MATH 1314

Section 4.4

Rational Functions

The objective in this section will be to identify the important features of a rational function and then to use them to sketch an accurate graph of the function.

A **rational function** can be expressed as $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial functions and $q(x) \neq 0$.

Example 1: Find the domain of $f(x) = \frac{x-2}{x^2-9}$

Denominator $\neq 0$

$x^2 - 9 \neq 0$

$\sqrt{x^2} \neq \sqrt{9}$

$x \neq \pm 3$

$(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

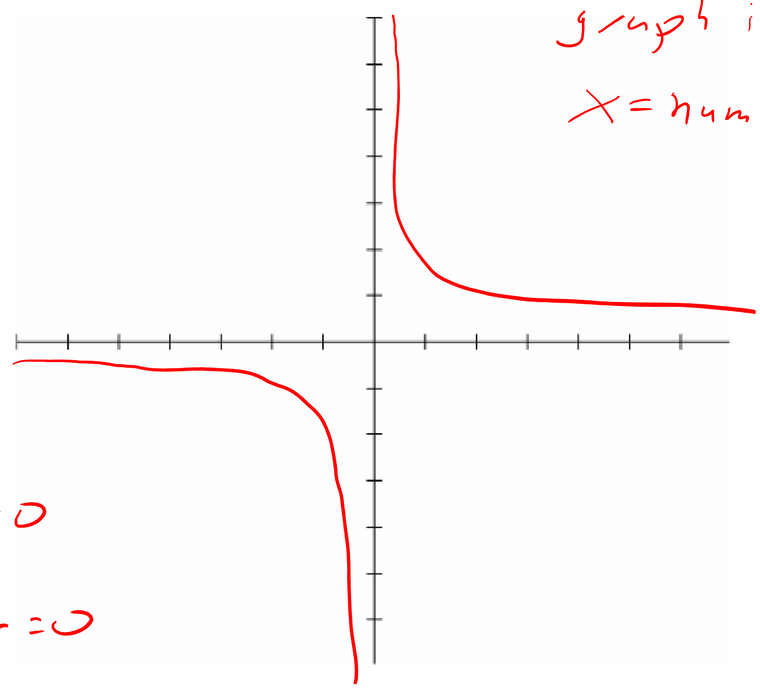
Vertical Asymptote of Rational Functions

The line $x = a$ is a **vertical asymptote** of the graph of a function f if $f(x)$ increases or decreases without bound as x approaches a.

Basic example is $f(x) = \frac{1}{x}$

vertical line that breaks the graph in 2 parts.

$x =$ number



x-values where:

Denominator $= 0$

and

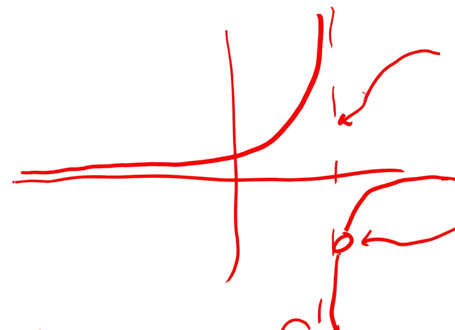
Numerator $\neq 0$

If Both

Numerator $= 0$

and

Denominator $= 0$



Hole: Does Not change the shape of the graph. Single (x, y) point that is undefined.

Factor the numerator and denominator. Look at each factor in the denominator.

- If a factor cancels with a factor in the numerator, then there is a hole where that factor equals zero.
- If a factor does not cancel, then there is a vertical asymptote where that factor equals zero.

Example 2: Find any vertical asymptote(s) and/or hole(s) of $f(x) = \dfrac{x^2 - 3x - 10}{x^2 - x - 6}$

$$f(x) = \frac{x^2 - 3x - 10}{x^2 - x - 6} = \frac{(x-5)(x+2)}{(x-3)(x+2)}$$

- Factors that appear in the denominator only: $x - 3 = 0$
  $x = 3 \rightarrow$ Vertical Asymptote

- Factors that appear in both numerator and denominator: $x + 2 = 0$
  $x = -2 \rightarrow$ Location of a hole.

[y-value of the hole: plug x-value into simplified function:

$$\frac{(x-5)\cancel{(x+2)}}{(x-3)\cancel{(x+2)}} = \frac{(-2-5)}{(-2-3)} = \frac{-7}{-5} = 7/5 \qquad (-2, 7/5)$$

Example 3: Find any vertical asymptote(s) and/or hole(s) of

$$f(x) = \frac{x^2 + 3x - 4}{x^2 - 9} = \frac{(x+4)(x-1)}{(x+3)(x-3)}$$

Vertical Asymptotes: VA: (Den Only)

$x + 3 = 0$ $\quad\quad$ $x - 3 = 0$

$x = -3$ $\quad\quad$ $x = 3$

2 vertical asymptotes: $x = -3$, $x = 3$

Holes: (Both Num & Den): None

Example 4: Find any vertical asymptote(s) and/or hole(s) of

$$f(x) = \frac{x^2 - 16}{x^2 - 2x - 8} = \frac{(x+4)(x-4)}{(x-4)(x+2)}$$

VA: (Den only): $x+2=0$

$x=-2$

Hole: (Num; Den): $x-4=0$

$x=4$

[Give a function that is identical to f(x) in all but one point. [Give simplified form of the function]

$g(x) = \frac{x+4}{x+2}$

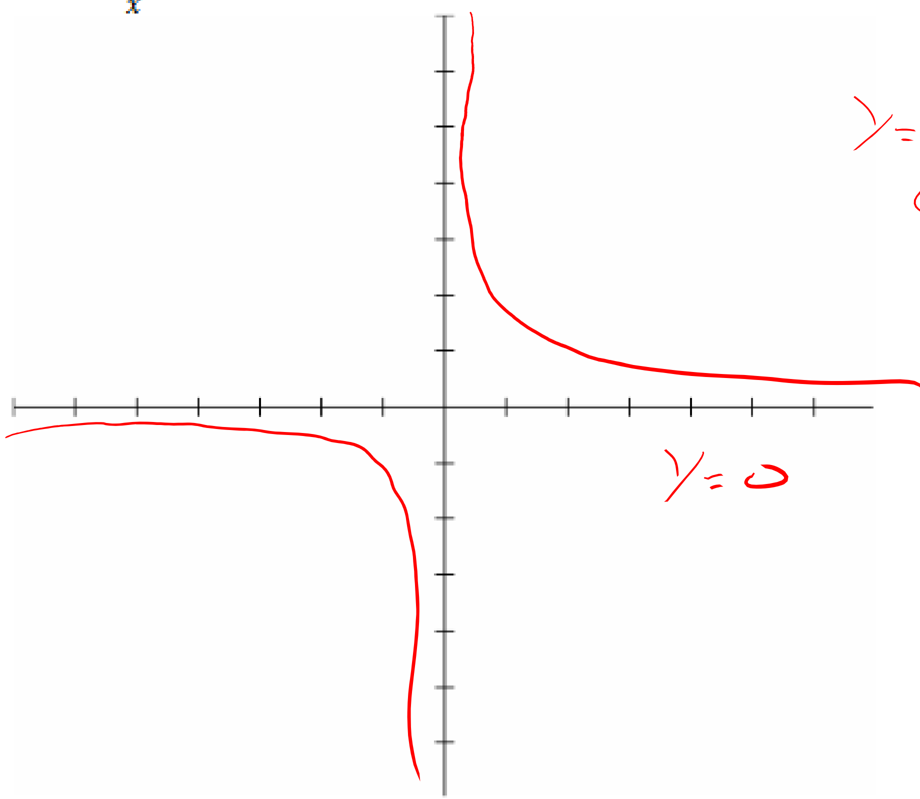
[y-value]: $\frac{(x+4)\cancel{(x-4)}}{\cancel{(x-4)}(x+2)} = \frac{(4+4)}{(4+2)} = \frac{8}{6} = 4/3$

Hole: $(4, 4/3)$

Horizontal Asymptote of Rational Functions

The line $y = b$ is a **horizontal asymptote** of the graph of a function f if $f(x)$ approaches b as x increases or decreases without bound.

Again we look at $f(x) = \frac{1}{x}$



y = number (if it exists)

Horizontal Line that the graph levels out to (far left and far right)

Horizontal asymptotes really have to do with what happens to the y-values as x becomes very large or very small. If the y-values approach a particular number at the far left and far right ends of the graph, then the function has a horizontal asymptote.

Note: A rational function may have several vertical asymptotes, but only at most one horizontal asymptote. Also, a graph cannot cross a vertical asymptote, but may cross a horizontal asymptote.

Locating Horizontal Asymptotes

To find the location of any horizontal asymptote, determine the degree of the numerator and the degree of the denominator. Then

- If the degree of the numerator is less than the degree of the denominator, the horizontal asymptote is $y = 0$.
- If the degree of the numerator is the same as the degree of the denominator, then the horizontal asymptote is $y = \frac{a}{b}$, where a and b are the leading coefficients of the numerator and denominator
- If the degree of the numerator is greater than the degree of the denominator, then there is no horizontal asymptote.

To summarize:

$f(x) = \frac{2x^3+5x+1}{x^2-7}$

$$\frac{Larger\ Degree}{Smaller\ Degree} : HA\ does\ not\ exist$$

$\frac{Deg\ 3}{Deg\ 2}$ HA: Does Not Exist

$g(x) = \frac{x+2}{x^2+5x+4}$

$$\frac{Smaller\ Degree}{Larger\ Degree} : HA\ is\ at\ y = 0$$

$\frac{Deg\ 1}{Deg\ 2}$ HA: $y=0$

$$\frac{Equal\ Degree}{Equal\ Degree} : HA\ exists: y = (quotient\ of\ leading\ coefficients)$$

$h(x) = \frac{2x^2+4x+1}{1x^2+5}$ $\frac{Deg\ 2}{Deg}$ $\frac{2x^2}{1x^2} \to \frac{2}{1}$ HA: $y=2$

Example 5: Find the horizontal, if there is one

$$f(x) = \frac{x+2}{x^2+6x+9}$$

$\frac{\text{Deg } 1}{\text{Deg } 2}$ $\frac{\text{small}}{\text{large}}$ HA: $y=0$

$$g(x) = \frac{\sqrt{2x^2+4x+1}}{x^2+4x+5} \rightarrow \sqrt{x^2} = x \rightarrow \frac{\text{Deg } 1}{\text{Deg } 2}$$

HA: $y=0$

Example6: Find the horizontal, if there is one

$$f(x) = \frac{\boxed{3x^4} + 12x^2 + 12}{\boxed{1x^4} + 7x^3 + 10} \qquad \frac{\text{Deg } 4}{\text{Deg } 4} \qquad \frac{\text{Equal}}{\text{Equal}}$$

$$\frac{3x^4}{1x^4} \to \frac{3}{1} \qquad \text{HA: } y = 3$$

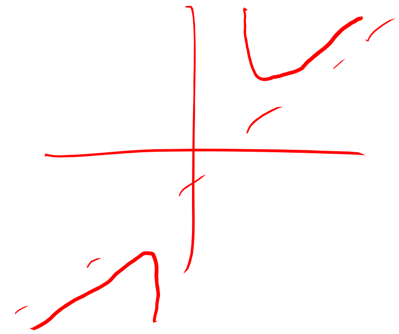
$$g(x) = \frac{7 - x^2}{2x^2 + 4x + 5} \qquad \frac{\text{Deg2}}{\text{Deg2}} \qquad \frac{\text{Equal}}{\text{Equal}}$$

$$\frac{-x^2}{2x^2} \to \frac{-1}{2} \qquad \text{HA: } y = -1/2$$

Example 7: Find the horizontal, if there is one

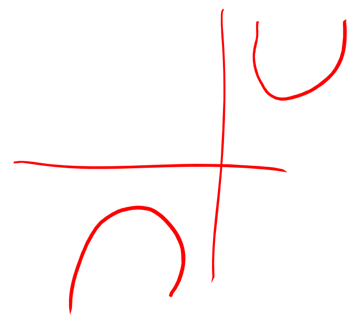
$$f(x) = \frac{x^2 + 3x + 2}{3x + 6}$$

$\frac{\text{Deg } 2}{\text{Deg } 1} \rightarrow \frac{\text{Large}}{\text{Small}}$ No HA



$g(x) = \frac{(x+2)^2}{x-1} = \frac{\text{Deg } 2}{\text{Deg } 1}$ $\frac{\text{Large}}{\text{Small}}$

No HA



Popper 24:

Answer the following for the function: $f(x) = \frac{3x^2-27}{x^2-x-6}$

$= \frac{3(x^2-9)}{x^2-x-6} = \frac{3(x+3)(x-3)}{(x-3)(x+2)}$

1. Determine the location (if any) of vertical asymptotes.

a. x = 2 b. x = 3 c. x = -2 d. None

Den Only

$x+2=0$

$x=-2$

2. Determine the location (if any) of any holes.

a. x = -2 b. x = 3 c. x = -3 d. None

Num ? Den

$x-3=0$

$x=3$

3. Determine the y-coordinate of the hole.

a. 18/5 b. 3 c. 27/5 d. 3/2

$\frac{3(x+3)\cancel{(x-3)}}{\cancel{(x-3)}(x+2)} = \frac{3(3+3)}{3+2} = \frac{3(6)}{5}$

$y = \frac{18}{5}$

4. Determine the location (if any) of horizontal asymptotes

a. y = 9/2 b. y = ½ c. y = 3 d. None

$\frac{\text{Deg } 2}{\text{Deg } 2} \rightarrow \frac{3x^2}{x^2} = \frac{3}{1}$

$y=3$