MATH 1314

Section 4.2

Dividing Polynomials

In this section, you'll learn two methods for dividing polynomials, long division and synthetic division. You'll also learn two theorems that will allow you to interpret results when you divide.

Suppose $P(x)$ and $D(x)$ are polynomial functions and $D(x) \neq 0$. Then there are unique polynomials $Q(x)$ (called the quotient) and $R(x)$ (called the remainder) such that $P(x) = D(x) * Q(x) + R(x)$.

We call $D(x)$ the divisor. The remainder function, $R(x)$, is either 0 or of degree less than the degree of the divisor.

$$\frac{32}{3} \rightarrow 3\overline{)32} \quad 10 \text{ R}2$$

$$\begin{array}{r} 32 \\ -3\downarrow \\ \hline 02 \\ -0 \\ \hline 2 \end{array}$$

$$10\tfrac{2}{3}$$

You can find the quotient and remainder using long division. Recall the steps you learned in elementary school to perform long division:

$$\frac{3x^5+2x^4-3x}{3x} = \frac{3x^5}{3x} + \frac{2x^4}{3x} - \frac{3x}{3x} = x^4 + \frac{2}{3}x^3 - 1$$

Polynomial, Divisor, Quotient, Remainder

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)} \rightarrow P(x) = D(x) \cdot Q(x) + R(x)$$

Example 1: Divide

$$\frac{x^6+4x^4+4x^2+16}{x^2+4}$$

Place holders

$$= \frac{x^6+0x^5+4x^4+0x^3+4x^2+0x+16}{x^2+4}$$

① Look for Placeholders in the numerator

② Create long division box

③ $x^2 \cdot \boxed{} = x^6$ → x^4

sign

number

power of x

④ Multiply 'new term' times entire outside (line up in columns)

⑤ Subtract

⑥ Repeat ③ to ⑤ until $R(x)=0$ or the Degree of $R(x) <$ Degree of $D(x)$

$Q(x)$: x^4+4

$$x^2+4 \,\overline{)\, x^6+0x^5+4x^4+0x^3+4x^2+0x+16}$$

$$\ominus\; x^6 \quad \ominus\; +4x^4$$

$$4x^2+0x+16$$

$$\ominus\; 4x^2 \qquad \ominus\; +16$$

$$0 \; \} R(x)$$

$Q(x) = x^4+4$

$R(x) = 0$

x^4+4

Example 2: Divide

$$\frac{12x^3 - x^2 - x + 0}{3x - 1} \rightarrow$$

$[3x \cdot 4x^2 = 12x^3]$ $[3x \cdot x = 3x^2]$

$$
\begin{array}{r}
4x^2 + x \\
3x-1 \enclose{longdiv}{12x^3 - x^2 - x + 0} \\
-(\cancel{12x^3} - 4x^2) \\
\hline
3x^2 - x + 0 \\
-(\cancel{3x^2} - x) \\
\hline
0
\end{array}
$$

$Q(x) = 4x^2 + x$

$R(x) = 0$

$\Bigg\} \; 4x^2 + x$

Popper 18 Question 1:

$P(x) = Q(x) \cdot D(x) + R(x)$

Example 3: If $D(x) = 2x - 5$, $Q(x) = 3x^2 + 5$ and $R(x) = 12$, find $P(x)$.

a. $6x^3 - 15x^2 + 34x - 85$

b. $6x^3 - 15x^2 + 10x - 13$

c. $3x^2 + 2x + 12$

d. $6x^3 - 15x^2 + 10x - 25$

$P(x) = (3x^2+5)(2x-5) + 12$

F O I L

$P(x) = 6x^3 - 15x^2 + 10x - 25 + 12$

$P(x) = 6x^3 - 15x^2 + 10x - 13$

Often it will be more convenient to use synthetic division to divide polynomials. This method is easy to use, as long as your divisor is $x \pm c$, for any real number c.

Dividing Polynomials Using Synthetic Division only possible if denominator is $x \pm$ Number

Example 4: Divide using synthetic division

$$\frac{6x^4 + x^3 - 10x^2 + 9}{x - 3} = \frac{6x^4 + x^3 - 10x^2 + 0x + 9}{x - 3}$$

opposite sign

	6	1	-10	0	9
3	↓	18	57	141	423
	6	19	47	141	432
		(3×19)	(3×47)	(3×141)	

← coefficients of numerator

P(x) was degree = 4 (4-1) so Q(x) is degree = 3

① Placeholders in Numerator

② Create synthetic division box

③ Carry Down 6

④ Multiply $3 \cdot 6 = 18 \rightarrow$ Next Column

⑤ Add Column $1 + 18 = 19$

⑥ Repeat ④, ⑤

$Q(x) = 6x^3 + 19x^2 + 47x + 141$ R(x)

$6x^3 + 19x^2 + 47x + 141 + \frac{432}{x-3}$ (combined answer)

Example 5: Divide using synthetic division

$$\frac{x^3 + 5x^2 - 7x + 2}{x - 1}$$

No place holders needed

$$\begin{array}{r|rrrr} 1 & 1 & 5 & -7 & 2 \\ & \downarrow & 1 & 6 & -1 \\ \hline & 1 & 6 & -1 & 1 \end{array}$$

$$Q(x) = 1x^2 + 6x - 1 \qquad R(x)$$

$$x^2 + 6x - 1$$

Combined Answer:

$$x^2 + 6x - 1 + \frac{1}{x-1} = Q(x) + \frac{R(x)}{D(x)}$$

Popper 18 Question 2:

Example 6: Divide using synthetic division

$$\frac{x^3+8}{x+2} = \frac{x^3+0x^2+0x+8}{x+2}$$

$$\begin{array}{r|rrrr} & 1 & 0 & 0 & 8 \\ -2 & \downarrow & -2 & 4 & -8 \\ \hline & 1 & -2 & 4 & 0 \end{array}$$

$Q(x) = x^2 - 2x + 4$ $\quad R(x)$

a. $x^2 - 2x + 4$

b. $x^2 + 2x - 4$

c. $x^2 - 2x + 4$, R 6

d. $x^2 + 2x + 4$, R 16

Here are two theorems that can be helpful when working with polynomials:

The Remainder Theorem: If $P(x)$ is divided by $x - c$, then the remainder is $P(c)$.

The Factor Theorem: c is a zero of $P(x)$ if and only if $x - c$ is a factor of $P(x)$, that is if the remainder when dividing by $x - c$ is zero.

You can use synthetic division and the remainder theorem to evaluate a function at a given value.

Remainder Theorem: $f(x) = x^2 + 4x - 2$ find $f(3)$

$f(3) = 3^2 + 4(3) - 2$

$9 + 12 - 2 = 19$

or

$$\begin{array}{r|rrr} & 1 & 4 & -2 \\ 3 & \downarrow & 3 & 21 \\ \hline & 1 & 7 & 19 \end{array}$$

Factor Theorem: If $R(x) = 0$ then $D(x)$ is a factor of $P(x)$

or c ($D(x) = x - c$) is a root of $P(x)$.

Example 7: Use synthetic division and the remainder theorem to find P(3) for $P(x) = 2x^3 - 5x^2 + 4x + 3$

$$\begin{array}{r|rrrr} & 2 & -5 & 4 & 3 \\ 3 & \downarrow & 6 & 3 & 21 \\ \hline & 2 & 1 & 7 & 24 \end{array}$$

$R(x) = P(3)$

$P(3) = 24$

Popper 18 Question 3:

Example 8: Determine if $x+2$ is a factor of $P(x) = x^3 + 6x^2 + 3x - 10$.

a. Yes

b. No

c. Cannot Be Determined

$$\begin{array}{r|rrrr} & 1 & 6 & 3 & -10 \\ -2 & \downarrow & -2 & -8 & 10 \\ \hline & 1 & 4 & -5 & 0 \end{array}$$

x^2+4x-5

$R(x)=0$

This means that $(x+2)$ is a factor

Meaning: $x^3+6x^2+3x-10 = (x+2)(x^2+4x-5)$

Popper 18 Question 4:

Use synthetic division to determine p(2) if

$p(x) = 5x^6 - 2x^5 + 7x^4 - 8x^3 + 2x - 3$

$0x^2$

a. -3

b. 305

c. -553

d. 273

$$\begin{array}{r|rrrrrrr} & 5 & -2 & 7 & -8 & 0 & 2 & -3 \\ 2 & \downarrow & 10 & 16 & 46 & 76 & 152 & 308 \\ \hline & 5 & 8 & 23 & 38 & 76 & 154 & 305 \end{array}$$

$$\begin{array}{r} 76 \\ \times\ 2 \\ \hline 152 \end{array} \qquad \begin{array}{r} 154 \\ \times\ 2 \\ \hline 308 \end{array}$$

$R(x) = 305$

$p(2) = 305$

And you may also need to work backwards.

Example 9: Find a polynomial with a degree of 4 with zeros at -3, 0, 2, 5.

$x=-3 \quad x=0 \quad x=2 \quad x=5$

$P(x)=(x+3)(x-0)\underbrace{(x-2)(x-5)}$

$(x+3)(x)(x-2)(x-5)$

$(x^2+3x)(x^2-5x-2x+10)$

$(x^2+3x)(x^2-7x+10)$

$P(x)=x^4-7x^3+10x^2+3x^3-21x^2+30x$

$P(x)=x^4-4x^3-11x^2+30x$

Popper 18 Question 5:

Example 10: Find a polynomial of degree 3 with zeros at 0, 2 and -3.

a. $p(x) = x\,(x + 2)\,(x - 3)$

b. $p(x) = x\,(2x)(-3x)$

c. $p(x) = x\,(x + 2)^3(x - 3)^3$

d. $p(x) = x\,(x - 2)(x + 3)$

$x=0 \quad x=2 \quad x=-3$

$(x-0)(x-2)\ (x+3)$

$x \quad (x-2)\ (x+3)$

If the polynomial $f(x) = x^3 + 3x^2 - 13x - 15$ has one root located at (-5, 0), determine all other roots.

$x=-5$ is a root, meaning $(x+5)$ is a factor

$$\begin{array}{r|rrrr} & 1 & 3 & -13 & -15 \\ -5 & \downarrow & -5 & 10 & 15 \\ \hline & 1 & -2 & -3 & 0 \end{array}$$

$0 \rightarrow R(x)=0$

(expected since we were told $x=-5$ is a root)

$$x^2-2x-3$$

$$(x-3)(x+1)$$

$x-3=0$ $\quad$ $x+1=0$

$x=3$ $\quad$ $x=-1$

All roots: $-5, -1, 3$

All "other" roots: $-1, 3$

A polynomial has roots of -1, 3, and 5, and a y-intercept at (0,-30). Determine its equation.

y-intercept or constant term

$x=-1$ $x=3$ $x=5$

$P(x)=a(x+1)(x-3)(x-5)$

$P(x)=a(x^2-3x+x-3)(x-5)$

$P(x)=a(x^2-2x-3)(x-5)$

$P(x)=a(x^3-2x^2-3x-5x^2+10x+15)$

constant term: 15
should be: -30

$P(x)=a(x^3-7x^2+7x+15)$

$15\cdot\square=-30$

$\downarrow$

$a=-2$

$P(x)=-2(x^3-7x^2+7x+15)$

$$\boxed{P(x)=-2x^3+14x^2-14x-30}$$