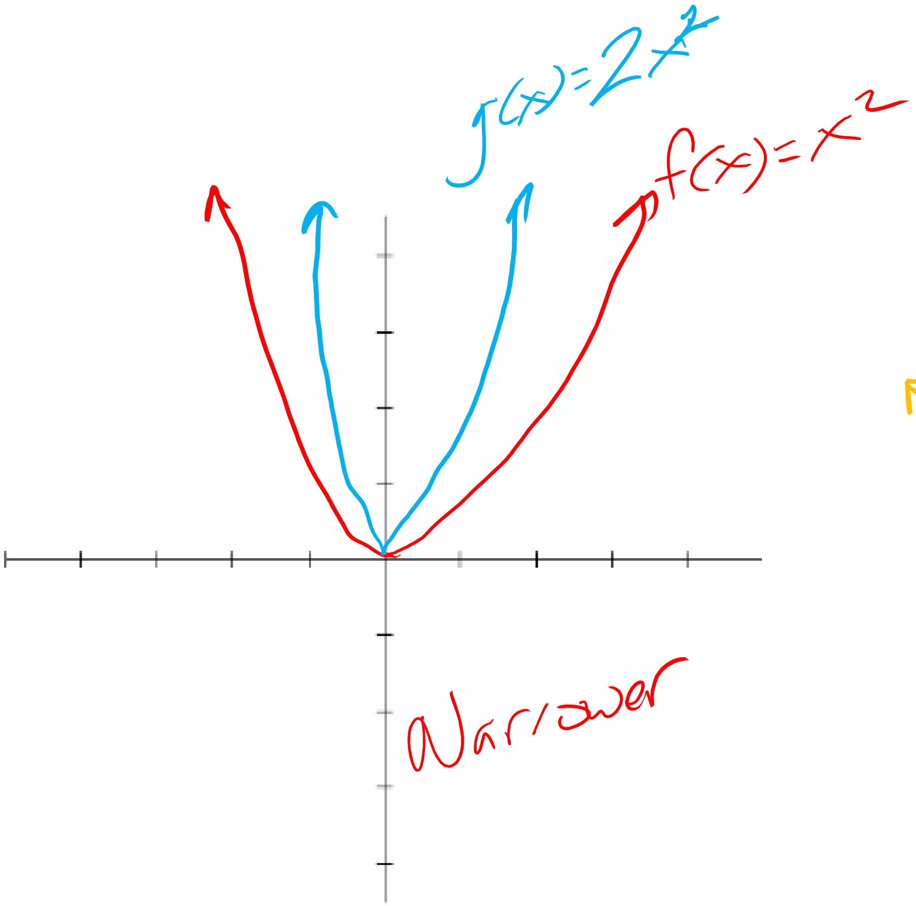
If $a > 0$ then the parabola will open upwards.

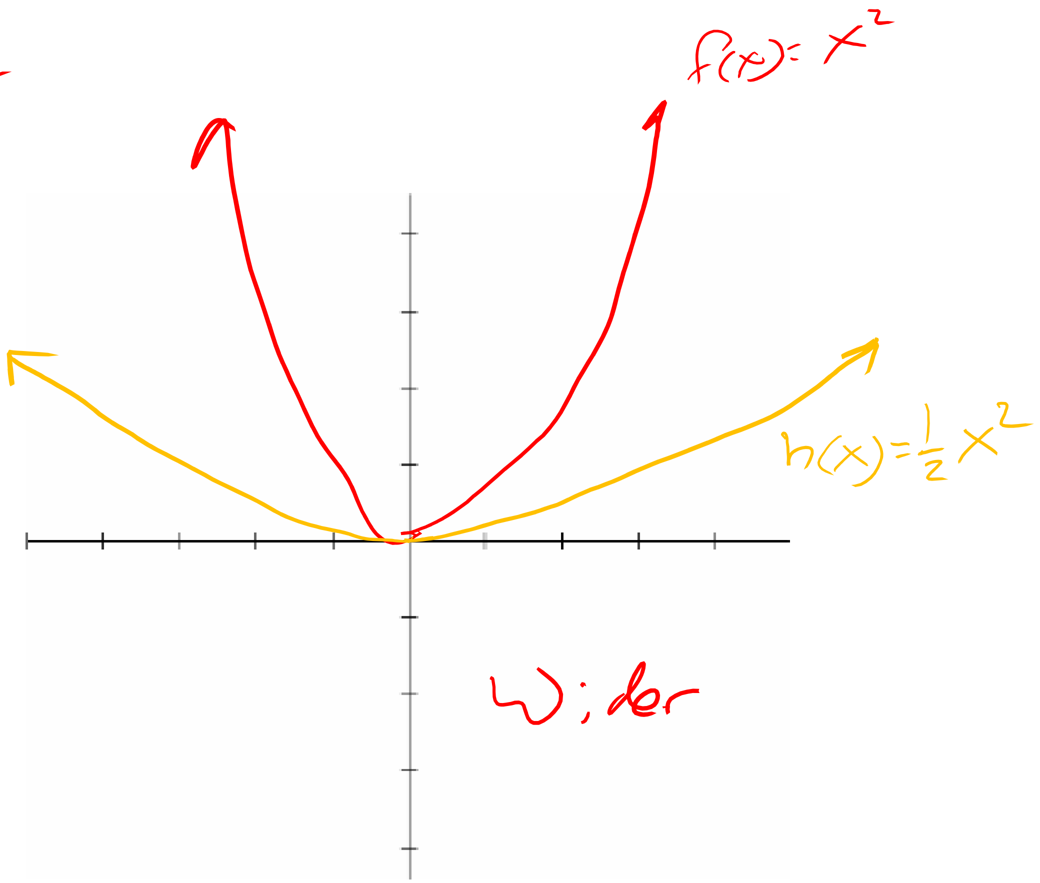


If $a < 0$ then the parabola will open downwards.



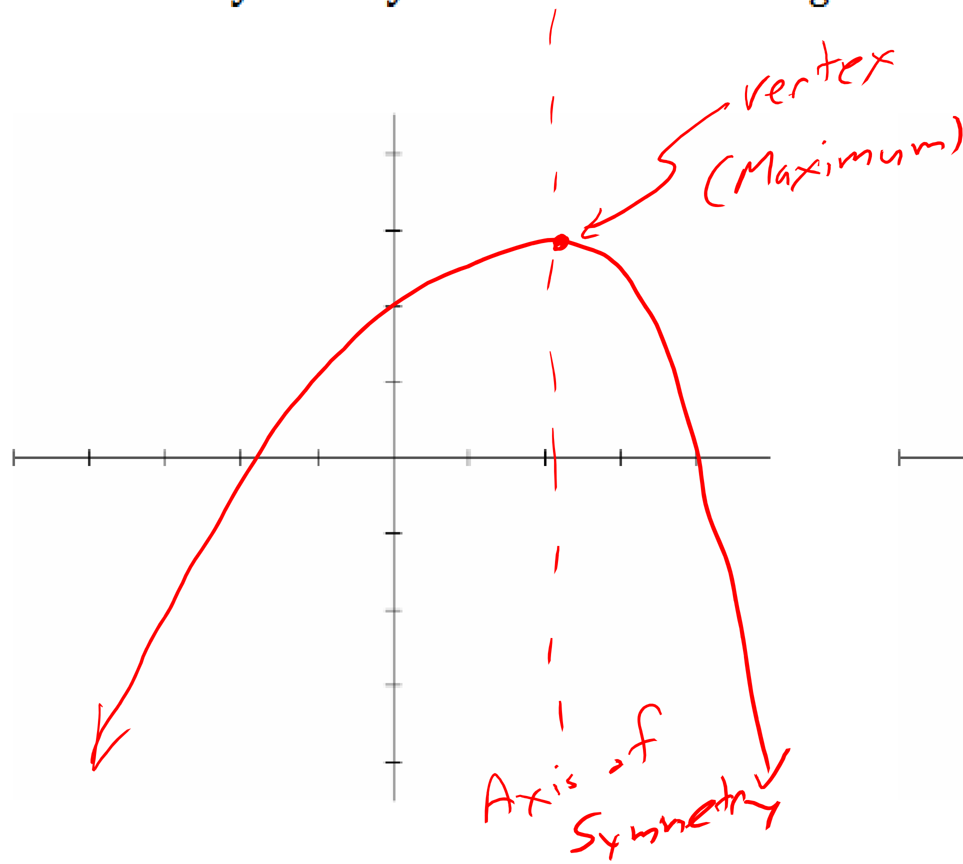
Note: The larger $|a|$, the narrower the parabola

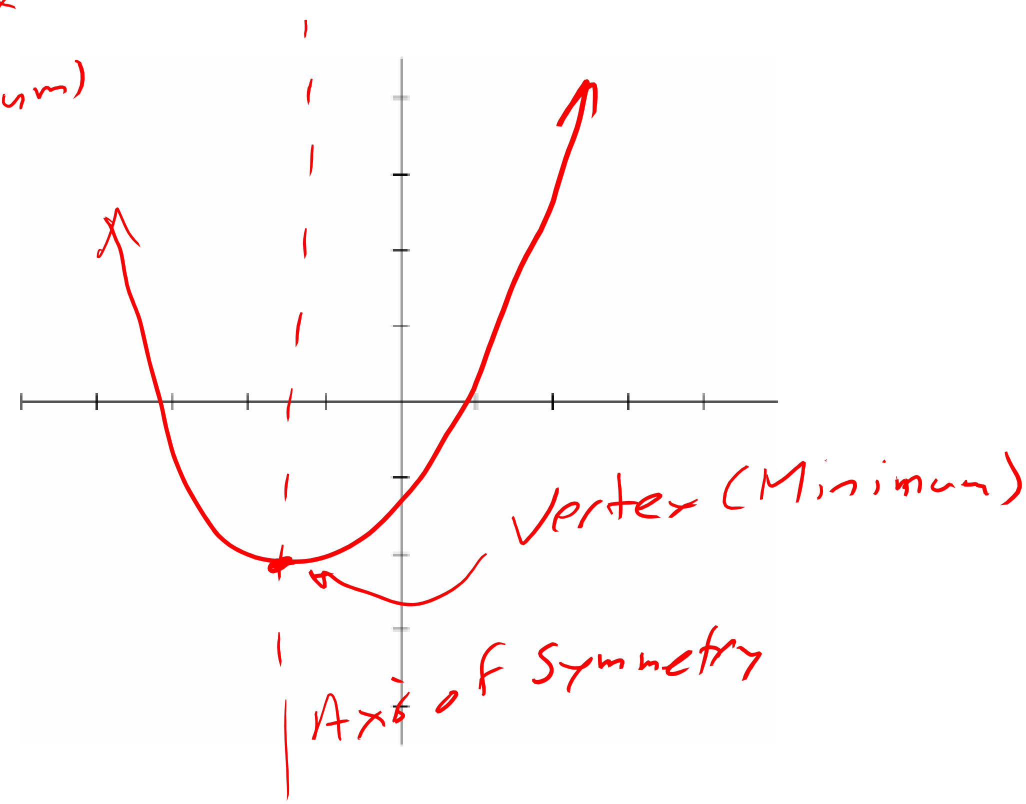




The **vertex** is the turning point of the parabola and is the **minimum point** on the graph when it opens upward and the **maximum point** on the graph when it opens downward. Every parabola has a maximum or minimum, but **NOT** both.

The **axis of symmetry** is a vertical line through the vertex that divides the graph in half.





The Standard form of a Quadratic Function

The quadratic function $\boxed{f(x) = a(x - h)^2 + k}$ is in **standard form**

x-int: set y=0 and solve.

y-int: set x=0 and solve.

The vertex is the point $(h,\ k)$ and the axis of symmetry is $x = h$

The domain is $(-\infty, \infty)$.

The range is $[k,\ \infty)$ if $a > 0$ or $(-\infty,\ k]$ if $a < 0$

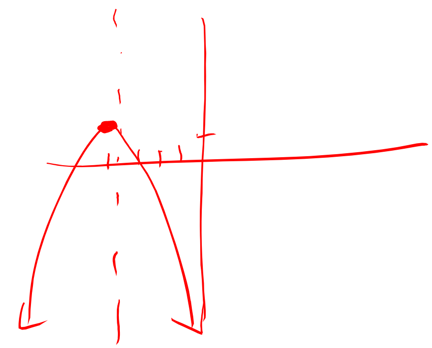
$f(x) = -2(x+4)^2 + 1$

$f(x) = a(x - \overset{-(-4)}{h})^2 + k$

$a = -2$

$h = -4$

$k = 1$



vertex: $(-4, 1)$

Axis of Symmetry: $x = -4$

Domain: $(-\infty, \infty)$

Direction: $a < 0$ open down

Range: $(-\infty, 1]$

Our first task will be to change a given quadratic function from the form $f(x) = ax^2 + bx + c$ to standard form. We'll complete the square to do this. Once the function is in standard form, we can sketch a graph using transformations and then read off the maximum or minimum value

General Form into Standard Form: Complete the Square

Standard Form into General: Simplify the expression

$f(x) = -2(x+4)^2 + 1$

$f(x) = -2(x+4)(x+4) + 1$

$f(x) = -2(x^2+4x+4x+16) + 1$

$f(x) = -2(x^2+8x+16) + 1$

$f(x) = -2x^2 - 16x - 32 + 1 \longrightarrow \boxed{f(x) = -2x^2 - 16x - 31}$

Example 1: Write the following quadratic in standard form. Then find the vertex and the axis of symmetry.

a. $f(x) = 3x^2 - 12x - 1$

$$f(x) = 3(x^2 - 4x) - 1 = \overset{a}{3}(x^2 - 4x + \overset{(\frac{b}{2})^2}{4}) \underbrace{- (3)(4)}_{-a \cdot (\frac{b}{2})^2} - 1$$

$b = -4$

$\frac{b}{2} = -2$

$(\frac{b}{2})^2 = 4$

$$f(x) = 3(x \underbrace{-2}_{b/2})^2 - 12 - 1$$

$$f(x) = 3(x - 2)^2 - 13$$

$$f(x) = a(x - h)^2 + k$$

$a = 3$

$h = 2$

$k = -13$

vertex: $(h, k) = \boxed{(2, -13)}$

Axis of Symmetry: $x = h$

$\boxed{x = 2}$

b. $f(x) = -x^2 + 2x + 3$

$$f(x) = -1\,(x^2 - 2x) + 3 = \overset{a}{-1}\,(x^2 - 2x + \overset{(b/2)^2}{1}) - \overbrace{(-1)(1)}^{-(a)(b/2)^2} + 3$$

$b = -2$

$\frac{b}{2} = -1$

$(b/2)^2 = 1$

$b/2$

$f(x) = -(x-1)^2 + 1 + 3$

$f(x) = -(x-1)^2 + 4$

$a = -1$

$h = 1$

$k = 4$

vertex: $(h, k) = (1, 4)$

axis of symmetry: $x = h \rightarrow x = 1$

Domain: $(-\infty, \infty)$

direction: opens down

Range: $(-\infty, k] \rightarrow (-\infty, 4]$

c. $f(x) = (-10x^2 + 60x)$

$f(x) = -10(x^2 - 6x) = -10(x^2 - 6x + 9) - (-10)(9)$

$b = -6$

$\frac{b}{2} = -3$

$\left(\frac{b}{2}\right)^2 = 9$

$f(x) = -10(x-3)^2 + 90$

vertex: $(h, k) \to (3, 90)$

Axis of Symmetry: $x = h \to x = 3$

What is the minimum/maximum value? Maximum of 90

opening down $(a = -10)$

what is: y-value

where is: x-value

Popper 13: $f(x) = (-2x^2 - 16x) - 27 =$

$-2(x^2 + 8x) - 27$

$b = 8$

$\frac{b}{2} = 4$

$(\frac{b}{2})^2 = 16$

$f(x) = -2(x^2 + 8x + 16) - (-2)(16) - 27$

$f(x) = -2(x+4)^2 + 5$

$a = -2$

$h = -4$

$k = 5$

1. Complete the square and rewrite in standard form:
   - a. $f(x) = -2(x + 9)^2$
   - b. $f(x) = 2(x + 4)^2 - 5$
   - c. $f(x) = -2(x + 4)^2 + 5$
   - d. $f(x) = 2(x - 4)^2 - 5$

2. Determine the direction of the parabola: $a = -2$
   - a. Opening Up
   - b. Opening Down

3. Determine the equation of the axis of symmetry: $x = 4$
   - a. $x = 4$
   - b. $x = -4$
   - c. $x = 5$
   - d. $x = -5$

4. Determine the coordinates of the vertex: (h, k)
   - a. (-4, 5)
   - b. (-5, 10)
   - c. (4, 5)
   - d. (-4, -5)

5. Is the vertex of this parabola a minimum or a maximum?
   - a. Minimum
   - b. Maximum
   - c. ~~Neither~~

Graphing Quadratic Functions with Equations in Standard Form

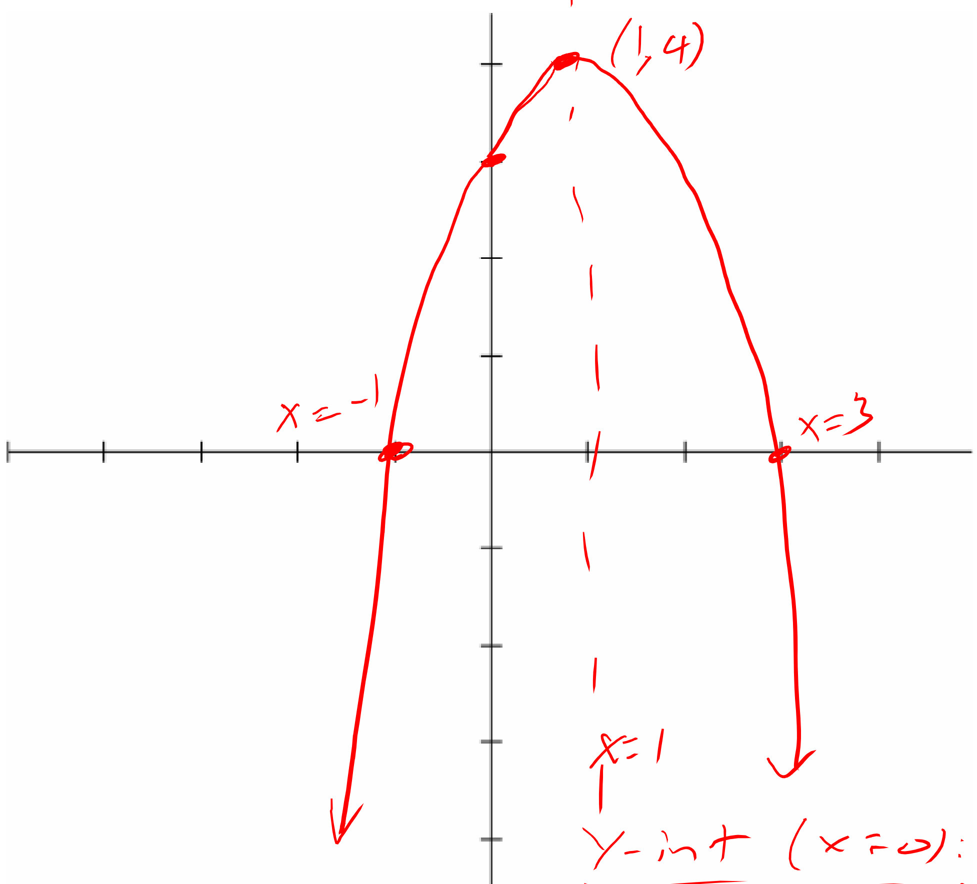
1. Determine whether the parabola opens upward or downward.
2. Determine the vertex.
3. Find any x-intercept by replacing $f(x)$ with 0 and then solving for x.
4. Find the y-intercept by replacing x with 0.
5. Plot the intercept(s) and vertex, sketch the graph and draw the axis of symmetry.

Example 2: Sketch the graph of $f(x) = -x^2 + 2x + 3$

$a=-1 \quad b=2 \quad c=3$

$x = \frac{-b}{2a} = \frac{-2}{2(-1)} = \frac{-2}{-2} = 1 \to h$

$f(1) = -(1)^2 + 2(1) + 3 = -1 + 2 + 3 = 4 \to k$



Standard form:

$f(x) = -1(x-1)^2 + 4$

Vertex: $(h,k) \to (1,4)$

Axis of Sym: $x = h \to x = 1$

Direction: $a = -1 \to$ Down

x-int: $(y=0)$

$0 = -(x-1)^2 + 4$

$-4 \qquad -4$

$-4 = -(x-1)^2$

$\div -1$

$\sqrt{4} = \sqrt{(x-1)^2}$

$x - 1 = \pm 2$

$+1 \qquad +1$

$x = 1-2,\ 1+2$

$x = -1$

$x = 3$

Y-int $(x=0)$:

$f(x) = \cancel{-x^2} \cancel{+2x} + 3$

$y = 3 \quad (y=c)$

Shortcut:

For $f(x) = ax^2 + bx + c$, the vertex is $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$. So the axis of symmetry is $x = -\frac{b}{2a}$.

Axis of symmetry: $x = \frac{-b}{2a} \longrightarrow h$

$f\left(\frac{-b}{2a}\right) \longrightarrow k$

$f(x) = a(x-h)^2 + k$

Example 3: Let $f(x) = 2x^2 + 4x + 7$. Determine, without graphing, whether the given quadratic function has a minimum or maximum value. Then find the coordinates of the minimum or maximum point

Min or Max? : $a = 2 > 0$ opens up

Minimum Value

$$x = \frac{-b}{2a} = \frac{-4}{2(2)} = \frac{-4}{4} = -1 \rightarrow h$$

$$f(-1) = 2(-1)^2 + 4(-1) + 7 = 2(1) - 4 + 7 = 2 - 4 + 7 = 5 \rightarrow k$$

vertex: $(h, k) \rightarrow \boxed{(-1, 5)}$

↳ Minimum

Example 4: Suppose $f\,(x) = 5x^2 - 30x + 41$. Write the equation in standard form. State the coordinates of the vertex. Determine, without graphing, whether the given quadratic function has a minimum or maximum value. Then find the coordinates of the minimum or maximum point.

Min or Max : $a = 5 > 0$ opens up $\rightarrow$ $\boxed{\text{Minimum}}$

$$x = \frac{-b}{2a} = \frac{-(-30)}{2(5)} = \frac{30}{10} = 3 \rightarrow h$$

$$f(3) = 5(3)^2 - 30(3) + 41 = 5(9) - 30(3) + 41 = 45 - 90 + 41$$

$$= -45 + 41 = -4$$

k

vertex: $(h, k) \rightarrow \boxed{(3, -4)}$

Finally, given the vertex of a quadratic function and one other point that lies on the graph of the quadratic function, you should be able to write the quadratic function.

Example 5: Find a quadratic function with vertex (2, 6) which passes through (-1, 4).

$f(x) = a(x-h)^2+k$

$(h,k) \to h=2$, $k=6$

plug in for x, y

$f(x) = a(x-2)^2+6$

$4 = a(-1-2)^2+6$

$4 = a(-3)^2+6$

$4 = 9a+6$

$-6 \quad\quad -6$

$\frac{-2}{9} = \frac{9a}{9}$

$a = -2/9$

$\boxed{f(x) = \frac{-2}{9}(x-2)^2+6}$ * (may have to go farther)

$f(x) = \frac{-2}{9}(x-2)(x-2)+6$

$f(x) = -\frac{2}{9}(x^2-4x+4)+6$

$\frac{6}{1}\cdot\frac{9}{9} = \frac{54}{9}$

$f(x) = -\frac{2}{9}x^2+\frac{8}{9}x-\frac{8}{9}+\frac{54}{9}$

$\boxed{f(x) = -\frac{2}{9}x^2+\frac{8}{9}x+\frac{46}{9}}$ *

Example 6: Find a quadratic function with vertex (3, -1) which passes through (5, 7).

$h = 3$

$k = -1$

$f(x) = a(x-h)^2 + k$

$f(x) = a(x-3)^2 - 1$

$7 = a(5-3)^2 - 1$

$7 = a(2)^2 - 1$

$7 = 4a \cancel{-1}$

$+1 \qquad \cancel{+1}$

$\frac{8}{4} = \frac{4a}{\cancel{4}}$

$2 = a$

$\boxed{f(x) = 2(x-3)^2 - 1}$

$f(x) = 2(x-3)(x-3) - 1$

$f(x) = 2(x^2 - 6x + 9) - 1$

$f(x) = 2x^2 - 12x + 18 - 1$

$\boxed{f(x) = 2x^2 - 12x + 17}$

Determine the equation of a parabola has x-intercepts of (5,0) and (-1,0) and a y-intercept of (0,-10).

$$\left.\begin{array}{l} x\text{-int}: \; x=5 \rightarrow x-5=0 \\ \qquad\quad x=-1 \rightarrow x+1=0 \end{array}\right\} f(x)=a(x-5)(x+1)$$

$f(x) = a(x-5)(x+1)$

plug in $(0,-10)$

$-10 = a(0-5)(0+1)$

$-10 = a(-5)(1)$

$\frac{-10}{-5} = \frac{-\cancel{5}a}{-5}$

$2 = a$

$$\boxed{f(x) = 2(x-5)(x+1)}$$

$f(x) = 2(x^2-4x-5)$

$$\boxed{f(x) = 2x^2-8x-10}$$