MATH 1314

Section 4.1

Polynomial Functions:

String of terms that are added or subtracted.

Each term: $2x^3$ ← positive, whole numbers.
(Example)
coefficient

A polynomial function is a function of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^n + \ldots + a_1 x^1 + a_n$$

where $a_n \neq 0$, $a_0, a_1, \ldots, a_n$ are real numbers and n is a whole number.

The degree of the polynomial function is n. We call the term $a_n x^n$ the leading term, and a_n is called the leading coefficient.

Degree: Largest Exponent

$$\boxed{P(0) = a_0}$$

Leading Term: Term with largest exponent

Constant term: Term without the x

Our objectives in working with polynomial functions will be, first, to gather information about the graph of the function and, second, to use that information to generate a reasonably good graph without plotting a lot of points. In later examples, we'll use information given to us about the graph of a function to write its equation.

$$P(x) = \underbrace{-3x^4} + 2x^3 - 5x \underbrace{+2}$$

Leading Term: $-3x^4$
(Degree: 4)

Constant Term: 2

Factored Form:

$p(x) = 2(x - 4)^2(x^2 + 1)^3$

Leading Term: "First" step of FOIL

$2(x)^2(x^2)^3$ [Each Largest Power of x and the outside exponent]

$2(x^2)(x^6)$

$2x^8$ Degree: 8

Constant term: → y-intercept

$p(0) = 2(0-4)^2(0^2+1)^3$

$= 2(-4)^2(1)^3$

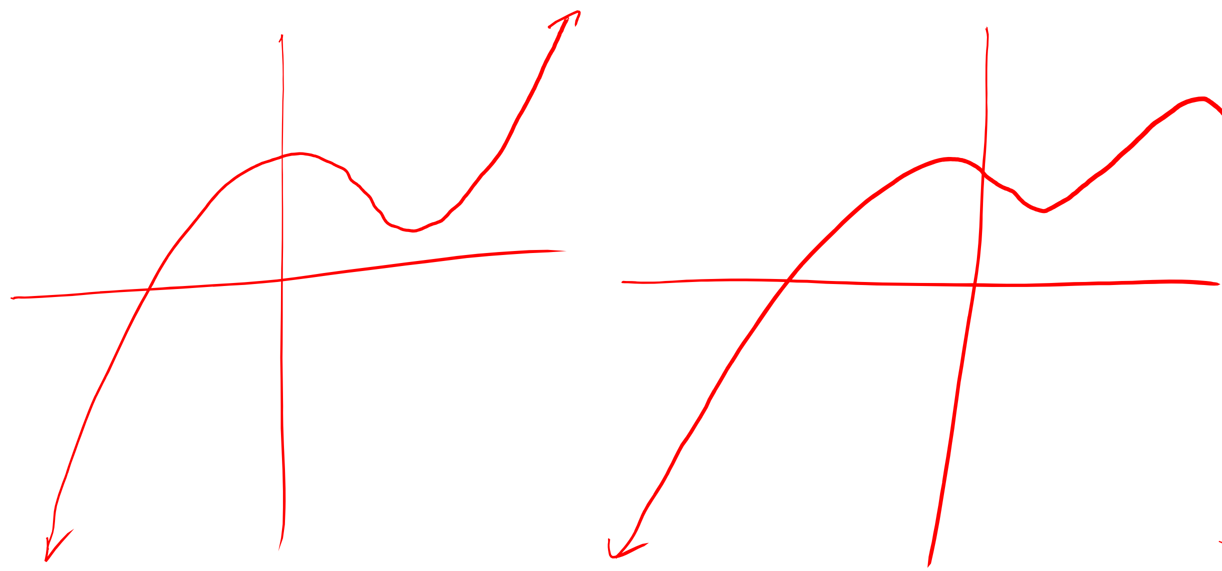
$= 2(16)(1) = 32$ (0, 32)

Graph Properties of Polynomial Functions

Let P be any nth degree polynomial function with real coefficients. The graph of P has the following properties.

1. P is continuous for all real numbers, so there are no breaks, holes, jumps in the graph. D: $(-\infty, \infty)$
2. The graph of P is a smooth curve with rounded corners and no sharp corners.
3. The graph of P has at most n x-intercepts.
4. The graph of P has at most $n - 1$ turning points.

degree of n



Example 1: Given the following polynomial functions, state the leading term, the degree of the polynomial and the leading coefficient, constant term

a. $P(x) = 6x^4 - 4x^3 + 7x - 2$

Leading Term: $6x^4$

constant term: -2

Degree: 4

Leading Coefficient: 6

b. $P(x) = (3x + 4)(x + 1)^2(x - 5)^3$

Leading Term: $(3x)(x)^2(x)^3 = 3x \cdot x^2 \cdot x^3 = 3x^6$

Degree: 6

Leading Coefficient: 3

Constant Term: $P(0) = (3 \cdot 0 + 4)(0+1)^2(0-5)^3 = 4(1)^2(-5)^3 = 4(1)(-125)$

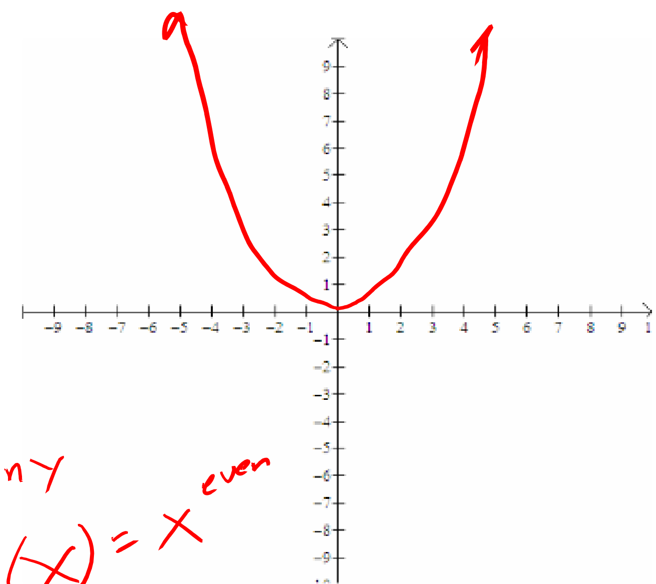
-500

We'll start with the shapes of the graphs of functions of the form $f(x) = x^n, n > 0$.

You should be familiar with the graphs of $f(x) = x^2$ and $g(x) = x^3$.

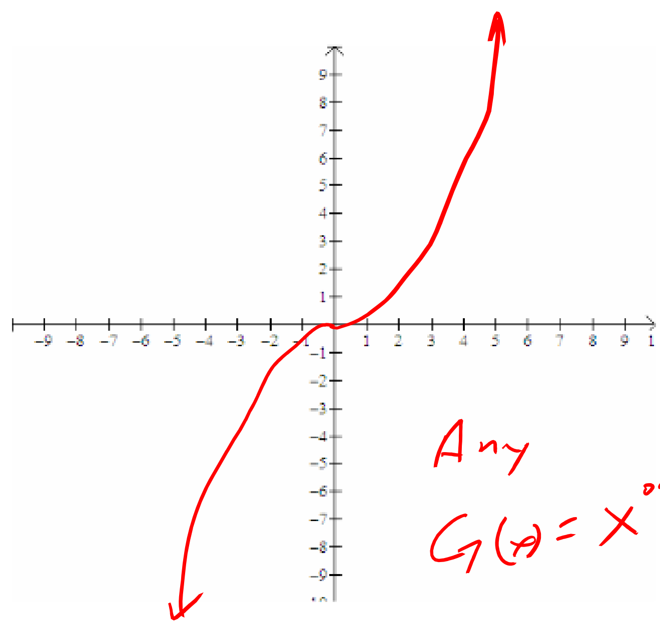
The graph of $f(x) = x^n, n > 0.$, n is even, will resemble the graph of $f(x) = x^2$, and the graph of $f(x) = x^n, n > 0$, n is odd, will resemble the graph of $f(x) = x^3$.

$f(x) = x^2$



Any $F(x) = x^{even}$

$g(x) = x^3$



Any $G(x) = x^{odd\,(>1)}$

Next, you will need to be able to describe the end behavior of a function.

End Behavior of Polynomial Functions

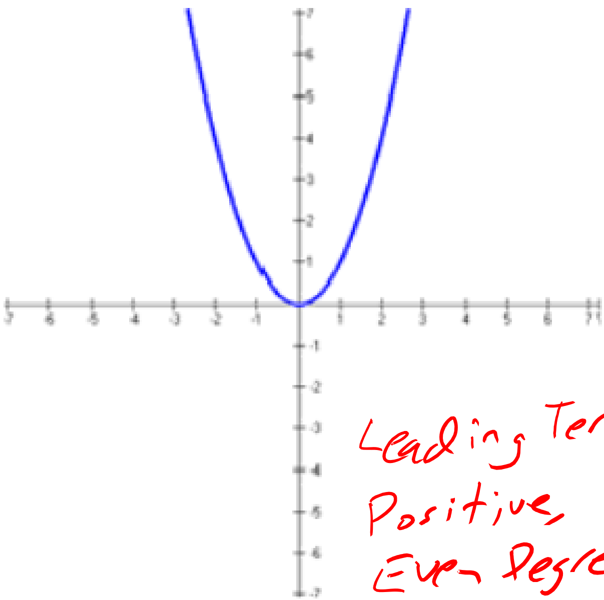
The behavior of a graph of a function to the far left or far right is called its **end behavior.**

The end behavior of a polynomial function is revealed by the leading term of the polynomial function.

End Behavior: what is happening at the far left and far right of the graph.

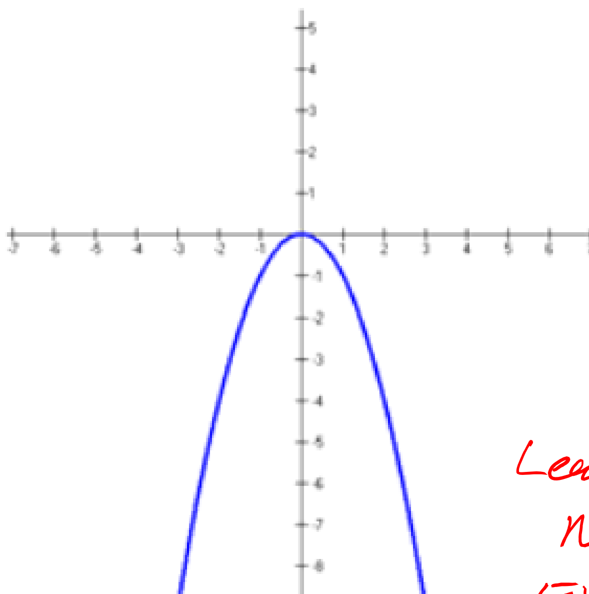
For Polynomials: Rising or Falling

1. Even-degree polynomials look like . $y = \pm x^2$



Leading Term: Positive, Even Degree

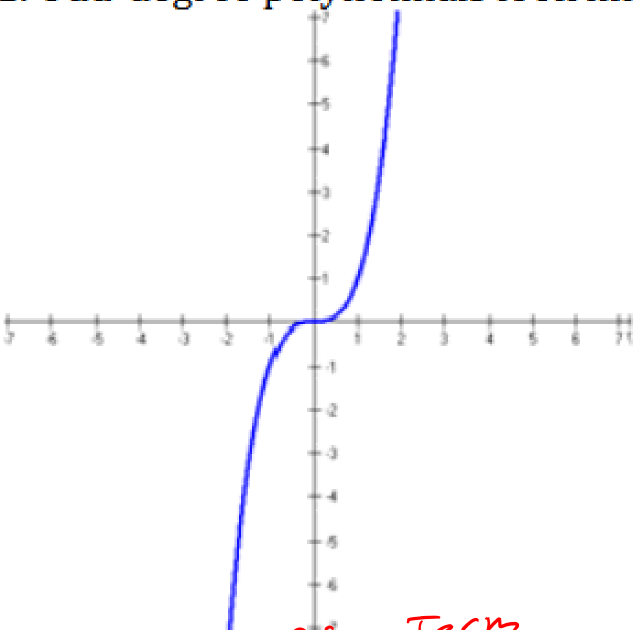
Left: Rising
Right: Rising

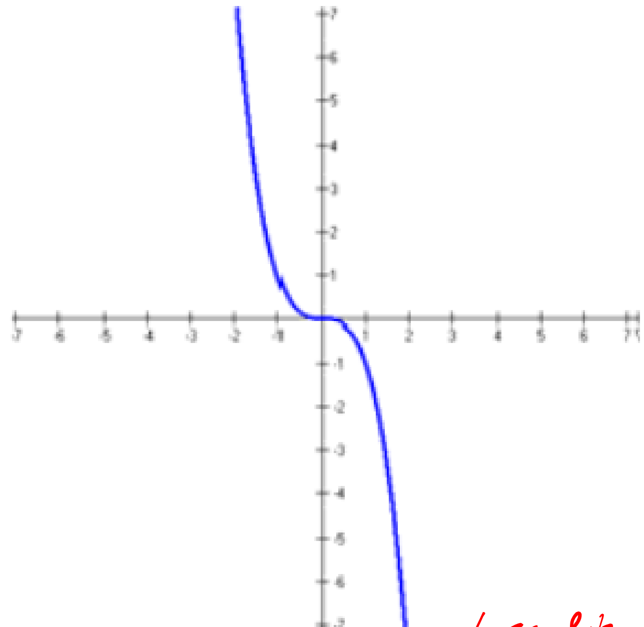


Leading Term: Negative, Even Degree

Left: Falling
Right: Falling

2. Odd-degree polynomials look like. $y = \pm x^3$



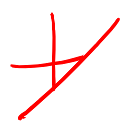


Leading Term positive, odd exponent

Left: Falling

Right: Rising

Think about a line with positive slope

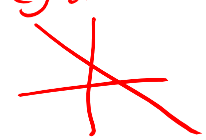


Leading Term: Negative odd exponent

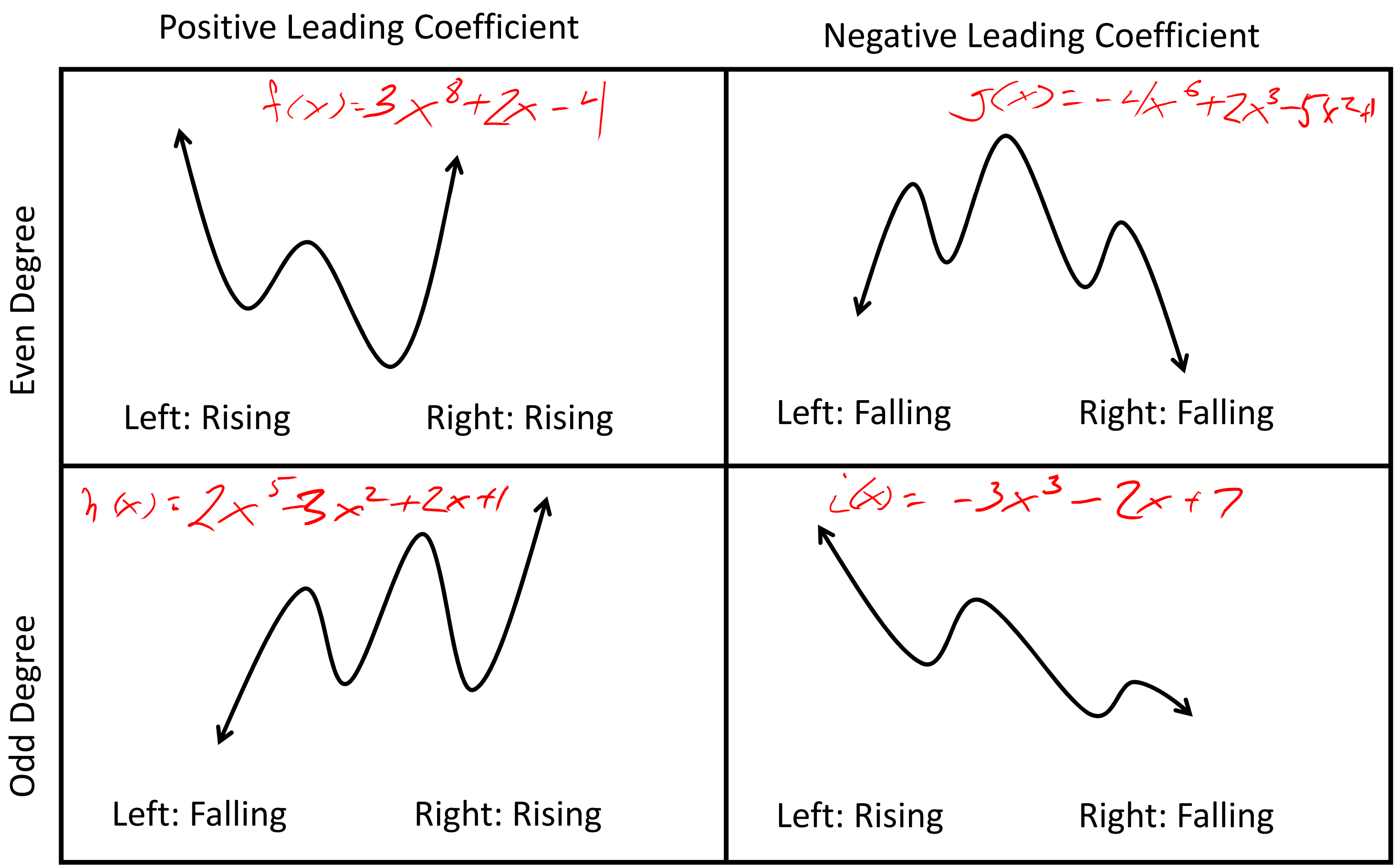
Left: Rising

Right: Falling

Think of a line with a negative slope



End behavior of Polynomials:



Popper 20

For the following, refer to $f(x) = -8x_3 - 2x_2 + 9x - 3$

1. What is the degree of the polynomial?

a. First   b. Sixth   c. Third   d. Eighth

2. What is the leading coefficient?

a. -8   b. -2   c. 9   d. -3

3. What is the end behavior on the left?

a. Rising   b. Falling

4. What is the end behavior on the right?

a. Rising   b. Falling

Leading Term: $-8x^3$

Leading Coefficient: -8

Degree: 3

Left: Rising

Right: Falling

Describe the End Behavior of the following:

$g(x) = (x^2 + 3)(x - 2)^4$

Leading Term: $(x^2)(x)^4 = x^2 \cdot x^4 = x^6$

Leading Coefficient: 1

Degree: 6

positive even

Left: Rising

Right: Rising

Next, you should be able to find the x intercept(s) and the y intercept of a polynomial function.

Zeros of Polynomial Functions (x-intercepts or Roots)

You will need to set the function equal to zero and then use the Zero Product Property to find the x-intercept(s). That means if $ab = 0$, then either $a = 0$ or $b = 0$. To find the y intercept of a function, you will find $f(0)$.

Example 2: Find the zeros of:

a. $f(x) = x^4 - x^2$

3 x-intercepts: $-1, 0, 1$

Y-int: $f(0) = 0^4 - 0^2 = 0$

$x^2(x^2-1)$

$x^2(x+1)(x-1)$ [Product of Linear Factors]

$x=0$ $\quad x+1=0$ $\quad x-1=0$ [Set insides equal to zero, ignore any outside exponents]

$x=-1$ $\quad x=+1$

b. $f(x) = -3x\left(x+\frac{1}{2}\right)(x-4)^3$ [Factored Form]

$-3x=0$ $\quad x+\frac{1}{2}=0$ $\quad x-4=0$

$x=0$ $\quad x=-1/2$ $\quad x=4$

3 x-intercepts: $-1/2, 0, 4$

y-intercept:

$f(0) = -3(0)(0+\frac{1}{2})(0-4)^3$

$= -3(0)(1/2)(-4)^3 = 0$

In some problems, one or more of the factors will appear more than once when the function is factored. The power of a factor is called its multiplicity. So given $\boxed{P(x) = x^3(x-3)^2(x+2)^1}$, then the multiplicity of the first factor is 3, the multiplicity of the second factor is and the multiplicity of the third factor is 1.

$P(0) = 0^3(0-3)^2(0+2)^1 = 0$

$x = 0$, M: 3

$x - 3 = 0$, $x = 3$, M: 2

$x + 2 = 0$, $x = -2$, M: 1

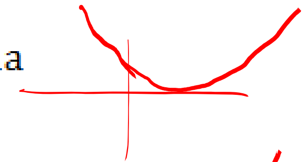
Degree = sum of multiplicities

Leading term: x^6

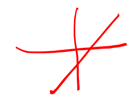
Description of the Behavior at Each *x*-intercept

1. Even Multiplicity: The graph touches the *x*-axis, but does not cross it. It looks like a parabola there.

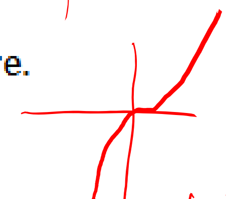
Tangent to the x-axis



2. Multiplicity of 1: The graph crosses the *x*-axis. It looks like a line there.



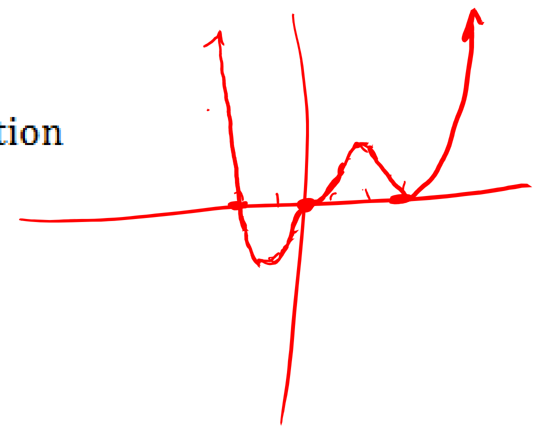
3. Odd Multiplicity greater than or equal 3: The graph crosses the *x*-axis. It looks like a cubic there.



You can use all of this information to generate the graph of a polynomial function.

- degree of the function
- end behavior of the function
- *x* and *y* intercepts (and multiplicities)
- behavior of the function through each of the *x* intercepts (zeros) of the function

End Behavior: L: ↑ Start R: ↑ End



(the best we can draw is a rough sketch of a polynomial)

Steps to graphing other polynomials:
1. Determine the **leading term**. Is the degree even or odd? Is the sign of the leading coefficient positive or negative?
2. Determine the **end behavior**. Which one of the 4 cases will it look like on the ends?
3. Factor the polynomial.
4. Make a table listing the factors, x intercepts, multiplicity, and describe the behavior at each x intercept.
5. Find the y- intercept.
6. Draw the graph, being careful to make a nice smooth curve with no sharp corners.

Note: without calculus or plotting lots of points, we don't have enough information to know how high or how low the turning points are

Example 3: Find the x and y intercepts. State the degree of the function. Sketch the graph of $f(x) = x^3 + 4x^2 + 4x$

Leading Term: x^3

Positive, Degree: 3

End Behavior: L: ↓ (Start), R: ↑ (End)

X-intercepts:

$x(x^2+4x+4)=0$

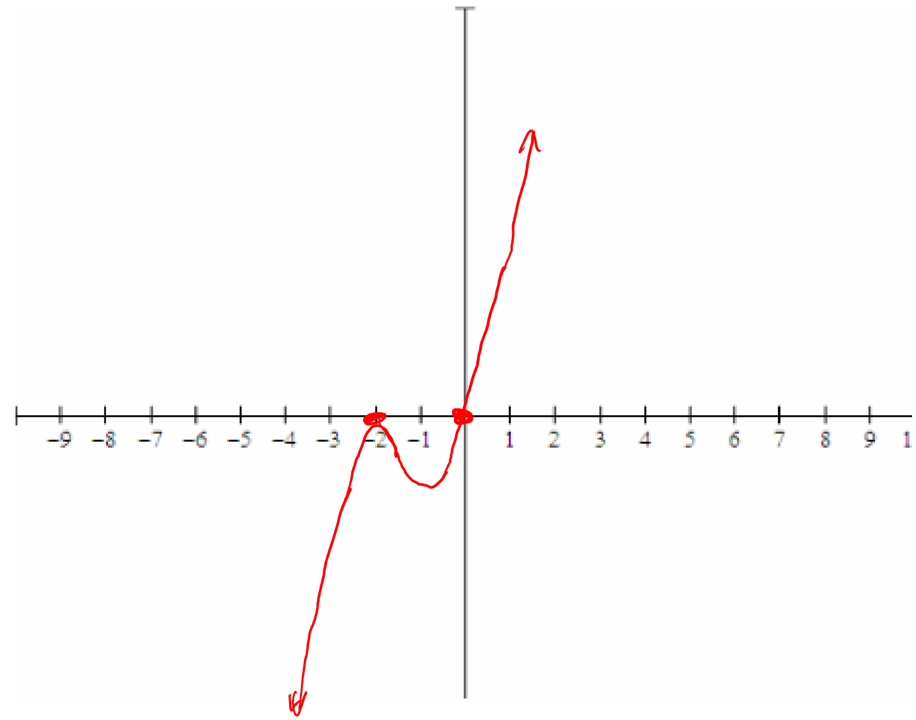
$x(x+2)(x+2)=0$

$x^1(x+2)^2=0$

$x=0$ $x+2=0$

M: 1 $x=-2$

M: 2



Y-int: $f(0) = 0^3+4(0)^2+4(0)=0$

Example 4: Find the x and y intercepts. State the degree of the function. Sketch the graph of
$P(x) = (x-3)^2(x+1)^5(x+2)^3$

Leading Term: $(x)^2(x)^5(x)^3 = x^{10}$

Positive, Degree: 10

End Behavior: L: ↑, R: ↑

x-intercepts:

$x-3=0$ $\quad$ $x+1=0$ $\quad$ $x+2=0$

$x=3$ $\quad$ $x=-1$ $\quad$ $x=-2$

M: 2 $\quad$ M: 5 $\quad$ M: 3

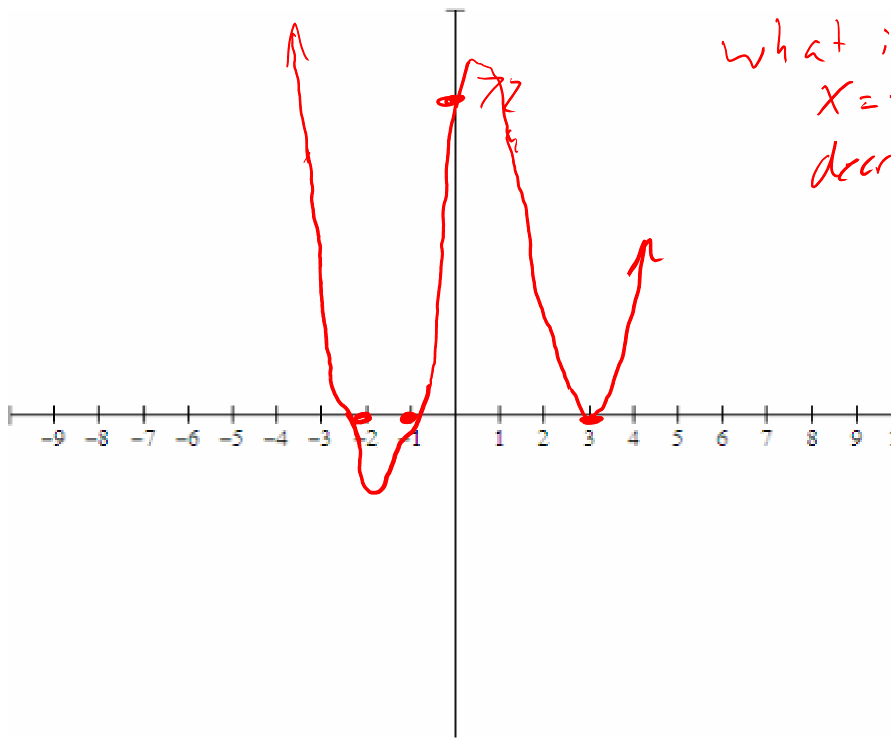
Y-int:

$P(0) = (0-3)^2(0+1)^5(0+2)^3$

$= (-3)^2(1)^5(2)^3$

$= 9 \cdot 1 \cdot 8 = 72$

$(0, 72)$



what is the behavior at $x=3$?

upwards parabola

what is the behavior at $x=-2$?

decreasing cubic

Example 5: Find the x and y intercepts. State the degree of the function. Sketch the graph of $g(x) = (3 - x)(x + 1)(x + 5)^2$

$g(x) = (-x+3)(x+1)(x+5)^2$

Leading Term: $(-x^1)(x^1)(x)^2 = -x^4$

Negative, Even Degree

End Behavior: L: ↓ R: ↓

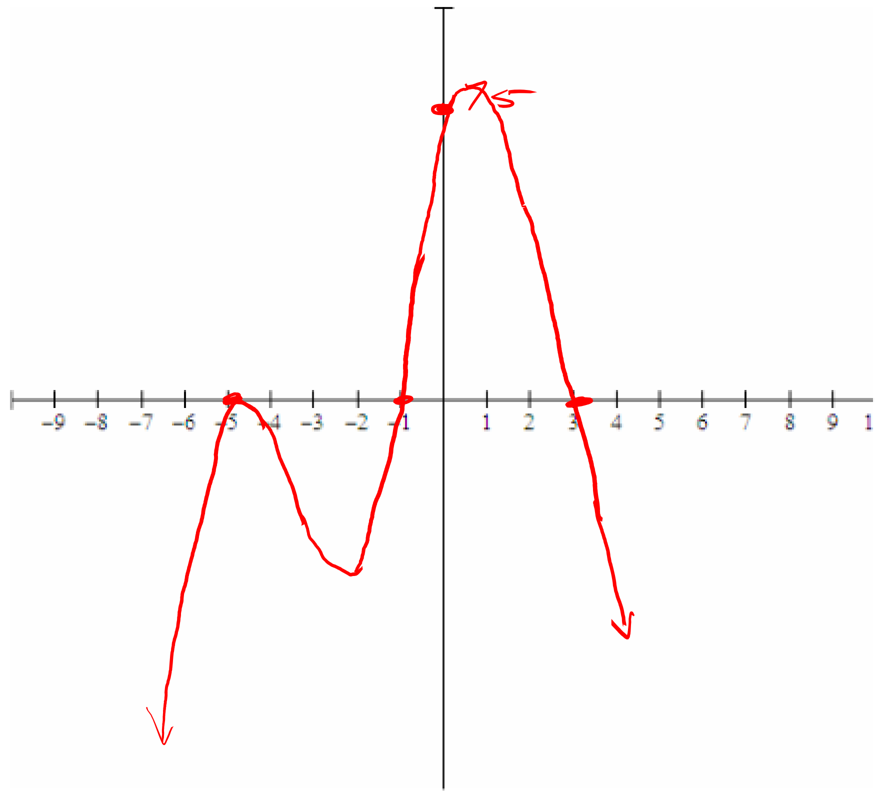
x-int:

$-x+3=0$ | $x+1=0$ | $x+5=0$

$-x=-3$ | $x=-1$ | $x=-5$

$x=3$ | M: 1 | M: 2

M: 1



Y-int:

$g(0) = (3-0)(0+1)(0+5)^2$

$(3)(1)(5)^2 = (3)(1)(25) = 75$

Example 6: Find the x and y intercepts. State the degree of the function. Sketch the graph of $f(x) = (x-2)^3(-x+1)^2(x+5)$

Leading Term: $(x)^3(-x)^2(x)^1 = x^3 \cdot x^2 \cdot x^1 = x^6$

Positive even

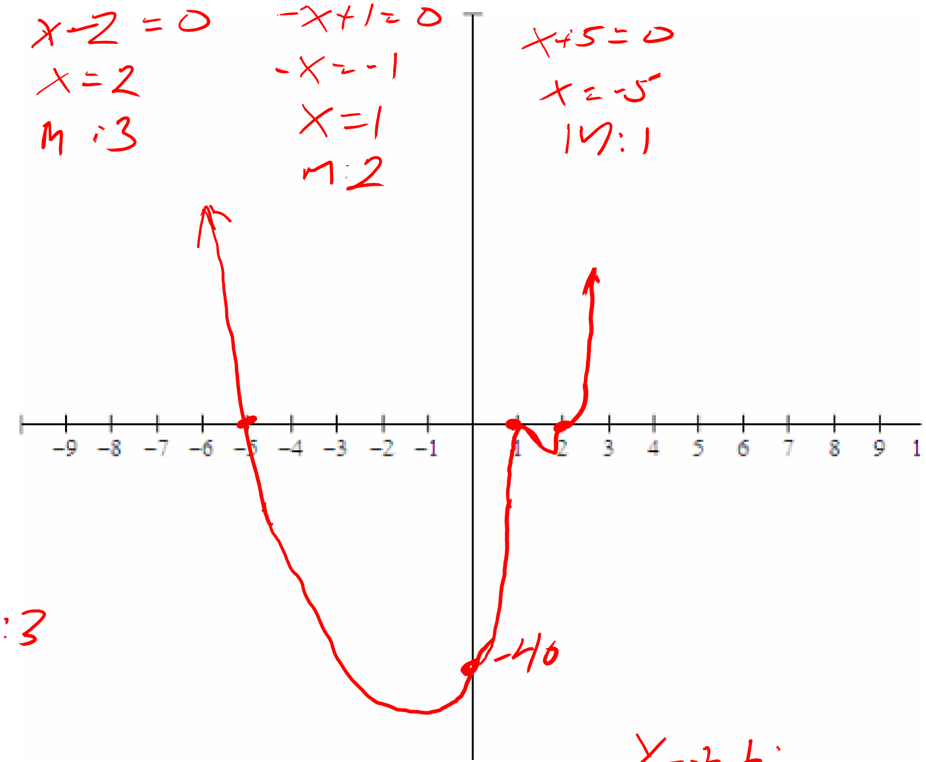
End Beh: L: ↑ R: ↑

x-int:

$x-2=0$, $x=2$, M: 3

$-x+1=0$, $-x=-1$, $x=1$, M: 2

$x+5=0$, $x=-5$, M: 1



y-int:

$f(0) = (-2)^3(1)^2(5)$

$= (-8)(1)(5) = -40$

Popper 21:

1. What are the x-intercepts:
   a. {-5, -1, -2}
   b. {-5, -1, 2}
   c. {-5, 1, 2}
   d. {-5}

2. What is the y-intercept:
   a. (0, -40)
   b. (0, -10)
   c. (0, 5)
   d. (0, 40)

3. What is the degree:
   a. Third
   b. Fifth
   c. Sixth
   d. Seventh

4. What is the behavior of the left-most x-intercept? M: 1
   a. linear
   b. quadratic
   c. cubic

5. What is the behavior of the middle x-intercept? M: 2
   a. linear
   b. quadratic
   c. cubic

6. What is the behavior at the right-most x-intercept? M: 3
   a. linear
   b. quadratic
   c. cubic

Example 7: Write the equation of the cubic polynomial $P(x)$ that satisfies the following conditions: zeros at $x = 3$, $x = -1$, and $x = 4$ and passes through the point (-3, 7)

$x = 3 \quad x = -1 \quad x = 4$

$x - 3 = 0 \quad x + 1 = 0 \quad x - 4 = 0$ → shortcut Each Factor: (x − "x-int")

$P(x) = a(x-3)(x+1)(x-4)$

plug in (−3, 7)

$7 = a(-3-3)(-3+1)(-3-4)$

$7 = a(-6)(-2)(-7)$

$\frac{7}{-84} = \frac{a(-\cancel{84})}{\cancel{-84}}$

$-\frac{1}{12} = a$

$P(x) = -\frac{1}{12}(x-3)(x+1)(x-4)$ ★

$P(x) = -\frac{1}{12}(x^2 + x - 3x - 3)(x-4)$

$P(x) = -\frac{1}{12}(x^2 - 2x - 3)(x-4)$

$P(x) = -\frac{1}{12}(x^3 - 2x^2 - 3x - 4x^2 + 8x + 12)$

$P(x) = -\frac{1}{12}(x^3 - 6x^2 + 5x + 12)$

$P(x) = -\frac{1}{12}x^3 + \frac{1}{2}x^2 - \frac{5}{12}x - 1$ ★

Example 8: Write the equation of the quartic function with y intercept 4 which is tangent to the x axis at the points (-1, 0) and (1, 0).

$\hookrightarrow 4^{th}$ Degree

$(0, 4)$

$x = -1 \quad x = 1$

$M: 2 \quad M: 2 \rightarrow 2 + 2 = 4$

$$f(x) = a(x+1)^2(x-1)^2$$

$$4 = a(0+1)^2(0-1)^2$$

$$4 = a(1)^2(-1)^2$$

$$4 = a$$

$$f(x) = 4(x+1)^2(x-1)^2$$