# MATH 1314

Section 5.4

# Properties of Logarithms

You will sometimes be asked to rewrite logarithmic expressions in either an expanded or contracted form. To do this, you will use the Laws of Logarithms. You must know all 8 of the Laws of Logarithms.

## Laws of Logarithms

If $m$, $n$ and $a$ are positive numbers, $a \neq 1$, then

1. $\log_a mn = \log_a m + \log_a n$

2. $\log_a \frac{m}{n} = \log_a m - \log_a n$

3. $\log_a m^n = n\log_a m$

4. $\log_a 1 = 0$

$\log_8 1 = 0$

5. $\log_a a = 1$

   ⑤ $\log_3 3 = 1$

6. $\log_a a^x = x$

7. $a^{\log_a x} = x$

8. $\log_a m = \frac{\log m}{\log a}$ (change of bases formula)

These properties are true for logs of any base, including common logs and natural logs.

Bases must be
$a > 0,\ a \neq 1$

① $\log_2(5x) = \log_2 5 + \log_2 x$

② $\ln\left(\frac{x}{3}\right) = \ln x - \ln 3$

③ $\log(x^5) = 5\log x$

⑥ $\log_4 4^x = x$

⑦ $e^{\ln x} = x$

$\log_5 8 = \frac{\log 8}{\log 5} = \frac{1}{\log_8 5}$

**Example 1:** Rewrite each of the following expressions in a form that has no logarithms of a product, quotient or power. (All numerators will be positive logs; All denominator values will be negative logs.)

a. $\log\left(\dfrac{3x}{y}\right) = \log(3x) - \log y = \log 3 + \log x - \log y$

b. $\ln(a^3 bc) = \ln(a^3) + \ln(b) + \ln(c) = 3\ln a + \ln b + \ln c$

c. $\log_4\left(\dfrac{4}{x}\right) = \log_4 4 - \log_4 x = 1 - \log_4 x$

**Example 1:** Rewrite each of the following expressions in a form that has no logarithms of a product, quotient or power.

d. $\log \sqrt{xy} = \log(xy)^{1/2} = \frac{1}{2}\log(xy) = \frac{1}{2}(\log x + \log y) = \frac{1}{2}\log x + \frac{1}{2}\log y$

e. $\ln\left(\dfrac{\sqrt[3]{x+4}}{(x+2)^4(x-5)^3}\right) = \ln \sqrt[3]{x+4} - \ln(x+2)^4 - \ln(x-5)^3$

$$= \frac{1}{3}\ln(x+4) - 4\ln(x+2) - 3\ln(x-5)$$

f. $\log \sqrt{\dfrac{(x-1)(x+2)^3}{x^2(x-2)}} = \frac{1}{2}\left(\log(x-1) + \log(x+2)^3 - \log x^2 - \log(x-2)\right)$

$$= \frac{1}{2}\left(\log(x-1) + 3\log(x+2) - 2\log x - \log(x-2)\right)$$

$$= \frac{1}{2}\log(x-1) + \frac{3}{2}\log(x+2) - \log x - \frac{1}{2}\log(x-2)$$

# Popper 25:

**Example 2:** Express each as a single logarithm:

1. $\log x - \underline{3} \log y = \log x - \log y^3 = \log\left(\frac{x}{y^3}\right)$

   a. $\log(x - y^3)$     b. $\log\left(\frac{x}{y^3}\right)$     c. $\log_3(x - y)$     d. $\log\left(\frac{x}{y}\right)^3$

2. $2\ln x + 3\ln(x + 2) - 5\ln y = \ln x^2 + \ln(x+2)^3 - \ln y^5 = \ln\left(\frac{x^2(x+2)^3}{y^5}\right)$

   a. $\ln\left(\frac{x^5 + 8x^2}{y^5}\right)$     b. $\ln\left(\frac{(x^2+2x)^5}{5y}\right)$     c. $\ln\left(\frac{x+2}{xy}\right)^5$     d. $\ln\left(\frac{x^2(x+2)^3}{y^5}\right)$

3. $\frac{1}{2}\ln(x + 2) - 3\ln(x^3 + 1) = \ln(x+2)^{1/2} - \ln(x^3+1)^3 = \ln\frac{\sqrt{x+2}}{(x^3+1)^3}$

   a. $\ln\left(\frac{\sqrt{x+2}}{(x^3+1)^3}\right)$     b. $\ln\sqrt{\frac{x+2}{(x^3+1)^3}}$     c. $\ln\left(\frac{\sqrt{x+2}}{x^9+1}\right)$     d. $\ln\left(\frac{x+2}{2(x^3+1)^3}\right)$

**Example 2:** Express each as a single logarithm:

d. $-2\log_4(x-5) - \log_4(x+1) + 2\log_4(x^2+1)$

$-\log_4(x-5)^2 - \log_4(x+1) + \log_4(x^2+1)^2 = \log_4 \dfrac{(x^2+1)^2}{(x-5)^2(x+1)}$

e. $\ln 18 - \ln 2 = \ln \dfrac{18}{2} = \ln 9 = \ln 3^2 = 2\ln 3$

Popper 25    Rewrite each as sums so that each logarithm contains a prime number.
(continued)    Simplify as much as possible

4.  $\ln 72 = \ln(2^3 \cdot 3^2) = \ln 2^3 + \ln 3^2 = 3\ln 2 + 2\ln 3$

   a. ln(2) ln(3)    b. 6ln(2)ln(3)    c. 3ln(2) + 2ln(3)    d. 2ln(36)

5.  $\log_2 96 = \log_2(2^5 \cdot 3) = \log_2 2^5 + \log_2 3 = 5\log_2 2 + \log_2 3 = 5 + \log_2 3$

   a. $5 + \log_2 3$    b. $\log_2 24 + 2$    c. $5\log_2 2 + \log_2 3$    d. $\log_2 24 + \log_2 4$

6.  $\log_7 147 - \log_7 3 = \log_7 \dfrac{147}{3} = \log_7 49 \rightarrow 7^{\square} = 49$

   a. 49    b. $\log_7 144$    c. 7    d. 2

   $\dfrac{147}{3} = \dfrac{120 + 27}{3}$
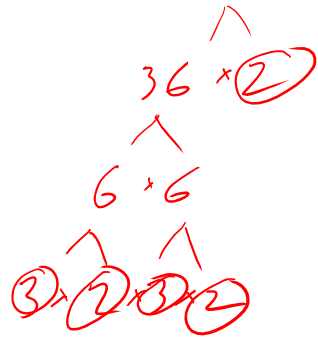
   $= \dfrac{120}{3} + \dfrac{27}{3}$

   $= 40 + 9$

   $= 49$

7.  $\log_4 32 + \log_4 2 = \log_4(32 \cdot 2) = \log_4 64 \rightarrow 4^{\boxed{3}} = 64$

   a. 2    b. 3    c. 4    d. 8

   $\dfrac{^2 16}{\times \dfrac{4}{64}}$

72

36 × ②

6 × 6

③ ② ③ ②

$$72 = 2^3 \times 3^2$$

96

4 8 × ②

24 × ②

4 × 6
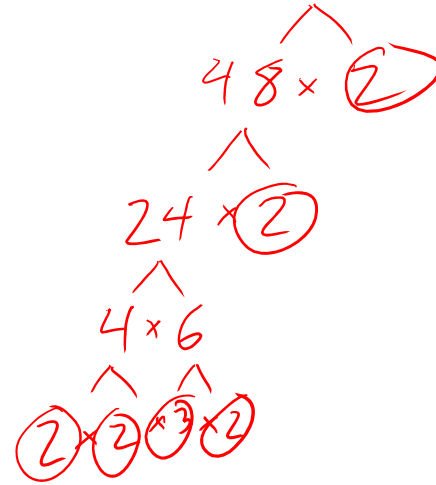
② ② ③ ②

$$96 = 2^5 \times 3$$

**Example 4:** Use the change of bases formula to solve $\log_8 12 = x$ and write in simplest form. Then use a calculator to evaluate to the nearest thousandth.

$$\log_8 12 = x \qquad \log_a m = \frac{\log_n m}{\log_n a}$$

change to base 2

$$\log_8 12 = \frac{\log_2 12}{\log_2 8} = \frac{\log_2(4 \cdot 3)}{\log_2(2^3)} = \frac{\log_2 4 + \log_2 3}{3 \log_2 2}$$

$$\log_2 4 \to 2^\square = 4 \to 2$$

$$\log_2 2 = 1$$

$$= \frac{2 + \log_2 3}{3} = \boxed{\frac{2}{3} + \log_2 \sqrt[3]{3}}$$

$$\frac{2}{3} + \frac{\log_2 3}{3} \to \frac{1}{3} \log_2 3$$

**Example 5:** Simplify each.

a. $\log_4 16^3 = \log_4 (4^2)^3 = \log_4 4^6 = 6$

b. $7^{3\log_7 3} = 7^{\log_7 3^3} = 3^3 = 27$

If it was $7^{\log_7 3}$  we can use inverse properties