

MATH 1314

Section 3.5

Maximum and Minimum Values

A quadratic equation is of the form $f(x) = ax^2 + bx + c$, where a , b , and c are real and $a \neq 0$

We have seen the graphs of **parabolas**.

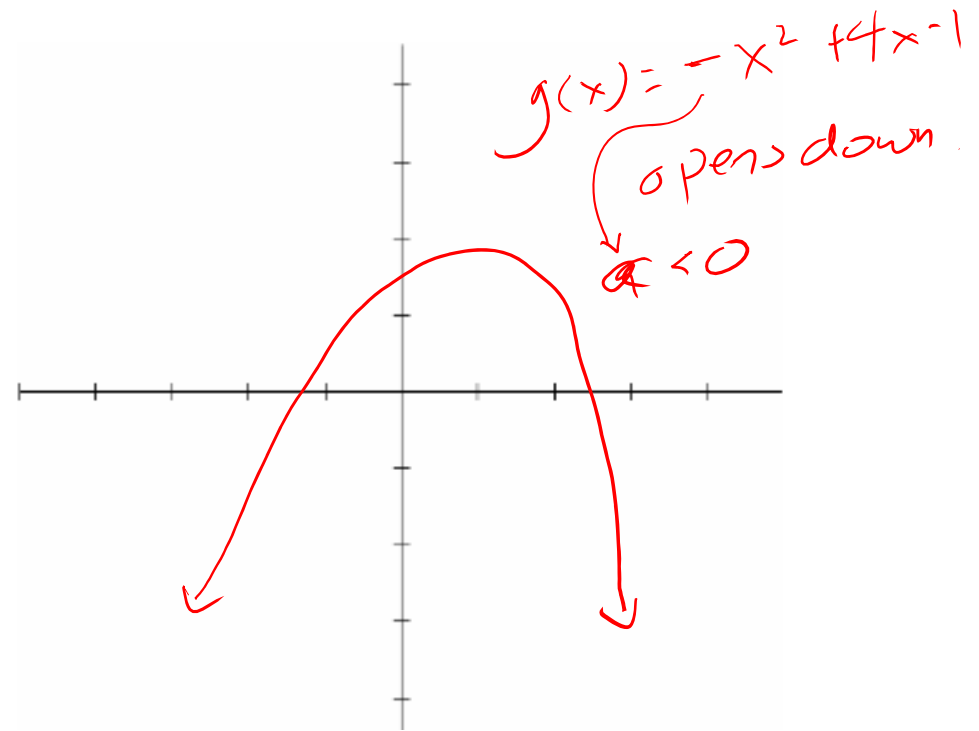
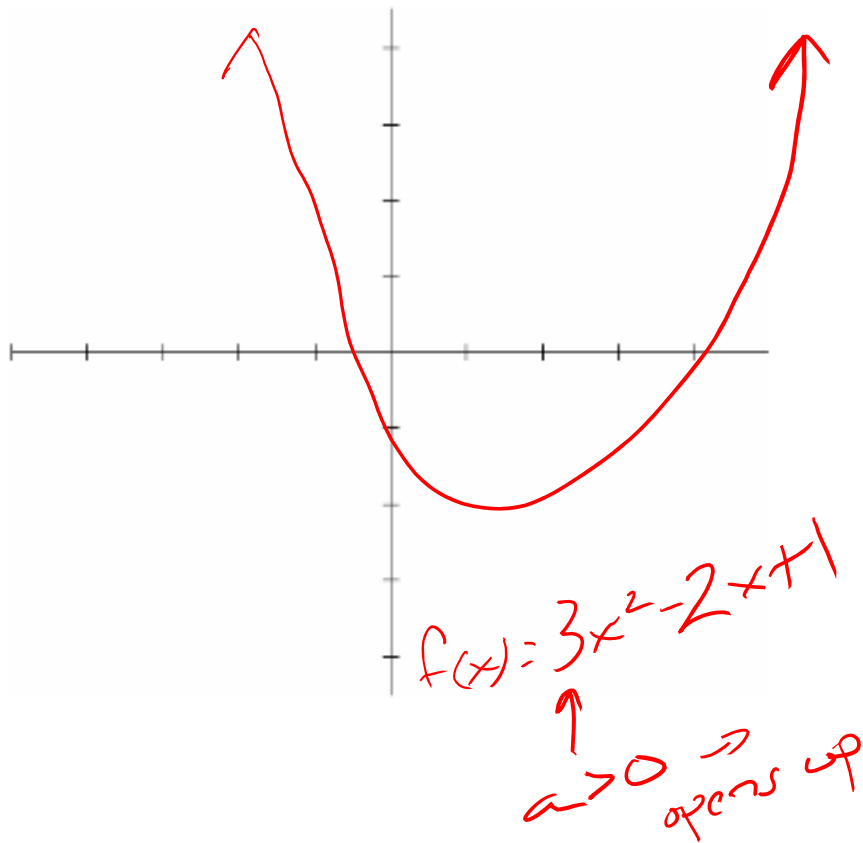
$$f(x) = ax^2 + bx + c \quad \text{Generic Form}$$

$$f(x) = -2x^2 - 3x + 1 \quad \text{Example}$$

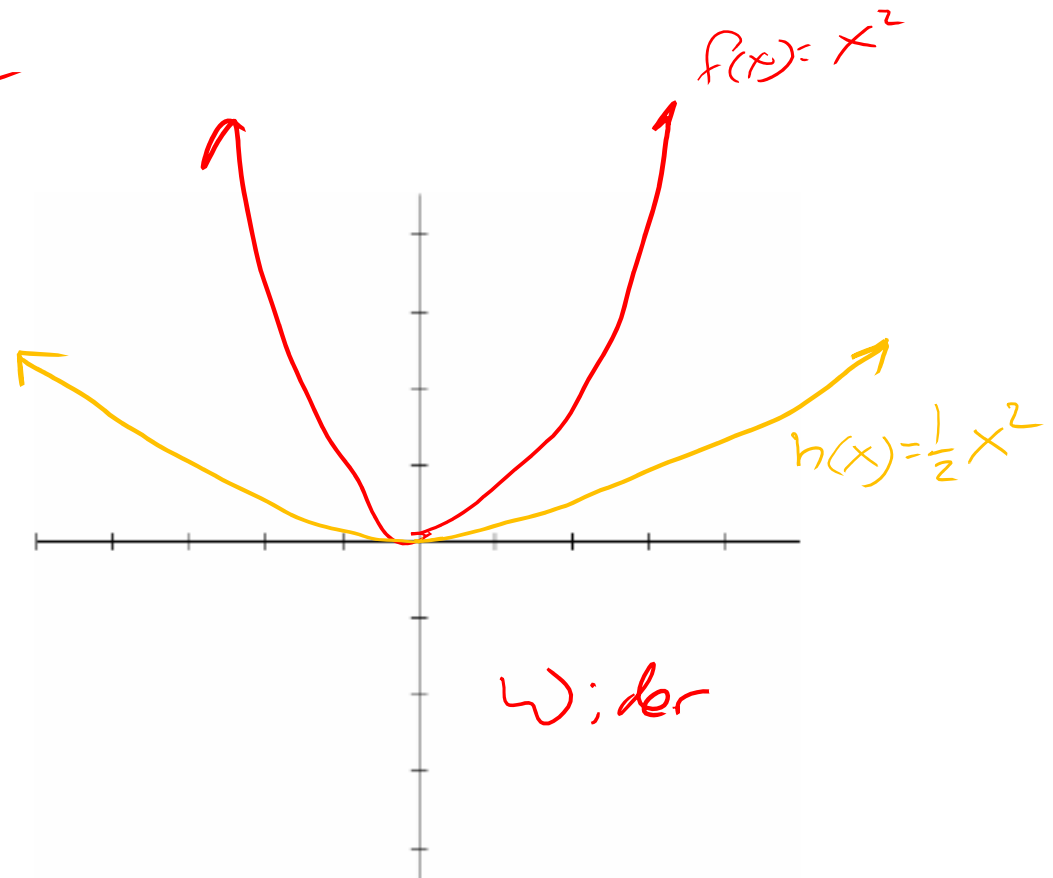
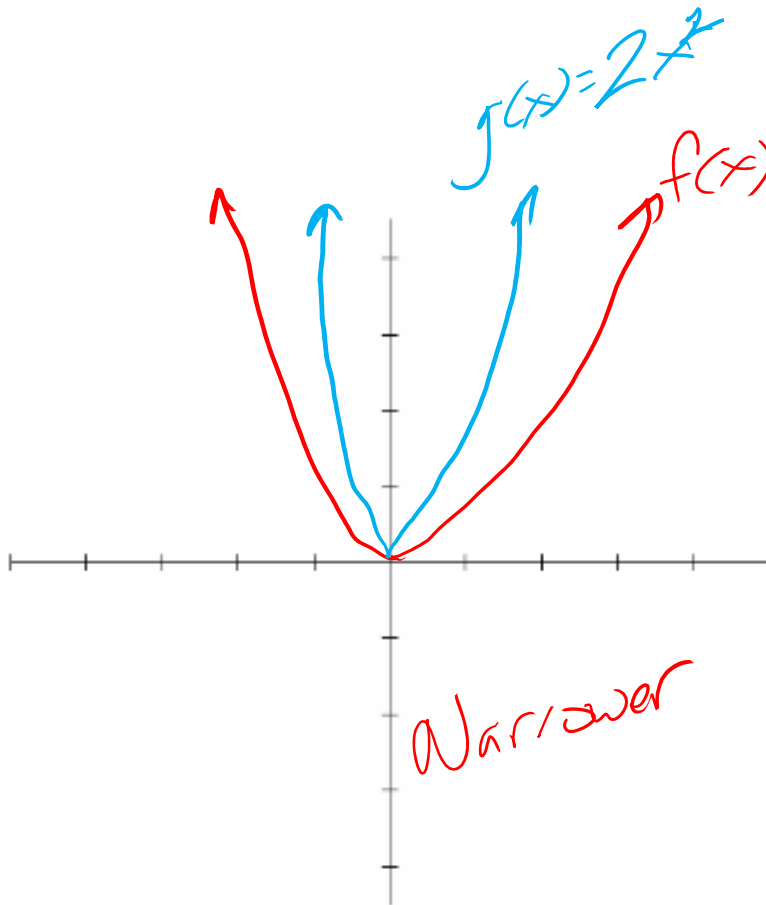
Opening Up or Opening Down

If $a > 0$ then the parabola will open upwards.

If $a < 0$ then the parabola will open downwards.

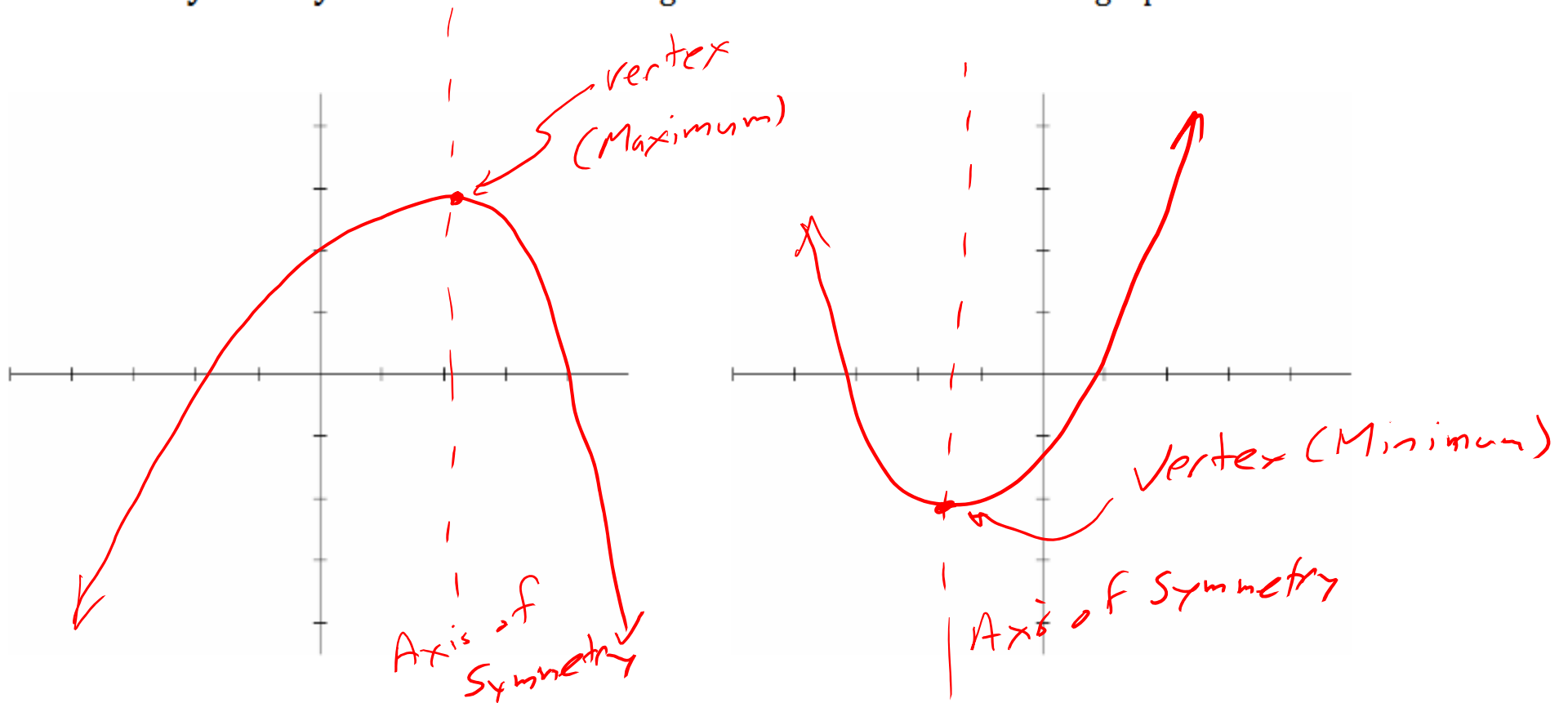


Note: The larger $|a|$, the narrower the parabola



The **vertex** is the turning point of the parabola and is the **minimum point** on the graph when it opens upward and the **maximum point** on the graph when it opens downward. Every parabola has a maximum or minimum, but **NOT** both.

The **axis of symmetry** is a vertical line through the vertex that divides the graph in half.



The Standard form of a Quadratic Function

The quadratic function $f(x) = a(x - h)^2 + k$ is in **standard form**

The vertex is the point (h, k) and the axis of symmetry is $x = h$

The domain is $(-\infty, \infty)$.

The range is $[k, \infty)$ if $a > 0$ or $(-\infty, k]$ if $a < 0$

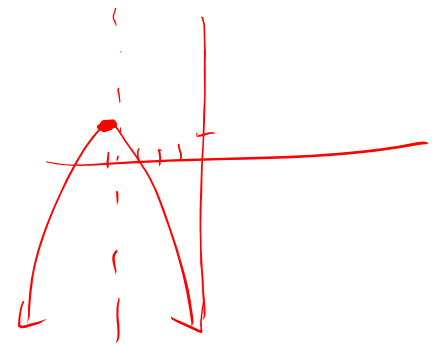
$$f(x) = -2(x + 4)^2 + 1$$

$$f(x) = a(x - h)^2 + k$$

$$a = -2$$

$$h = -4$$

$$k = 1$$



X-int: set $y = 0$ and solve.

Y-int: set $x = 0$ and solve.

vertex: $(-4, 1)$

Axis of Symmetry: $x = -4$

Domain: $(-\infty, \infty)$

Direction: $a < 0$ open down

Range: $(-\infty, 1]$

Our first task will be to change a given quadratic function from the form $f(x) = ax^2 + bx + c$ to standard form. We'll complete the square to do this. Once the function is in standard form, we can sketch a graph using transformations and then read off the maximum or minimum value

General Form into Standard Form:
Complete the Square

Standard Form into General: simplify the expression

$$f(x) = -2(x+4)^2 + 1$$

$$f(x) = -2(x+4)(x+4) + 1$$

$$f(x) = -2(x^2 + 4x + 4x + 16) + 1$$

$$f(x) = -2(x^2 + 8x + 16) + 1$$

$$f(x) = -2x^2 - 16x - 32 + 1 \rightarrow$$

$$f(x) = -2x^2 - 16x - 31$$

Example 1: Write the following quadratic in standard form. Then find the vertex and the axis of symmetry.

a. $f(x) = (3x^2 - 12x) - 1$

$$f(x) = 3(x^2 - 4x) - 1 = 3(x^2 - 4x + 4) - (3)(4) - 1$$

$\begin{matrix} a \\ \downarrow \\ 3 \end{matrix}$
 $\begin{matrix} (\frac{b}{2})^2 \\ \downarrow \\ 4 \end{matrix}$
 $-a \cdot (\frac{b}{2})^2$

$$b = -4$$

$$\frac{b}{2} = -2$$

$$\left(\frac{b}{2}\right)^2 = 4$$

$$f(x) = 3(x - 2)^2 - 12 - 1$$

$$f(x) = 3(x - 2)^2 - 13$$

$$f(x) = a(x - h)^2 + k$$

$$a = 3$$

$$h = 2$$

$$k = -13$$

$$\text{vertex: } (h, k) = (2, -13)$$

$$\text{Axis of Symmetry: } x = h$$

$$x = 2$$

$$\text{b. } f(x) = (-x^2 + 2x) + 3$$

$$f(x) = -1(x^2 - 2x) + 3 = -1(x^2 - 2x + 1) - (-1)(1) + 3$$

a
 \downarrow

$(\frac{b}{2})^2$
 \downarrow

$-\underbrace{(a)(\frac{b}{2})^2}$

$$b = -2$$

$$\frac{b}{2} = -1$$

$$(\frac{b}{2})^2 = 1$$

$$f(x) = -(x-1)^2 + 1 + 3$$

$\frac{b}{2} \rightarrow$

$$f(x) = -(x-1)^2 + 4$$

$$a = -1$$

$$h = 1$$

$$k = 4$$

$$\text{vertex: } (h, k) = (1, 4)$$

axis of symmetry: $x = h \rightarrow x = 1$

Domain: $(-\infty, \infty)$

Range: $(-\infty, k] \rightarrow (-\infty, 4]$

direction: opens
down

c. $f(x) = (-10x^2 + 60x)$

$$f(x) = -10(x^2 - 6x) = -10(x^2 - 6x + 9) - (-10)(9)$$

$$b = -6$$

$$\frac{b}{2} = -3$$

$$\left(\frac{b}{2}\right)^2 = 9$$

$$f(x) = -10(x-3)^2 + 90$$

vertex: $(h, k) \rightarrow (3, 90)$

Axis of Symmetry: $x = h \rightarrow x = 3$

what is : y-value
where is : x-value

What is the minimum/maximum value? Maximum of 90 opening down ($a = -10$)

Popper 14:

$$f(x) = (-2x^2 - 16x) - 27 =$$

1. Complete the square and rewrite in standard form:

a. $f(x) = -2(x + 9)^2$

b. $f(x) = 2(x + 4)^2 - 5$

c. $f(x) = -2(x + 4)^2 + 5$

d. $f(x) = 2(x - 4)^2 - 5$

$$-2(x^2 + 8x) - 27$$

$$b = 8$$

$$\frac{b}{2} = 4$$

$$\left(\frac{b}{2}\right)^2 = 16$$

2. Determine the direction of the parabola:

$$a = -2$$

a. Opening Up

b. Opening Down

3. Determine the equation of the axis of symmetry:

$$x = 4$$

a. $x = 4$

b. $x = -4$

c. $x = 5$

d. $x = -5$

$$f(x) = -2(x^2 + 8x + 16) - (-2)(16) - 27$$

$$f(x) = -2(x + 4)^2 + 5$$

$$a = -2$$

$$h = -4$$

$$k = 5$$

4. Determine the coordinates of the vertex:

$$(h, k)$$

a. $(-4, 5)$

b. $(-5, 10)$

c. $(4, 5)$

d. $(-4, -5)$

5. Is the vertex of this parabola a minimum or a maximum?

a. Minimum

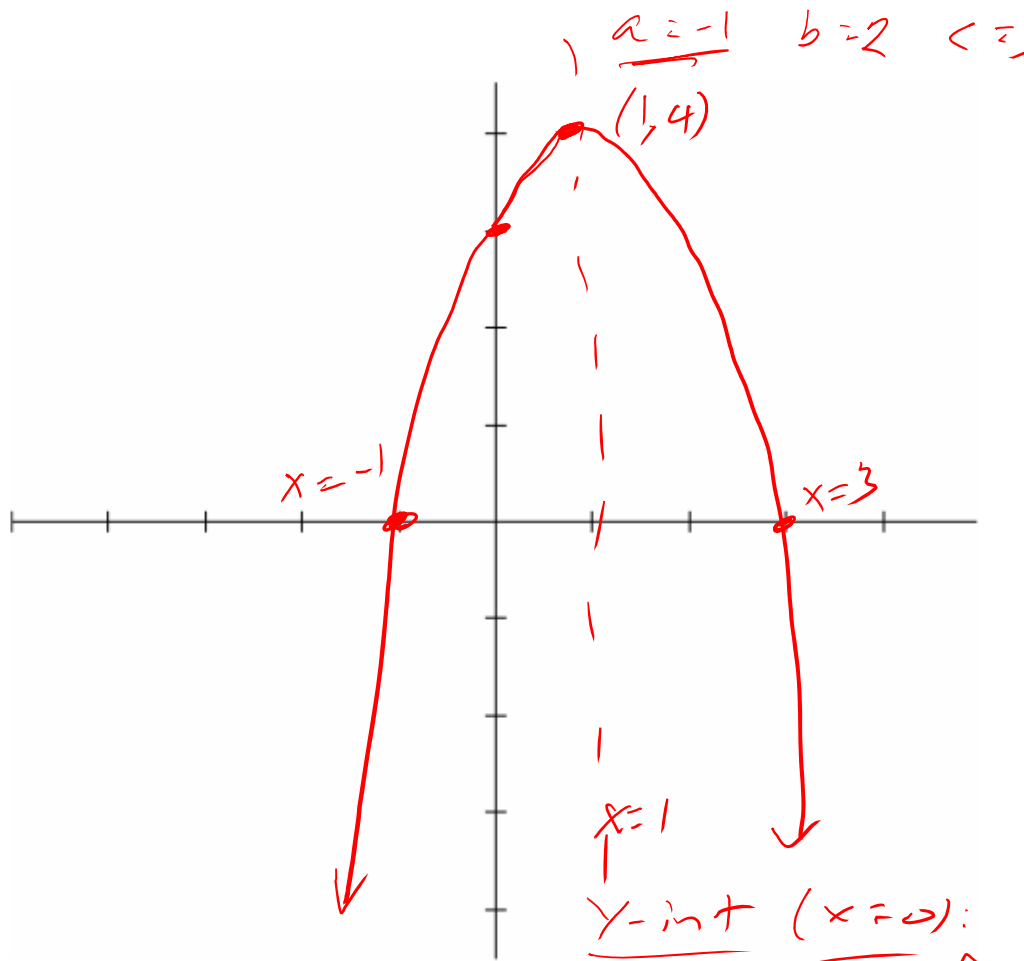
b. Maximum

c. ~~Neither~~

Graphing Quadratic Functions with Equations in Standard Form

1. Determine whether the parabola opens upward or downward.
2. Determine the vertex.
3. Find any x -intercept by replacing $f(x)$ with 0 and then solving for x .
4. Find the y -intercept by replacing x with 0.
5. Plot the intercept(s) and vertex, sketch the graph and draw the axis of symmetry.

Example 2: Sketch the graph of $f(x) = -x^2 + 2x + 3$



$a = -1$ $b = 2$ $c = 3$

Vertex: $(1, 4)$

Y-int ($x=0$):
 $f(x) = -x^2 + 2x + 3$
 $y = 3$ ($y = c$)

$x = \frac{-b}{2a} = \frac{-2}{2(-1)} = \frac{-2}{-2} = 1 \rightarrow h$

$f(1) = -(1)^2 + 2(1) + 3 = -1 + 2 + 3 = 4$

Standard form:

$f(x) = -1(x-1)^2 + 4$

Vertex: $(h, k) \rightarrow (1, 4)$

Axis of Sym: $x = h \rightarrow x = 1$

Direction: $a = -1 \rightarrow$ Down

X-int: ($y=0$)

$0 = -(x-1)^2 + 4$ $x = -1$
 $-4 = -(x-1)^2$ $x = 3$

$-4 = -(x-1)^2$
 $4 = (x-1)^2$
 $\sqrt{4} = \sqrt{(x-1)^2}$
 $2 = |x-1|$
 $x-1 = \pm 2$
 $x = 1-2, 1+2$

Shortcut:

For $f(x) = ax^2 + bx + c$, the vertex is $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$. So the axis of symmetry is $x = -\frac{b}{2a}$.

Axis of symmetry: $x = -\frac{b}{2a} \rightarrow \underline{h}$
 $f\left(-\frac{b}{2a}\right) \rightarrow \underline{k}$

$$f(x) = \underline{a} \left(x - \underline{h}\right)^2 + \underline{k}$$

Example 3: Let $f(x) = 2x^2 + 4x + 7$. Determine, without graphing, whether the given quadratic function has a minimum or maximum value. Then find the coordinates of the minimum or maximum point

Min or Max? : $a = 2 > 0$ opens up 

Minimum Value

$$x = \frac{-b}{2a} = \frac{-4}{2(2)} = \frac{-4}{4} = -1 \rightarrow h$$

$$f(-1) = 2(-1)^2 + 4(-1) + 7 = 2(1) - 4 + 7 = 2 - 4 + 7 = 5 \rightarrow k$$

vertex: $(h, k) \rightarrow \boxed{(-1, 5)}$

\hookrightarrow Minimum

Example 4: Suppose $f(x) = 5x^2 - 30x + 41$. Write the equation in standard form. State the coordinates of the vertex. Determine, without graphing, whether the given quadratic function has a minimum or maximum value. Then find the coordinates of the minimum or maximum point.

Min or Max: $a = 5 > 0$ opens up \rightarrow Minimum

$$x = \frac{-b}{2a} = \frac{-(-30)}{2(5)} = \frac{30}{10} = 3 \rightarrow h$$

$$f(3) = 5(3)^2 - 30(3) + 41 = 5(9) - 30(3) + 41 = 45 - 90 + 41 = -45 + 41 = -4$$

k

vertex: $(h, k) \rightarrow$ $(3, -4)$

Finally, given the vertex of a quadratic function and one other point that lies on the graph of the quadratic function, you should be able to write the quadratic function.

Example 5: Find a quadratic function with vertex (2, 6) which passes through (-1, 4).

$$f(x) = a(x-h)^2 + k$$

$$(h, k) \rightarrow h = 2$$

$$k = 6$$

Plug in for x & y

$$f(x) = a(x-2)^2 + 6$$

$$4 = a(-1-2)^2 + 6$$

$$4 = a(-3)^2 + 6$$

$$4 = 9a + 6$$

$$\frac{-2}{9} = \frac{9a}{9} \quad a = -\frac{2}{9}$$

$$f(x) = -\frac{2}{9}(x-2)^2 + 6$$

* (may have to justify)

$$f(x) = -\frac{2}{9}(x-2)(x-2) + 6$$

$$f(x) = -\frac{2}{9}(x^2 - 4x + 4) + 6$$

$$\frac{6 \cdot 9}{1 \cdot 9} = \frac{54}{9}$$

$$f(x) = -\frac{2}{9}x^2 + \frac{8}{9}x - \frac{8}{9} + \frac{54}{9}$$

$$f(x) = -\frac{2}{9}x^2 + \frac{8}{9}x + \frac{46}{9} *$$

$$7 = a(5-3)^2 - 1 \quad \frac{7 = 4a + 1}{+1}$$

$$7 = a(2)^2 - 1 \quad \frac{8 = 4a}{\div 4}$$

$$2 = a$$

11C 2C 3A 4D 5C

Example 6: Find a quadratic function with vertex (3, -1) which passes through (5, 7).

Popper 15:

1. Substitute into the standard form equation:

a. $f(x) = a(x + 3)^2 - 1$ b. $f(x) = a(x + 3)^2 + 1$

c. $f(x) = a(x - 3)^2 - 1$

d. $f(x) = a(x + 1)^2 + 3$

2. Substitute the additional point:

a. $5 = a(7 - 3)^2 - 1$ b. $7 = a(5 + 3)^2 - 1$

c. $7 = a(5 - 3)^2 - 1$

d. $7 = a(5 + 3)^2 - 1$

3. Solve for a:

a. 2 b. $1/8$

c. $3/2$

d. 8

4. Quadratic Function in Standard Form:

a. $f(x) = 1/8(x + 3)^2 - 1$ b. $f(x) = 2(x + 3)^2 - 1$

c. $f(x) = 3/2(x - 3)^2 - 1$

d. $f(x) = 2(x - 3)^2 - 1$

5. Quadratic Function in General Form:

a. $f(x) = 3/2 x^2 + 9x + 1/8$ b. $f(x) = 2x^2 - 12x - 19$

c. $f(x) = 2x^2 - 12x + 17$

d. $f(x) = 1/8 x^2 - 5/8 x + 1$

$$f(x) = 2(x - 3)(x - 3) - 1$$

$$f(x) = 2(x^2 - 6x + 9) - 1$$

$$f(x) = 2x^2 - 12x + 18 - 1$$

$$f(x) = 2x^2 - 12x + 17$$

Determine the equation of a parabola has x-intercepts of (5,0) and (-1,0) and a y-intercept of (0,-10).

$$\left. \begin{array}{l} x\text{-int: } x=5 \rightarrow x-5=0 \\ x=-1 \rightarrow x+1=0 \end{array} \right\} f(x) = a(x-5)(x+1)$$

$$f(x) = a(x-5)(x+1)$$

Plug in (0, -10)

$$-10 = a(0-5)(0+1)$$

$$-10 = a(-5)(1)$$

$$\frac{-10}{-5} = \frac{-5a}{-5}$$

$$2 = a$$

$$f(x) = 2(x-5)(x+1)$$

$$f(x) = 2(x^2 - 4x - 5)$$

$$f(x) = 2x^2 - 8x - 10$$