MATH 1314

Section 4.4

Rational Functions

The objective in this section will be to identify the important features of a rational function and then to use them to sketch an accurate graph of the function.

A **rational function** can be expressed as $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial functions and $q(x) \neq 0$.

Example 1: Find the domain of $f(x) = \frac{x-2}{x^2-9}$

Denominator $\neq 0$

$x^2 - 9 \neq 0$

$\sqrt{x^2} \neq \sqrt{9}$

$x \neq \pm 3$

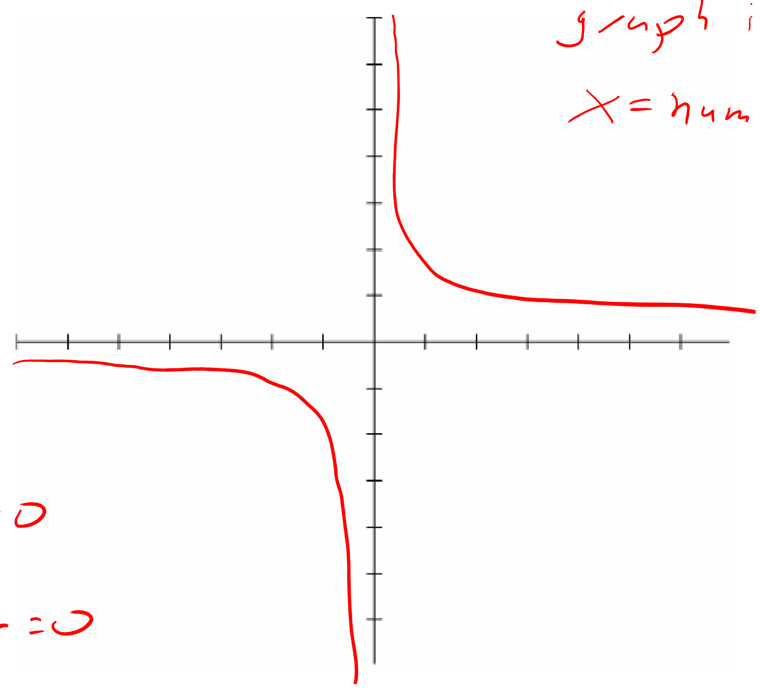
$(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

Vertical Asymptote of Rational Functions

The line $x = a$ is a **vertical asymptote** of the graph of a function f if $f(x)$ increases or decreases without bound as x approaches a.

Basic example is $f(x) = \frac{1}{x}$

vertical line that breaks the graph in 2 parts.



$x = $ number

x-values where:

Denominator $= 0$

and

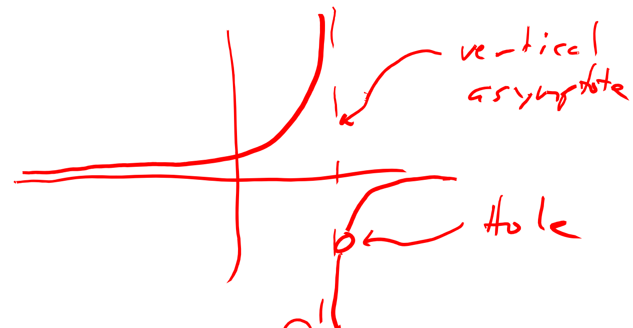
Numerator $\neq 0$

If Both

Numerator $= 0$

and

Denominator $= 0$



Hole: Does Not change the shape of the graph. Single (x,y) point that is undefined.

Factor the numerator and denominator. Look at each factor in the denominator.

- If a factor cancels with a factor in the numerator, then there is a hole where that factor equals zero.
- If a factor does not cancel, then there is a vertical asymptote where that factor equals zero.

Example 2: Find any vertical asymptote(s) and/or hole(s) of $f(x) = \dfrac{x^2 - 3x - 10}{x^2 - x - 6}$

$$f(x) = \frac{x^2 - 3x - 10}{x^2 - x - 6} = \frac{(x-5)(x+2)}{(x-3)(x+2)}$$

- Factors that appear in the denominator only: $x - 3 = 0$

  $x = 3 \rightarrow$ Vertical Asymptote

- Factors that appear in both numerator and denominator: $x + 2 = 0$

  $x = -2 \rightarrow$ Location of a hole.

[y-value of the hole: plug x-value into simplified function:

$$\frac{(x-5)\cancel{(x+2)}}{(x-3)\cancel{(x+2)}} = \frac{(-2-5)}{(-2-3)} = \frac{-7}{-5} = 7/5 \qquad (-2, 7/5)$$

Example 3: Find any vertical asymptote(s) and/or hole(s) of

$$f(x) = \frac{x^2+3x-4}{x^2-9} = \frac{(x+4)(x-1)}{(x+3)(x-3)}$$

Vertical Asymptotes: VA: (Den Only)

$x+3=0$ $\quad$ $x-3=0$

$x=-3$ $\quad$ $x=3$

2 vertical asymptotes: $x=-3$, $x=3$

Holes: (Both Num & Den): None

Example 4: Find any vertical asymptote(s) and/or hole(s) of

$$f(x) = \frac{x^2 - 16}{x^2 - 2x - 8} = \frac{(x+4)(x-4)}{(x-4)(x+2)}$$

VA: (Den only): $x+2=0$

$x=-2$

Give a function that is identical to f(x) in all but one point.
[Give simplified form of the function]

$g(x) = \frac{x+4}{x+2}$

Hole: (Num; Den): $x-4=0$

$\underline{\underline{x=4}}$

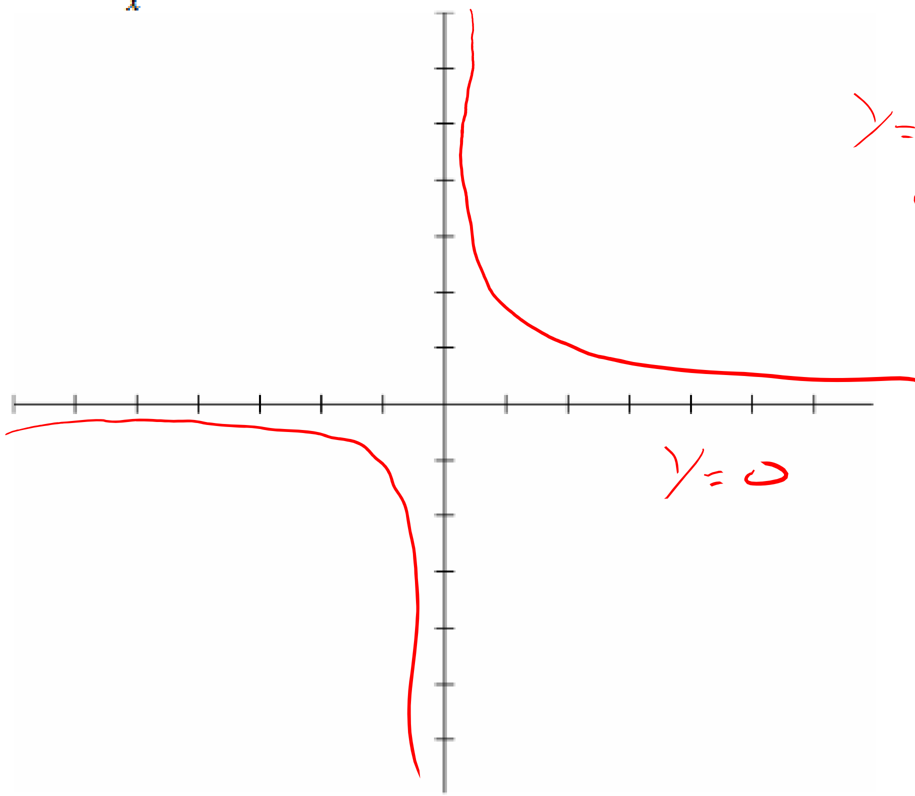
[y-value]: $\frac{(x+4)\cancel{(x-4)}}{\cancel{(x-4)}(x+2)} = \frac{(4+4)}{(4+2)} = \frac{8}{6} = 4/3$

Hole: $(4, 4/3)$

Horizontal Asymptote of Rational Functions

The line $y = b$ is a **horizontal asymptote** of the graph of a function f if $f(x)$ approaches b as x increases or decreases without bound.

Again we look at $f(x) = \frac{1}{x}$



y = number
(if it exists)

Horizontal Line
that the graph
levels out to
(far left and far
right)

Horizontal asymptotes really have to do with what happens to the y-values as x becomes very large or very small. If the y-values approach a particular number at the far left and far right ends of the graph, then the function has a horizontal asymptote.

Note: A rational function may have several vertical asymptotes, but only at most one horizontal asymptote. Also, a graph cannot cross a vertical asymptote, but may cross a horizontal asymptote.

Locating Horizontal Asymptotes

To find the location of any horizontal asymptote, determine the degree of the numerator and the degree of the denominator. Then

- If the degree of the numerator is less than the degree of the denominator, the horizontal asymptote is $y = 0$.
- If the degree of the numerator is the same as the degree of the denominator, then the horizontal asymptote is $y = \frac{a}{b}$, where a and b are the leading coefficients of the numerator and denominator
- If the degree of the numerator is greater than the degree of the denominator, then there is no horizontal asymptote.

To summarize:

$f(x) = \frac{2x^3+5x+1}{x^2-7}$

$$\frac{Larger\ Degree}{Smaller\ Degree}: HA\ does\ not\ exist$$

$\frac{Deg\ 3}{Deg\ 2}$ HA: Does Not Exist

$g(x) = \frac{x+2}{x^2+5x+4}$

$$\frac{Smaller\ Degree}{Larger\ Degree}: HA\ is\ at\ y = 0$$

$\frac{Deg\ 1}{Deg\ 2}$ HA: $y=0$

$$\frac{Equal\ Degree}{Equal\ Degree}: HA\ exists: y = (quotient\ of\ leading\ coefficients)$$

$h(x) = \frac{2x^2+4x+1}{1x^2+5}$ $\frac{Deg\ 2}{Deg}$ $\frac{2x^2}{1x^2} \rightarrow \frac{2}{1}$ HA: $y=2$

Example 5: Find the horizontal, if there is one

$$f(x) = \frac{x+2}{x^2+6x+9}$$

$\frac{\text{Deg } 1}{\text{Deg } 2}$ $\frac{\text{small}}{\text{large}}$ HA: $y=0$

$$g(x) = \frac{\sqrt{2x^2+4x+1}}{x^2+4x+5} \rightarrow \sqrt{x^2} = x \rightarrow \frac{\text{Deg } 1}{\text{Deg } 2}$$

HA: $y=0$

Example6: Find the horizontal, if there is one

$$f(x) = \frac{\boxed{3x^4} + 12x^2 + 12}{\boxed{1x^4} + 7x^3 + 10} \qquad \frac{\text{Deg } 4}{\text{Deg } 4} \qquad \frac{\text{Equal}}{\text{Equal}}$$

$$\frac{3x^4}{1x^4} \to \frac{3}{1} \qquad HA: y = 3$$

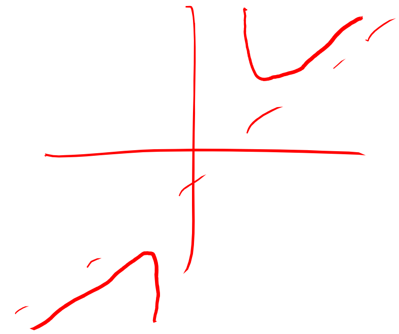
$$g(x) = \frac{7 - x^2}{2x^2 + 4x + 5} \qquad \frac{\text{Deg2}}{\text{Deg2}} \qquad \frac{\text{Equal}}{\text{Equal}}$$

$$\frac{-x^2}{2x^2} \to \frac{-1}{2} \qquad HA: y = -1/2$$

Example 7: Find the horizontal, if there is one

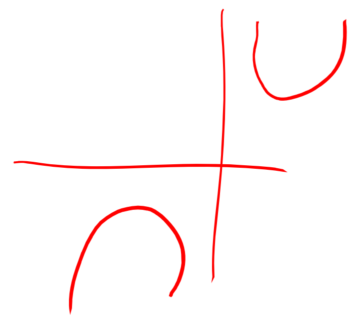
$$f(x) = \frac{x^2 + 3x + 2}{3x + 6}$$

$\frac{\text{Deg } 2}{\text{Deg } 1} \rightarrow \frac{\text{Large}}{\text{Small}}$ No HA



$$g(x) = \frac{(x+2)^2}{x-1} = \frac{\text{Deg } 2}{\text{Deg } 1} \quad \frac{\text{Large}}{\text{Small}}$$

No HA



Popper 20:

Answer the following for the function: $f(x) = \frac{3x^2-27}{x^2-x-6}$ $= \frac{3(x^2-9)}{x^2-x-6} = \frac{3(x+3)(x-3)}{(x-3)(x+2)}$

1. Determine the location (if any) of vertical asymptotes.

a. x = 2 b. x = 3 c. x = -2 d. None

Den Only

$x+2=0$

$x=-2$

2. Determine the location (if any) of any holes.

a. x = -2 b. x = 3 c. x = -3 d. None

Num & Den

$x-3=0$

$x=3$

3. Determine the y-coordinate of the hole.

a. 18/5 b. 3 c. 27/5 d. 3/2

$\frac{3(x+3)\cancel{(x-3)}}{\cancel{(x-3)}(x+2)} = \frac{3(3+3)}{3+2} = \frac{3(6)}{5}$

$y = \frac{18}{5}$

4. Determine the location (if any) of horizontal asymptotes

a. y = 9/2 b. y = ½ c. y = 3 d. None

$\frac{\text{Deg } 2}{\text{Deg } 2} \to \frac{3x^2}{x^2} = \frac{3}{1}$

$y=3$

Steps to Graphing a Rational Function

1. Factor numerator and denominator. If a factor in the numerator cancels with a factor in the denominator then there is a **hole** in the graph when that cancelled factor equal zero.
2. Find x-intercept(s) by setting numerator equal to zero. Factor in Numerator Only
3. Find y-intercept (if there is one) by substituting $x = 0$ in the function. $f(0)$
4. Find horizontal asymptote (if there is one).
5. Find vertical asymptote, if any, by setting the denominator equal to zero.
6. Use the x-intercept(s) and vertical asymptote(s) to divide the x-axis into intervals. Choose a test point in each interval to determine if the function is positive or negative there. This will tell you whether the graph approaches the vertical asymptote in an upward or downward direction.
7. Graph! *Except for the breaks at the vertical asymptotes, the graph should be a nice smooth curve with no sharp corners.*

Quick Summary of Rational Functions:

Domain: Exclude any values that make denominator equal zero

Vertical Asymptote: x = k, when (x – k) appears in denominator only

Hole: x = m, when (x – m) appears in both numerator and denominator

[To find y-value of hole, simplify the function and plug in m]

Horizontal Asymptote: compare degree of numerator and denominator: n > d, no HA; n < d, HA is y = 0; n = d, HA is ratio of leading coefficients.

x-intercepts: x = n, where (x – n) appears in numerator only

y-intercept: the value of f(0)

Note: There is no guarantee all of these will appear in each rational function.

Example 8: Sketch the graph of

$$f(x) = \frac{1}{x-1}$$

Domain: $x - 1 \neq 0$

$x \neq 1$

$(-\infty, 1) \cup (1, \infty)$

VA: Den only:

$x - 1 = 0$

$x = 1$

HA: $\frac{\text{Deg } 0}{\text{Deg } 1}$

$y = 0$

Hole: Num & Den:

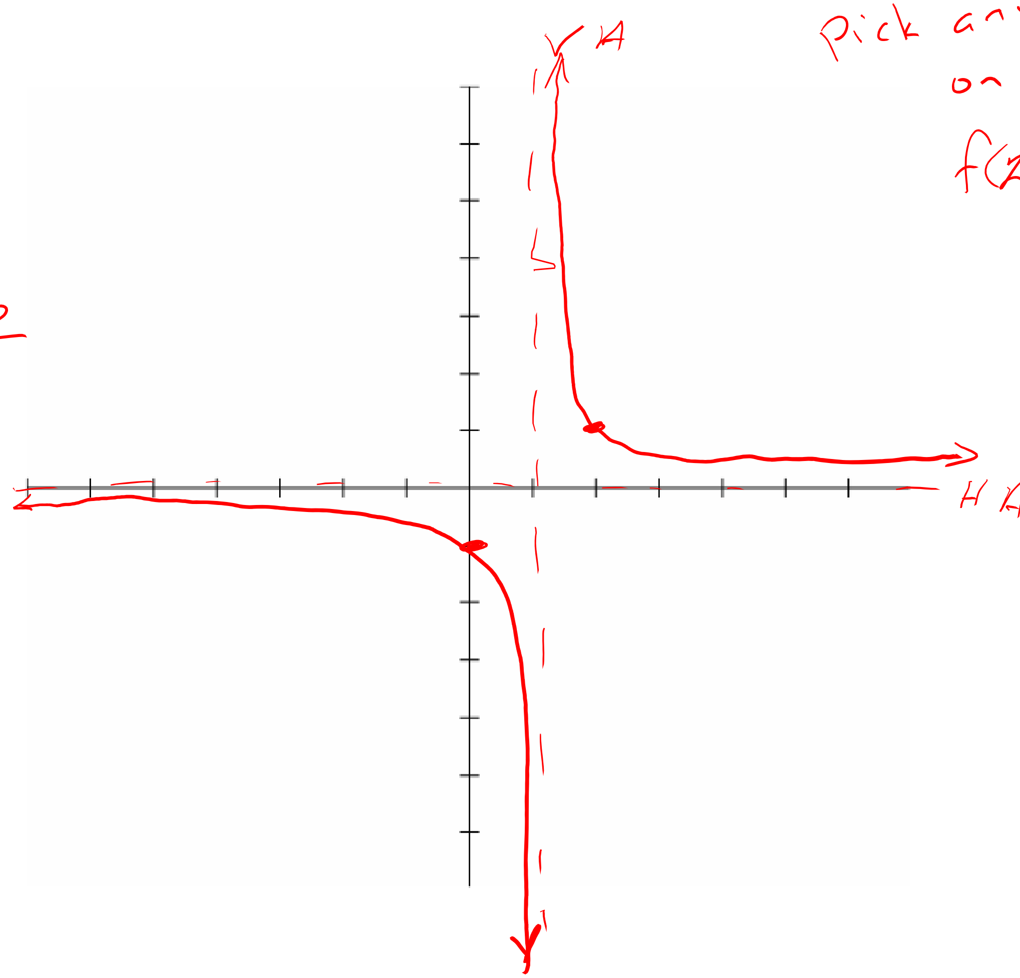
None

x-int: Num only

None

y-int: $f(0) = \frac{1}{0-1} = \frac{1}{-1} = -1$

$(0, -1)$



Pick any x-value on the right of VA. $(x = 2)$

$f(2) = \frac{1}{2-1} = \frac{1}{1} = 1$

$(2, 1)$

Example 9: Sketch the graph of

$$f(x) = \frac{x^2 + x - 6}{x^2 + 3x - 10} = \frac{(x+3)(x-2)}{(x+5)(x-2)}$$

Domain:
$x+5 \neq 0$ $\quad$ $x-2 \neq 0$
$x \neq -5$ $\quad$ $x \neq 2$

$(-\infty, -5) \cup (-5, 2) \cup (2, \infty)$

VA: Den Only: $x+5=0$
$x=-5$

Hole: Num & Den: $x-2=0$
$x=2$

HA: $\frac{\text{Deg } 2}{\text{Deg } 2}$ $\quad$ $\frac{1x^2}{1x^2}$ $\quad$ $y = \frac{1}{1} = 1$

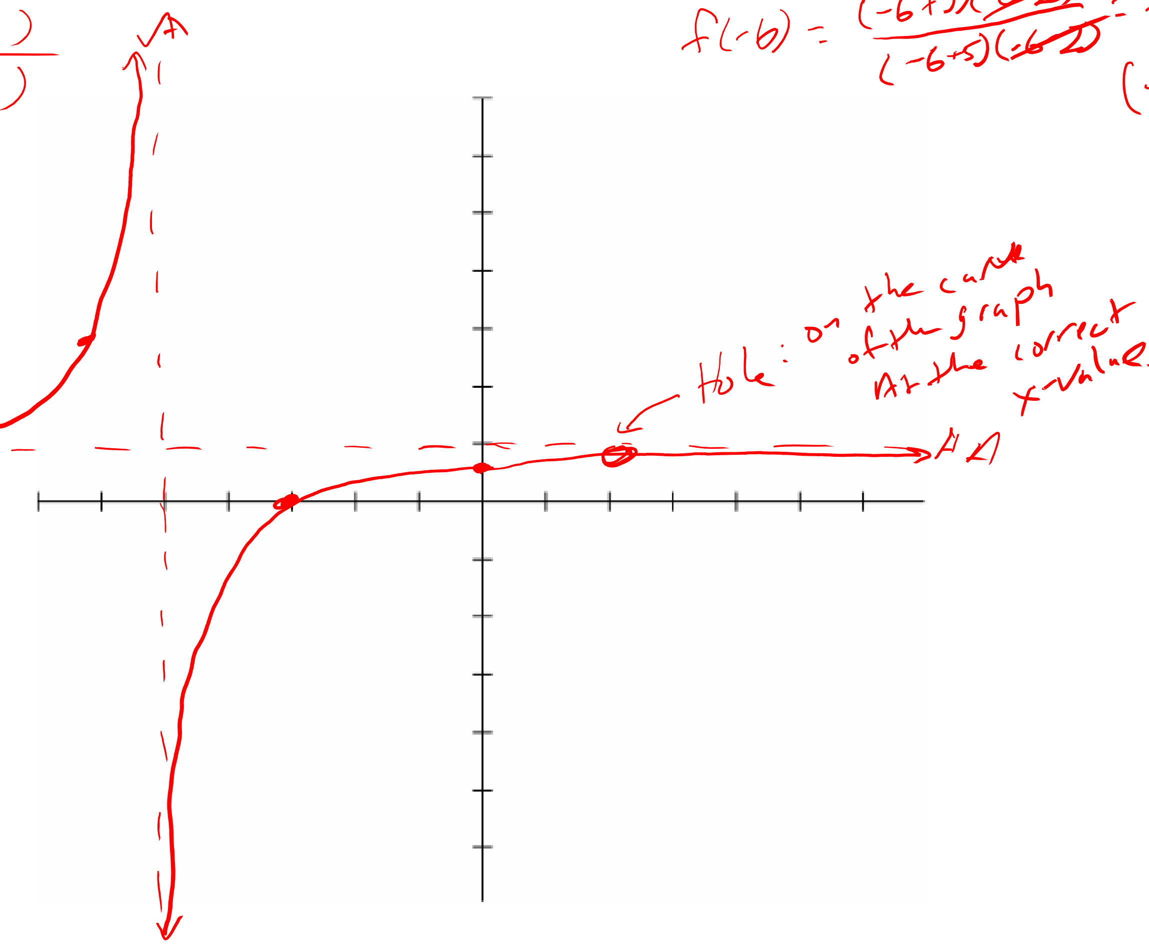
x-int: Num Only: $x+3=0$
$x=-3$

y-int: $f(0) = \frac{-6}{-10} = \frac{3}{5}$

Pick an x left of VA

$f(-6) = \frac{(-6+3)\cancel{(-6-2)}}{(-6+5)\cancel{(-6-2)}} = \frac{-3}{-1} = 3$

$(-6, 3)$



Example 10: Sketch the graph of

$$f(x) = \frac{x^2+5}{x^2-4} = \frac{x^2+5}{(x+2)(x-2)}$$

Domain: $x+2 \neq 0$ $x \neq -2$ $x-2 \neq 0$ $x \neq 2$

$(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

VA: (Den only) $x+2=0$ $x=-2$ $x-2=0$ $x=2$

Hole: (Num & Den): None

HA: $\frac{\text{Deg } 2}{\text{Deg } 2}$ $\frac{1x^2}{1x^2} \to y=1$

x-int: (Num Only) $x^2+5=0$ $x^2=-5$ $x=\pm\sqrt{-5}$

No x-int (Real)

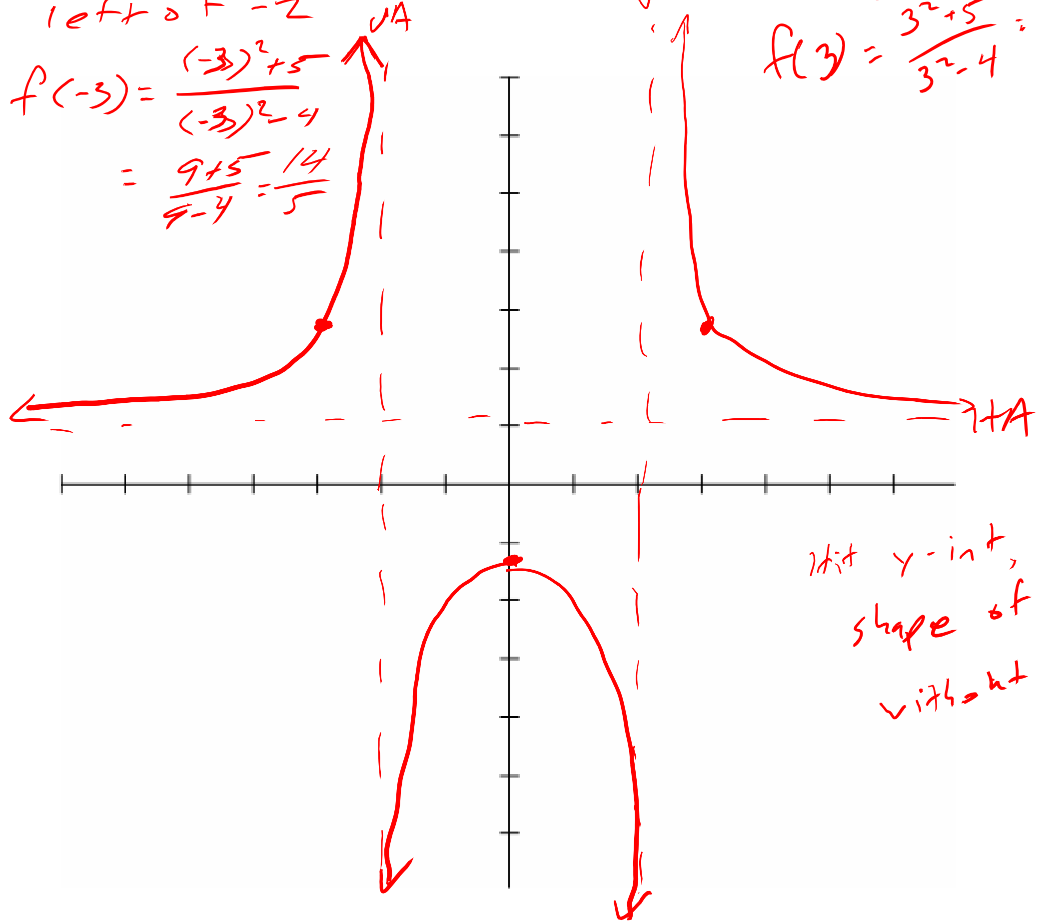
Y-int: $f(0) = \frac{5}{-4} = -5/4 = -1.25$

Pick an x left of -2

$f(-3) = \frac{(-3)^2+5}{(-3)^2-4} = \frac{9+5}{9-4} = \frac{14}{5}$

Pick an x right of 2

$f(3) = \frac{3^2+5}{3^2-4} = \frac{9+5}{9-4} = \frac{14}{5}$



Hit y-int, Match the shape of 2 VAs without hitting x-axis

$$f(x)=\frac{x^2+4x+3}{x^2-5x-6}=\frac{(x+1)(x+3)}{(x-6)\cancel{(x+1)}}$$

Domain: $x-6\neq 0 \quad x+1\neq 0$

$x\neq 6 \quad x\neq -1$

$(-\infty,-1)\cup(-1,6)\cup(6,\infty)$

VA: $x-6=0$

$x=6$

Hole: $x+1=0$

$x=-1$

x-int: $x+3=0$

$x=-3$

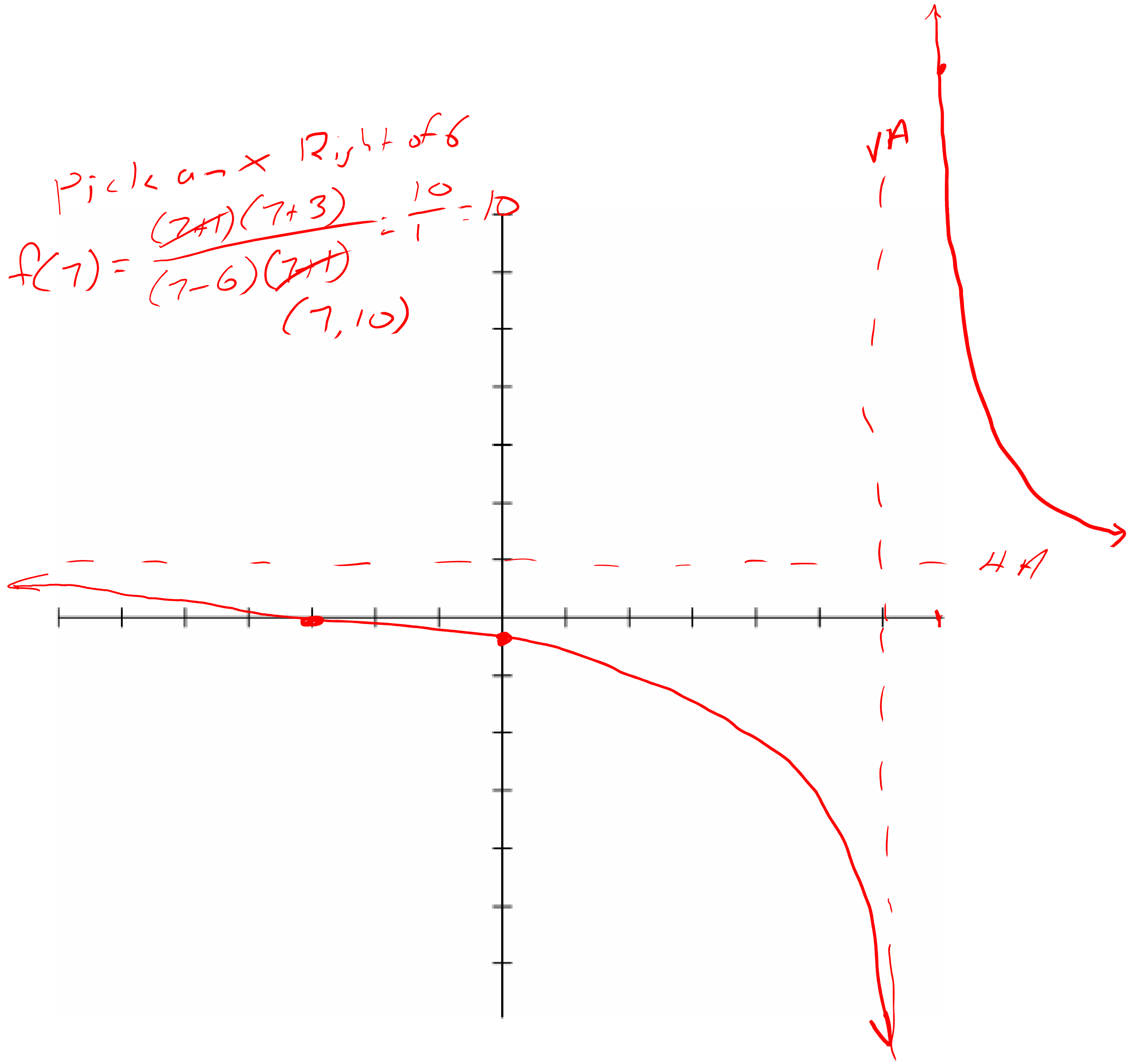
y-int: $f(0)=\frac{3}{-6}=-\frac{1}{2}$

HA: $\frac{\text{Equal Deg}}{\text{Equal Deg}}=\frac{1x^2}{1x^2}\to y=1$

Pick an x Right of 6

$$f(7)=\frac{\cancel{(7+1)}(7+3)}{(7-6)\cancel{(7+1)}}=\frac{10}{1}=10$$

$(7,10)$



Popper 20 continued: $f(x) = \frac{2x^2-4x-70}{x^2-49}$

$\frac{2(x^2-2x-35)}{x^2-49}$

$\frac{2(x-7)(x+5)}{(x-7)(x+7)}$

$x-7 \neq 0 \quad x+7 \neq 0$

$x \neq 7 \quad x \neq -7$

5. Find the domain

   a. (-∞, 7) b. (-∞, -7) U (7, ∞)

   c. (-∞, -7) U (-7, 7) U (7, ∞) d. (-∞, ∞)

6. Find any holes

   $x-7=0$

   $x=7$

   a. x = -7, 7 b. x = -7 c. x = 7 d. None

7. Find any vertical asymptotes

   $x+7=0$

   $x=-7$

   a. x = -7, 7 b. x = -7 c. x = 7 d. None

8. Find any horizontal asymptotes

   $\frac{\text{Deg } 2}{\text{Deg } 2}$ $\frac{2x^2}{1x^2}$ $y=\frac{2}{1}=2$

   a. y = 0 b. y = -5 c. y = 2 d. None

9. Find any x-intercepts

   $x+5=0$

   $x=-5$

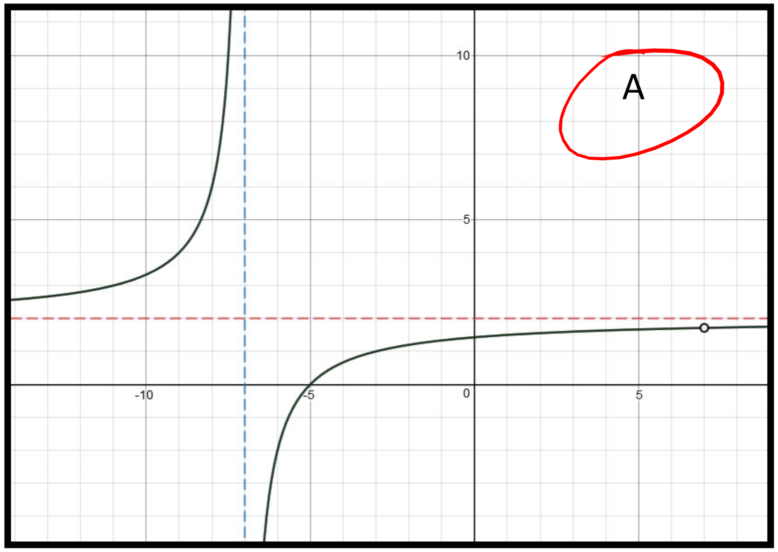
   a. x = -5, 7 b. x = -5 c. x = 7 d. None

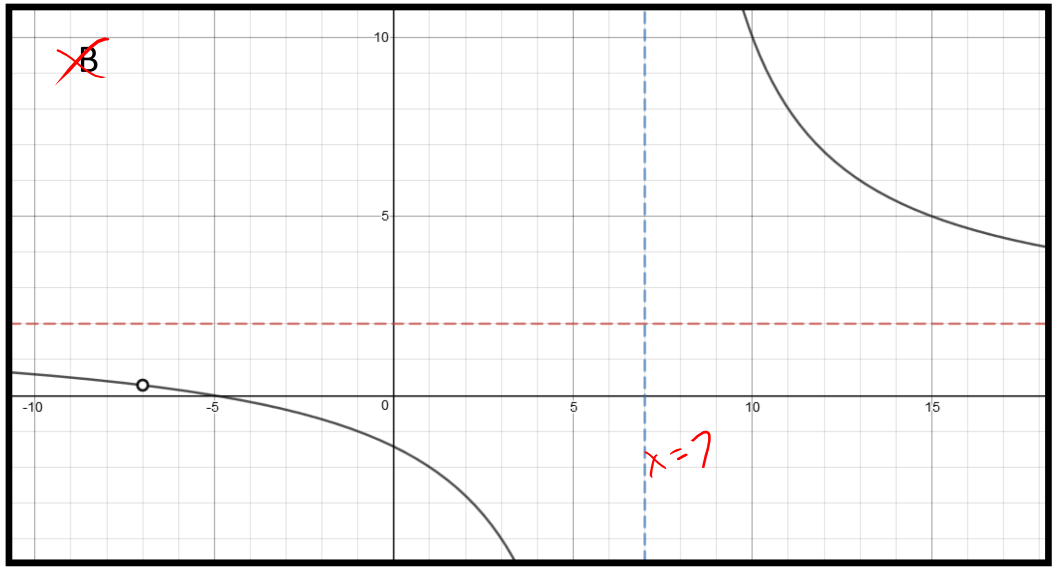
10. Find any y-intercepts

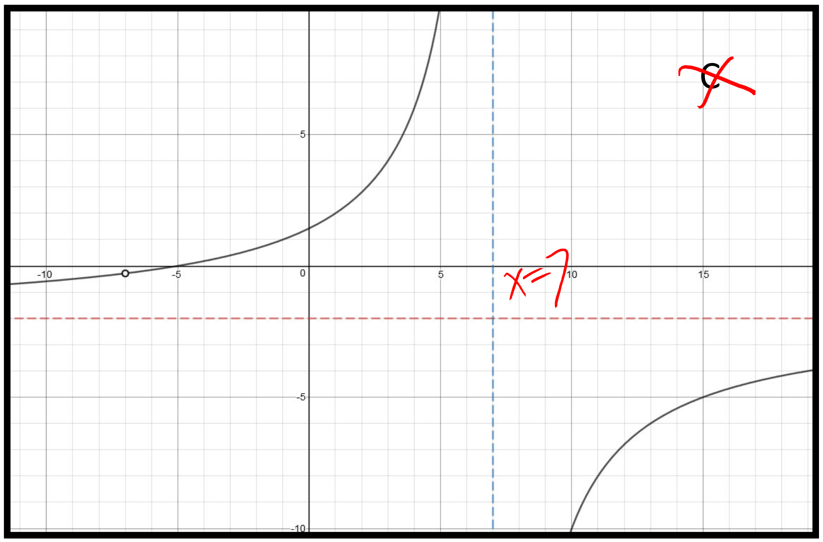
   $f(0)=\frac{-70}{-49}=10/7$

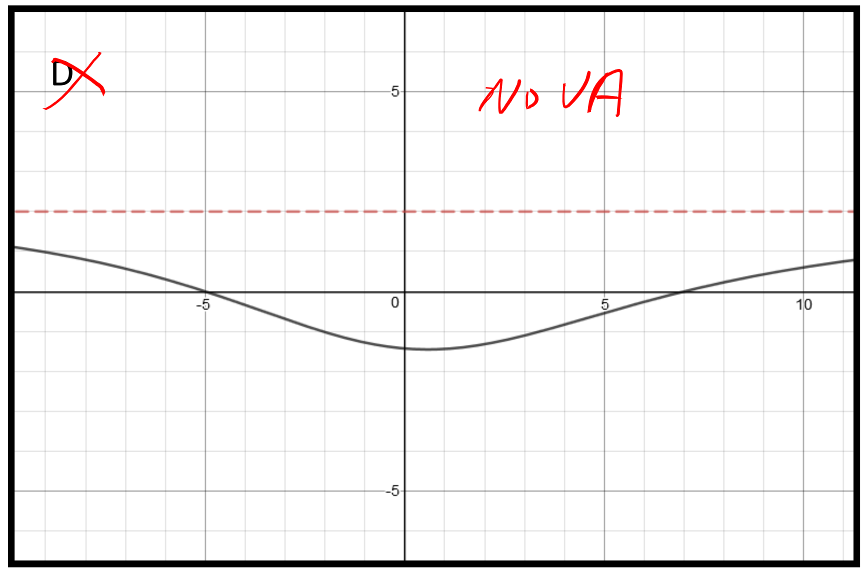
   a. y = -70 b. y = -10/7 c. y = 10/7 d. None

11. Sketch (choices on next slide) A


A
10
5
-10
-5
0
5


B
10
5
-10
-5
0
5
10
15
x=7


C
5
-10
-5
0
5
10
15
-5
-10
x=7


D
No VA
5
-5
0
5
10
-5