

MATH 1314

Section 4.4

Rational Functions

The objective in this section will be to identify the important features of a rational function and then to use them to sketch an accurate graph of the function.

A **rational function** can be expressed as $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial functions and $q(x) \neq 0$.

Example 1: Find the domain of $f(x) = \frac{x-2}{x^2-9}$

Denominator $\neq 0$

$$x^2 - 9 \neq 0$$

$$\sqrt{x^2} \neq \sqrt{9}$$

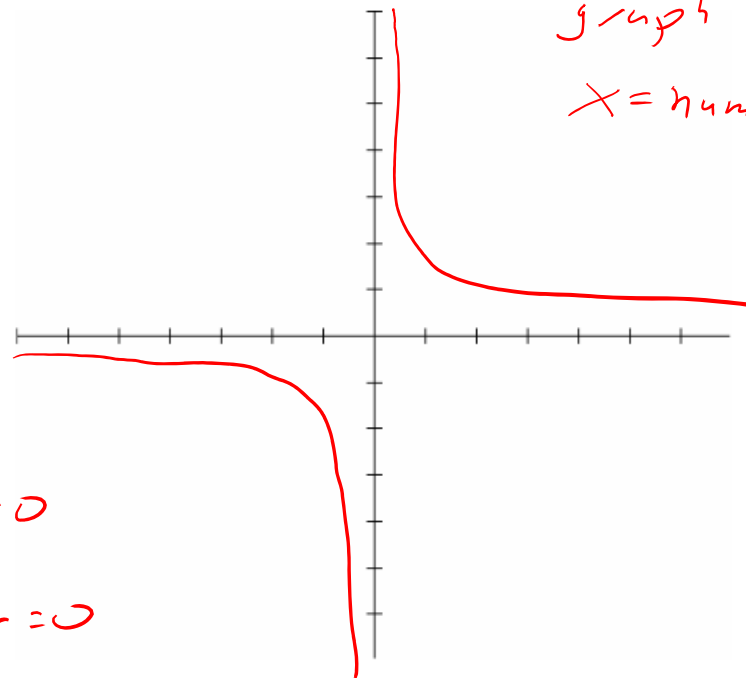
$$x \neq \pm 3$$

$$(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$$

Vertical Asymptote of Rational Functions

The line $x = a$ is a **vertical asymptote** of the graph of a function f if $f(x)$ increases or decreases without bound as x approaches a .

Basic example is $f(x) = \frac{1}{x}$



vertical line that breaks the graph in 2 parts.

$x = \text{number}$

x -values where:

Denominator = 0

and

Numerator $\neq 0$

If Both

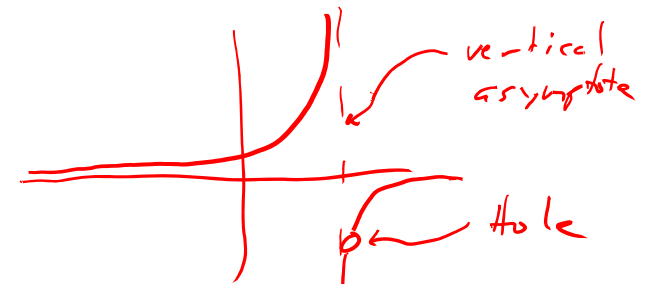
Numerator = 0

and

Denominator = 0

Hole: Does not change the

shape of the graph. Single (x, y) point that is undefined.



vertical asymptote

hole

Factor the numerator and denominator. Look at each factor in the denominator.

- If a factor cancels with a factor in the numerator, then there is a hole where that factor equals zero.
- If a factor does not cancel, then there is a vertical asymptote where that factor equals zero.

Example 2: Find any vertical asymptote(s) and/or hole(s) of $f(x) = \frac{x^2 - 3x - 10}{x^2 - x - 6}$

$$f(x) = \frac{x^2 - 3x - 10}{x^2 - x - 6} = \frac{(x-5)(x+2)}{(x-3)(x+2)}$$

• Factors that appear in the denominator only: $x-3=0$
 $x=3 \rightarrow$ Vertical Asymptote

• Factors that appear in both numerator and denominator:

[y-value of the hole: plug x-value into simplified function]

$$x+2=0$$

$x=-2 \rightarrow$ Location of a hole

$$\frac{(x-5)(x+2)}{(x-3)(x+2)} = \frac{(-2-5)}{(-2-3)} = \frac{-7}{-5} = 7/5 \quad (-2, 7/5)$$

Example 3: Find any vertical asymptote(s) and/or hole(s) of

$$f(x) = \frac{x^2 + 3x - 4}{x^2 - 9} = \frac{(x+4)(x-1)}{(x+3)(x-3)}$$

Vertical Asymptotes: VA: (Den Only)

$$x+3=0$$

$$x=-3$$

$$x-3=0$$

$$x=3$$

2 vertical asymptotes: $x=-3$, $x=3$

Holes: (Both Num; Den). None

Example 4: Find any vertical asymptote(s) and/or hole(s) of

$$f(x) = \frac{x^2 - 16}{x^2 - 2x - 8} = \frac{(x+4)(x-4)}{(x-4)(x+2)}$$

V.A. (Den only): $x+2=0$
 $x=-2$

Hole: (Num; Den): $x-4=0$
 $x=4$

Give a function that is identical to $f(x)$ in all but one point.
[Give simplified form of the function]

$$g(x) = \frac{x+4}{x+2}$$

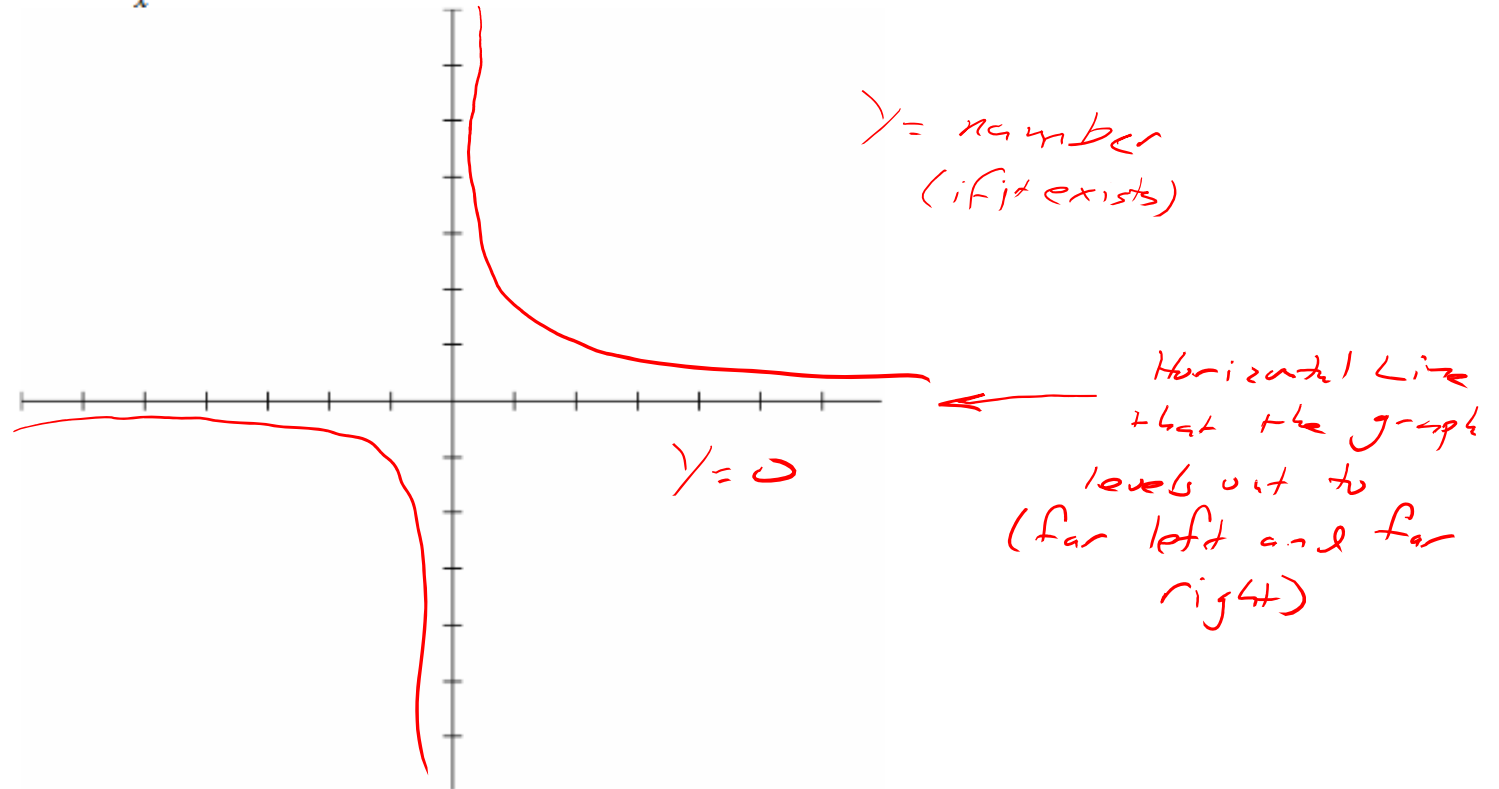
[y-value]: $\frac{(x+4)\cancel{(x-4)}}{\cancel{(x-4)}(x+2)} = \frac{(4+4)}{(4+2)} = \frac{8}{6} = \frac{4}{3}$

Hole: $(4, \frac{4}{3})$

Horizontal Asymptote of Rational Functions

The line $y = b$ is a **horizontal asymptote** of the graph of a function f if $f(x)$ approaches b as x increases or decreases without bound.

Again we look at $f(x) = \frac{1}{x}$



Horizontal asymptotes really have to do with what happens to the y -values as x becomes very large or very small. If the y -values approach a particular number at the far left and far right ends of the graph, then the function has a horizontal asymptote.

Note: A rational function may have several vertical asymptotes, but only at most one horizontal asymptote. Also, a graph cannot cross a vertical asymptote, but may cross a horizontal asymptote.

Locating Horizontal Asymptotes

To find the location of any horizontal asymptote, determine the degree of the numerator and the degree of the denominator. Then

- If the degree of the numerator is less than the degree of the denominator, the horizontal asymptote is $y = 0$.
- If the degree of the numerator is the same as the degree of the denominator, then the horizontal asymptote is $y = \frac{a}{b}$, where a and b are the leading coefficients of the numerator and denominator
- If the degree of the numerator is greater than the degree of the denominator, then there is no horizontal asymptote.

To summarize:

$$f(x) = \frac{2x^3 + 5x + 1}{x^2 - 7} \quad \frac{\text{Larger Degree}}{\text{Smaller Degree}} : \text{HA does not exist}$$

$\frac{\text{Deg } 3}{\text{Deg } 2}$ HA: Does Not Exist

$$g(x) = \frac{x + 2}{x^2 + 5x + 4} \quad \frac{\text{Smaller Degree}}{\text{Larger Degree}} : \text{HA is at } y = 0$$

$\frac{\text{Deg } 1}{\text{Deg } 2}$ HA: $y = 0$

$\frac{\text{Equal Degree}}{\text{Equal Degree}} : \text{HA exists: } y = (\text{quotient of leading coefficients})$

$$h(x) = \frac{2x^2 + 4x + 1}{1x^2 + 5} \quad \frac{\text{Deg } 2}{\text{Deg } 2} \quad \frac{2x^2}{1x^2} \rightarrow \frac{2}{1} \quad \text{HA: } y = 2$$

Example 5: Find the horizontal, if there is one

$$f(x) = \frac{x+2}{x^2+6x+9} \quad \begin{array}{l} \text{Deg 1} \\ \text{Deg 2} \end{array} \quad \begin{array}{l} \text{small} \\ \text{large} \end{array} \quad \text{HA: } y=0$$

$$g(x) = \frac{\sqrt{2x^2+4x+1} \rightarrow \sqrt{x^2=x} \rightarrow \text{Deg 1}}{x^2+4x+5} \quad \begin{array}{l} \text{Deg 1} \\ \text{Deg 2} \end{array} \quad \text{HA: } y=0$$

Example 6: Find the horizontal, if there is one

$$f(x) = \frac{3x^4 + 12x^2 + 12}{x^4 + 7x^3 + 10}$$

Deg 4 Equal
Deg 4 Equal

$$\frac{3x^4}{x^4} \rightarrow \frac{3}{1} \quad \text{HA: } y = 3$$

$$g(x) = \frac{7 - x^2}{2x^2 + 4x + 5}$$

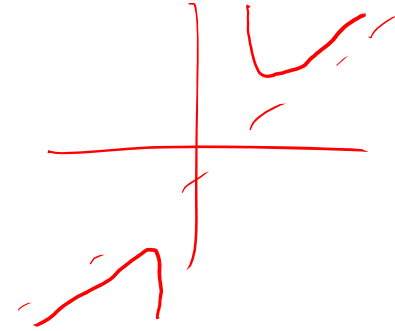
Deg 2 Equal
Deg 2 Equal

$$\frac{-x^2}{2x^2} \rightarrow \frac{-1}{2} \quad \text{HA: } y = -\frac{1}{2}$$

Example 7: Find the horizontal, if there is one

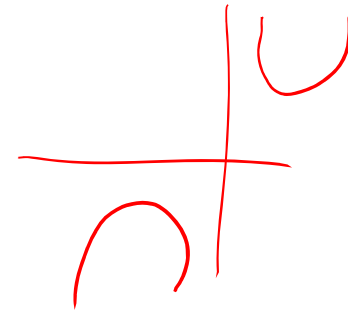
$$f(x) = \frac{x^2 + 3x + 2}{3x + 6}$$

$\frac{\text{Deg 2}}{\text{Deg 1}} \rightarrow \frac{\text{Large}}{\text{Small}} \quad \text{No HA}$



$$g(x) = \frac{(x+2)^2}{x-1} = \frac{\text{Deg 2}}{\text{Deg 1}} \quad \frac{\text{Large}}{\text{Small}}$$

No HA



Popper 20:

Answer the following for the function: $f(x) = \frac{3x^2 - 27}{x^2 - x - 6} = \frac{3(x^2 - 9)}{x^2 - x - 6} = \frac{3(x+3)(x-3)}{(x-3)(x+2)}$

1. Determine the location (if any) of vertical asymptotes.

- a. $x = 2$ b. $x = 3$ c. $x = -2$ d. None

Den Only

$$x + 2 = 0$$

$$x = -2$$

2. Determine the location (if any) of any holes.

- a. $x = -2$ b. $x = 3$ c. $x = -3$ d. None

Nu = Den

$$x - 3 = 0$$

$$x = 3$$

3. Determine the y-coordinate of the hole.

- a. $18/5$ b. 3 c. $27/5$ d. $3/2$

$$\frac{3(x+3)(\cancel{x-3})}{(\cancel{x-3})(x+2)} = \frac{3(3+3)}{3+2} = \frac{3(6)}{5} \\ y = \frac{18}{5}$$

4. Determine the location (if any) of horizontal asymptotes

- a. $y = 9/2$ b. $y = 1/2$ c. $y = 3$ d. None

$$\frac{\text{Deg } 2}{\text{Deg } 2} \rightarrow \frac{3x^2}{x^2} = \frac{3}{1} \\ y = 3$$