MATH 1314

Section 3.7

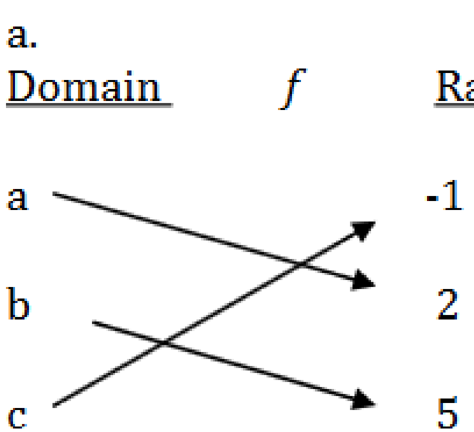
Inverse Functions

Definition: Function: every x corresponds to one y

Let *f* be a function with domain A. *f* is said to be **one-to-one** if no two elements in A have the same image.

one to one Function: Every x corresponds to one y and every y corresponds to one x.

Example 1: Determine if the following function is one-to-one.



one to one function.

$f(a)=2$

$f(b)=5$

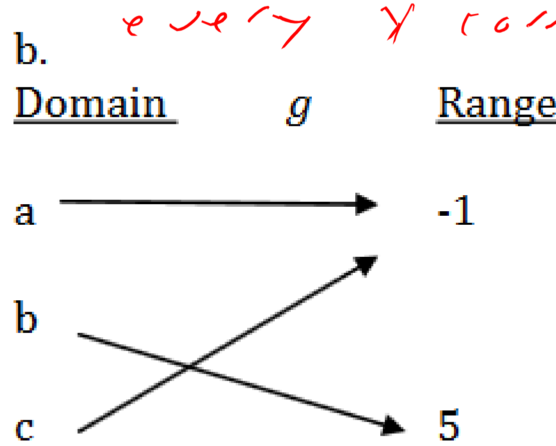
$f(c)=-1$

3 unique pairs:

(a, 2)

(b, 5)

(c, -1)



$g(a)=-1$

$g(b)=5$

$g(c)=-1$

2 x-values have the same answer.

Not one to one.

A one-to-one function has an inverse. The inverse function reverses whatever the first function did. These two statements mean exactly the same thing:

1. f is one-to-one (1-1)
2. f has an inverse function

Any 1 to 1 function has an inverse function.

The inverse of a function f is denoted by f^{-1}, read "f-inverse".

Note: $f^{-1}(x) \neq \frac{1}{f(x)}$ like $x^{-3} = \frac{1}{x^3}$

$f(x)$ is a function, then its inverse $f^{-1}(x)$.

Domain and Range

Suppose f is a one-to-one function with domain A and range B. The inverse function has domain B and range A. If $f(x)$ has $D: [0, \infty)$, $R: (-\infty, \infty)$ then $f^{-1}(x)$ has $D: (-\infty, \infty)$, $R: [0, \infty)$

Example 1: Suppose f and g are inverse functions. If $f(3) = -1$ and $f(-1) = 4$, then find $g(-1)$.

x y

If $f(x)$ passes through (a, b), then $f^{-1}(x)$ will pass through (b, a).

$f(x)$		$g(x)$
$(3, -1)$	$\rightarrow$	$(-1, 3)$
$(-1, 4)$	$\rightarrow$	$(4, -1)$

(since these are inverses)

$g(-1) = 3$

x

Property of Inverse Functions

Let f and g be two functions such that $\boxed{(f \circ g)(x) = x}$ for every x in the domain of g and $\boxed{(g \circ f)(x) = x}$ for every x in the domain of f then **f and g are inverses of each other.**

Example 2: Show that the following functions are inverses of each other.

$f(x) = 3x + 7$ and $g(x) = \frac{x}{3} - \frac{7}{3}$ [we must show $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$]

✓ $(f \circ g)(x) = f(g(x)) = f\left(\frac{x}{3} - \frac{7}{3}\right) = 3\left(\frac{x}{3} - \frac{7}{3}\right) + 7 = x - \cancel{7} + \cancel{7} = x$

✓ $(g \circ f)(x) = g(f(x)) = g(3x+7) = \frac{3x+7}{3} - \frac{7}{3} = \frac{\cancel{3}x}{\cancel{3}} + \cancel{\frac{7}{3}} - \cancel{\frac{7}{3}}$

$= x$

These are inverse functions.

Example 3: Determine whether the following pair of functions are inverses of each other.

$f(x) = 2x - 1$ and $g(x) = \frac{x}{2} + 1$

$(f \circ g)(x) = f(g(x)) = f\left(\frac{x}{2} + 1\right) = 2\left(\frac{x}{2} + 1\right) - 1 = x + 2 - 1 = x + 1 \neq x$

Not Inverse Functions

How to find the equation of the inverse function of a one-to-one function:

1. Replace $f(x)$ by y.
2. Exchange x and y.
3. Solve for y.
4. Replace y by $f^{-1}(x)$
5. Verify.

Example 4: Write the equation of the inverse function for $f(x) = 3x - 3$

$y = 3x - 3$

$x = 3y - \cancel{3}$

$+3 \qquad \cancel{+3}$

$\frac{x+3}{3} = \frac{\cancel{3}y}{\cancel{3}}$

$\frac{x+3}{3} = y$

$\boxed{y = \frac{x+3}{3}} = \frac{x}{3} + \frac{3}{3} = \frac{1}{3}x + 1$

$\boxed{f^{-1}(x) = \frac{1}{3}x + 1}$

Example 5: Write the equation of the inverse for $f(x) = \frac{6}{4-x}$

$$y = \frac{6}{4-x}$$

$$\frac{x}{1} = \frac{6}{4-y}$$

$$6 = x(4-y)$$

$$6 = 4x - xy$$

$$-4x \quad -\cancel{4x}$$

$$\frac{6-4x}{-x} = \frac{-\cancel{x}y}{-\cancel{x}}$$

$$\boxed{\frac{6-4x}{-x} = y}$$

$$y = \frac{(6-4x)(-1)}{(-x)(-1)} = \frac{-6+4x}{x}$$

$$\boxed{f^{-1}(x) = \frac{4x-6}{x}}$$

Determine the inverse of the following: $f(x) = \frac{2x-3}{x+5}$

$y = \frac{2x-3}{x+5}$

$\frac{x}{1} = \frac{2y-3}{y+5}$

$2y - \cancel{3} = \cancel{xy} + 5x$

$-xy + \cancel{3} \quad -\cancel{xy} \quad + 3$

[All y-terms left, All Non-y-terms, right]

$2y - xy = 5x+3$

$2y - xy = 5x+3$

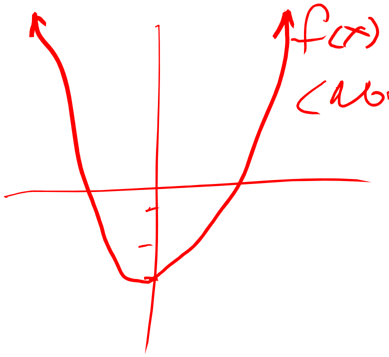
$\frac{y(2-x)}{2-x} = \frac{5x+3}{2-x}$

$\boxed{y = \frac{5x+3}{2-x}} = \frac{5x+3}{-x+2} = \frac{5x+3}{-1(x-2)}$

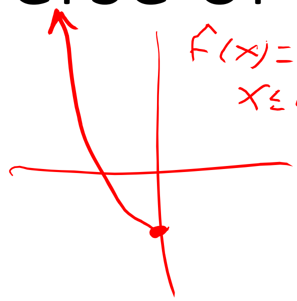
$\boxed{f^{-1}(x) = -\frac{5x+3}{x-2}}$

Domain Restriction

Determine the inverse of f(x) = $x^2 - 3$ for $x \le 0$



$f(x)=x^2-3$ (not 1 to 1)



$f(x)=x^2-3$, $x\le 0$

(ignore until the last step)

$f(x)$: D: $(-\infty, 0]$

meaning

$f^{-1}(x)$: R: $(-\infty, 0]$

(y-values must be negative)

$$y = x^2 - 3$$

$$x = y^2 - \cancel{3}$$

$$+3 \qquad \cancel{+3}$$

$$\sqrt{x+3} = \sqrt{y^2}$$

$$\pm\sqrt{x+3} = y$$

$$y = \pm\sqrt{x+3}$$

$$\boxed{f^{-1}(x) = -\sqrt{x+3}}$$

Domain of $f(x)$ must be the Range of $f^{-1}(x)$.

Popper 15: Question 1:

Example 6: Write the equation of the inverse for $\cancel{f(x)}$ $y =$ $= (x+1)^3 + 1$ $\rightarrow$ $x = (y+1)^3 + 1$

$-1 \qquad -1$

$x - 1 = (y+1)^3$

$\sqrt[3]{x-1} = y+1$

$-1 \qquad -1$

$\sqrt[3]{x-1} - 1$

a. $\sqrt[3]{x-2}$ b. $\sqrt[3]{x-1}+1$ c. $\sqrt[3]{x-1}-1$ d. $\sqrt[3]{x-1}$

Question 2:

Example 7: Write the equation of the inverse for $\cancel{f(x)}$ $y =$ $= \sqrt[3]{x+4}$

a. $\sqrt[3]{x-4}$ b. $x^3 - 4$ c. $x^3 + 4$ d. $\sqrt[3]{x} - 2$

$x = \sqrt[3]{y+4}$

$x^3 = y+4$

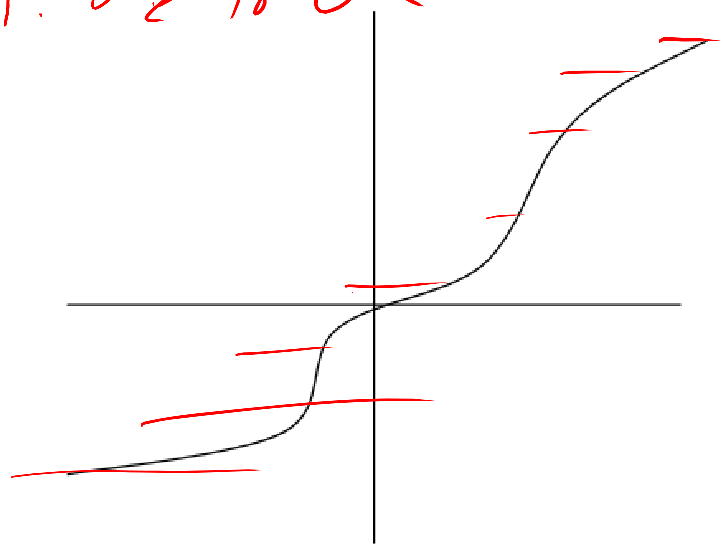
$x^3 - 4 = y$

It is easiest to determine if a function is one-to-one by looking at its graph. We can use the Horizontal Line Test to determine if a function is one-to-one.

Horizontal Line Test: A function is one-to-one if no horizontal line intersects its graph in more than one point.

VLT: Function

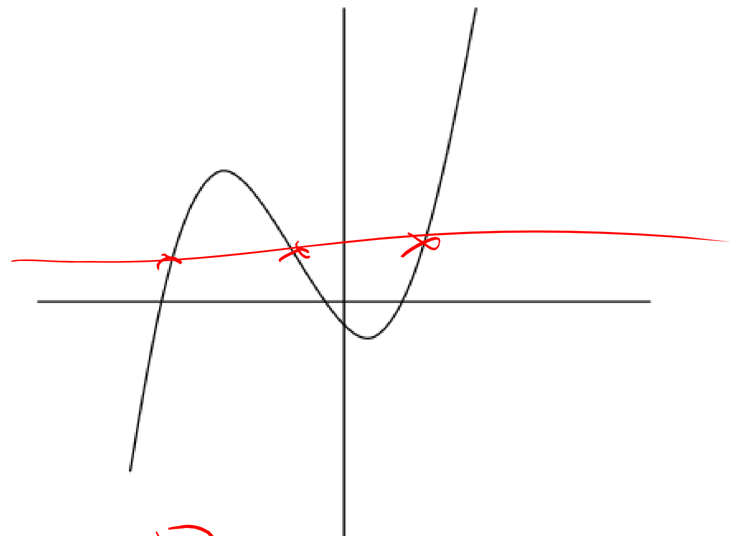
HLT: One-to-One



Has an inverse function

VLT: Function

HLT: Not one-to-one.



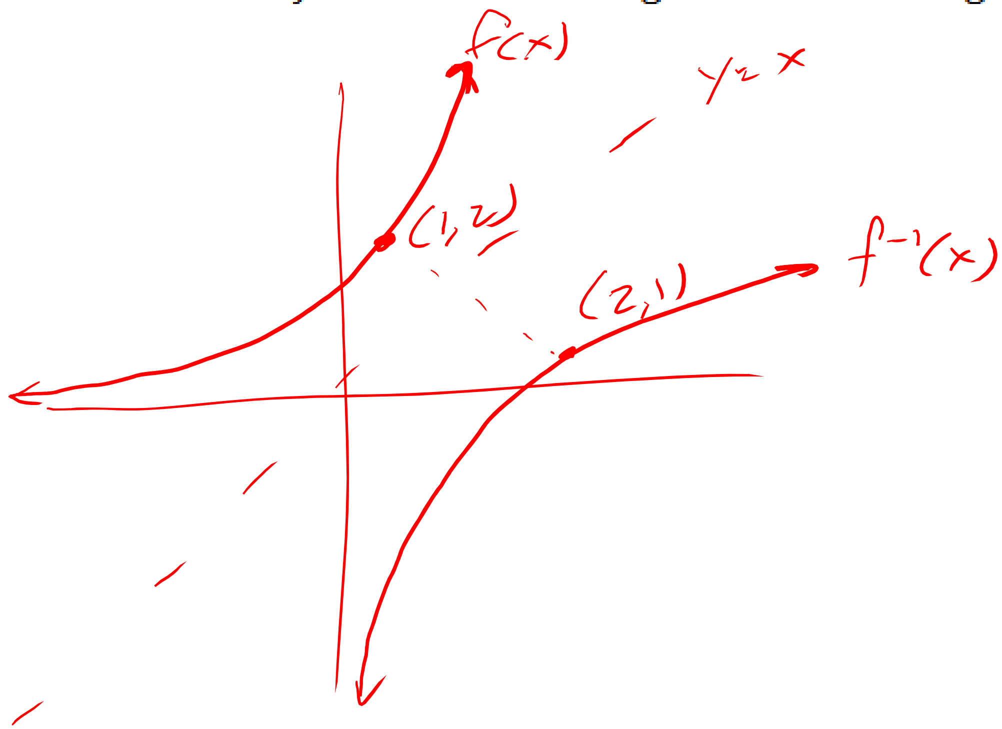
Does Not a Inverse Function.

Graphing the Inverse Function

Given that *f* is 1-1, the graph of f^{-1} is a reflection of the graph of *f* about the line $y = x$

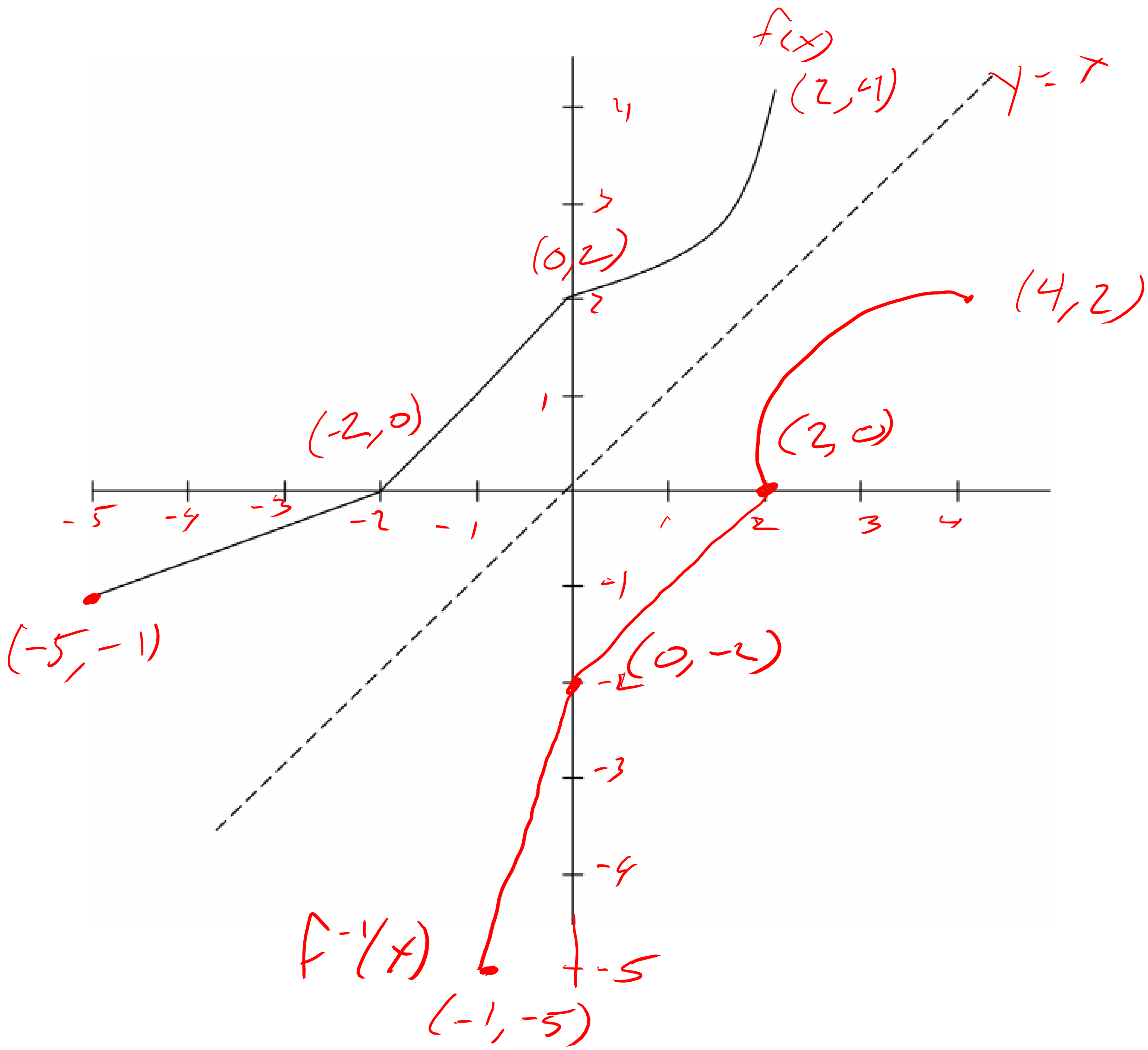
Remember:

1. The inverse function reverses whatever the first function did.
2. The Domain of *f* becomes the Range of and the Range of *f* becomes the Domain of f^{-1}.



Reflect over $y=x$

any (a,b) point of $f(x)$ becomes a (b,a) point on $f^{-1}(x)$

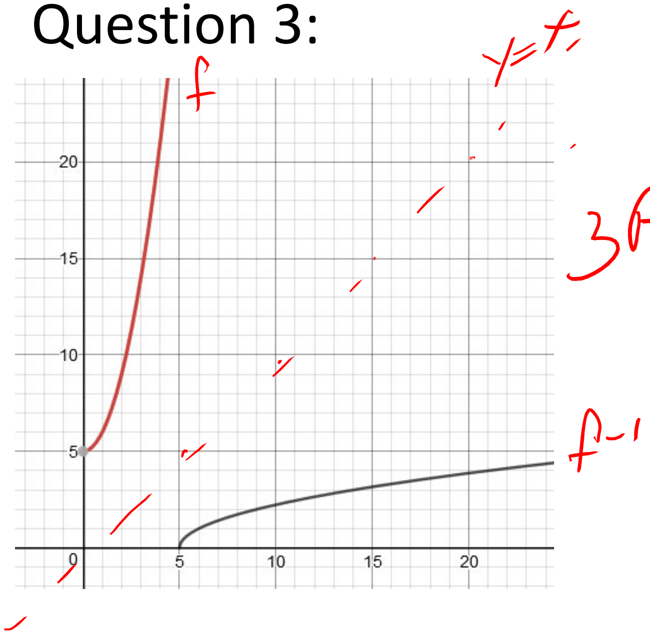

f(x)
(2,4)
y = x
(0,2)
(4,2)
(-2,0)
(2,0)
(-5,-1)
(0,-2)
f⁻¹(x)
(-1,-5)
-5
-4
-3
-2
-1
1
2
3
4

Popper 15…continued:

Are the following pairs of graphs (on the same axis) inverses to one another? (a) Yes, they are inverses (b) No, they are not

Are they reflections through $y=x$

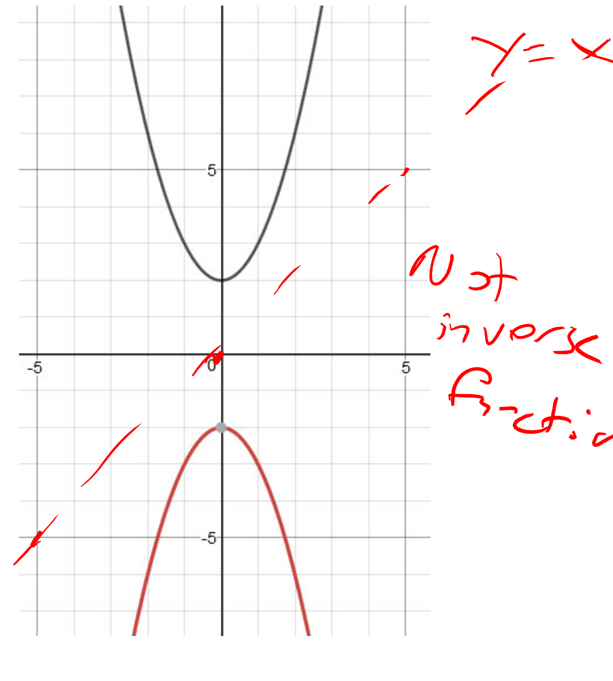
Question 3:



3A

Question 4:

4B



Not inverse functions