MATH 1314

Section 5.5

Solving Exponential and Logarithmic Equations

We'll start with exponential equations.
An exponential equation is an equation in which the variable appears in the exponent. To solve these equations, isolate the exponential expression on one side of the equation, then take the logarithm of both sides of the equation to solve for the variable. You can use either natural logarithms or common logarithms. Read the directions carefully, as theymay instruct you as to which to use.

$$3^x + \cancel{2} = 11$$
$$\quad -\cancel{2} \quad -2$$
$$3^x = 9$$

$x = 2$

(since $3^2 = 9$)

$$3^x + \cancel{2} = 13$$
$$\quad \cancel{-2} \quad -2$$
$$3^x = 11$$

option1: use Base in equation

$$3^x = 11$$
$$\cancel{\log_3} 3^x = \log_3 11$$
$$\boxed{x = \log_3 11}$$

$$\log_3 11 = \frac{\ln 11}{\ln 3}$$

(change of Base formula)

option2: use Base e

$$3^x = 11$$
$$\ln 3^x = \ln 11$$
$$\frac{x \cdot \cancel{\ln 3}}{\cancel{\ln 3}} = \frac{\ln 11}{\ln 3}$$
$$\boxed{x = \frac{\ln 11}{\ln 3}}$$

Example 1: Solve for x: $5^{3x} = 9$. (a) Give the exact value using natural logarithms. (b) Use a calculator to approximate the solution to the nearest thousandth. (c) Rewrite the results from part (a) so that each logarithm contains a prime number.

$5^{3x} = 9$

$\ln 5^{3x} = \ln 9$

$\frac{\cancel{3}x \cdot \cancel{\ln 5}}{\cancel{3 \ln 5}} = \frac{\ln 9}{3 \ln 5}$

$x = \frac{\ln 9}{3 \ln 5} = \frac{1}{3} \cdot \frac{\ln 9}{\ln 5}$

$x \approx 0.455$

$x = \frac{1}{3} \cdot \frac{\ln 9}{\ln 5} = \frac{1}{3} \cdot \frac{\ln 3^2}{\ln 5} = \frac{2}{3} \frac{\ln 3}{\ln 5}$

change of Base:

$x = \frac{2}{3} \log_5 3$

Popper 26:

Example 2: Solve for x: $4e^{(x+5)} + 5 = 7$. (a) Give the exact value using natural logarithms. (b) Use a calculator to approximate the solution to the nearest thousandth. (c) Rewrite the results from part (a) so that each logarithm contains a prime number.

$4e^{x+5} + 5 = 7$

1. Simplify the equation:

   a. $e^{(x+5)} = 2$ b. $e^{(x+5)} = ½$ c. $e^{(x+5)} = 3$ d. $e^{(x+5)} = 48$

$4e^{x+5} = 2$

$e^{x+5} = \frac{1}{2}$

2. Solve the equation:

   a. x = ln(½) + 5 b. x = log(½) – 5 c. x = ln(½) – 5 d. x = ln(½)

$\cancel{\ln e}^{x+5} = \ln(1/2)$

$x+5 = \ln(1/2)$

$x = \ln(1/2) - 5$

3. Approximate the answer (Calculator):

   a. x ≈ 4.317 b. x ≈ -5.301 c. x ≈ -5.693 d. x ≈ -0.693

$x = \ln(1) - \ln(2) - 5$

$x = 0 - \ln(2) - 5$

4. Simplify the logarithmic answer (Question 2):

   a. x = ln(-3) b. x = ln(-4.5) c. x = ln(½) – 5 d. x = -ln(2) – 5

$x = -\ln(2) - 5$

$$4e^{x+5} + 5 = 7$$

Example 3: Solve for x: $e^{2x} - 9e^{x} + 20 = 0$. (a) Give the exact value using natural logarithms. (b) Use a calculator to approximate the solution to the nearest thousandth. (c) Rewrite the results from part (a) so that each logarithm contains a prime number.

$$e^{2x} - 9e^{x} + 20 = 0$$

$$u = e^{x}, \quad u^2 = (e^x)^2 = e^{2x}$$

$$u^2 - 9u + 20 = 0$$

$$(u-4)(u-5) = 0$$

$$u - 4 = 0 \qquad u - 5 = 0$$

$$u = 4 \qquad u = 5$$

$$e^x = 4 \qquad e^x = 5$$

$$\cancel{\ln} e^x = \ln 4 \qquad \cancel{\ln e}^x = \ln 5$$

$$x = \ln 4 \qquad x = \ln 5$$

$$x = \ln 4 \qquad \boxed{x = \ln 5}$$

$$x \approx 1.386 \qquad x \approx 1.609$$

$$x = \ln 4$$

$$x = \ln 2^2$$

$$\boxed{x = 2\ln 2}$$

Popper 26 Continued:

Solve the following equation: $3\,(2)^{x-7} + 6 = 12$

5. Simplify the equation:

a. $(2)^{x-7} = 2$ b. $(2)^{x-7} = 6$ c. $(2)^{x-7} = -3$ d. $(2)^{x-7} = 12$

6. Solve in base 2:

a. $\log_2 2 - 7$ b. $\log_2 2 + 13$ c. $\log_2 2$ d. $\log_2 2 + 7$

7. Solve in base e:

a. $\ln 2 + 7$ b. $\ln 2 - 7$ c. $\frac{\ln 2}{\ln 2} + 7$ d. $\frac{\ln 7}{\ln 2} + 2$

8. Simplify either answer:

a. 7/2 b. 2/7 c. 5 d. 8

$$3(2)^{x-7} + 6 = 12$$

$$\frac{\cancel{3}(2)^{x-7}}{\cancel{3}} = \frac{6}{3}$$

$$2^{x-7} = 2^1$$

$$x - \cancel{7} = 1$$
$$\underline{\;+7 \qquad +7\;}$$
$$x = 8$$

solve in base e:

$$\ln 2^{x-7} = \ln 2$$

$$\frac{(x-7)\cancel{\ln 2}}{\cancel{\ln 2}} = \frac{\ln 2}{\ln 2}$$

$$x - 7 = \frac{\ln 2}{\ln 2}$$

$$x = \frac{\ln 2}{\ln 2} + 7$$

simplify:

$$x = 1 + 7 = 8$$

Solve in base 2:

$$\cancel{\log_2} 2^{x-7} = \log_2 2$$

$$x - 7 = \log_2 2$$

$$x = \log_2 2 + 7$$

simplify:

$$x = 1 + 7 = 8$$

Example 4: Solve for x: $25^{3x-2} = \frac{1}{(\sqrt{125})^x}$ (a) Give the exact value using natural logarithms. (b) Use a calculator to approximate the solution to the nearest thousandth. (c) Rewrite the results from part (a) so that each logarithm contains a prime number.

$$25^{3x-2} = \frac{1}{\sqrt{125}^x}$$

$$25^{3x-2} = \left(\frac{1}{\sqrt{125}}\right)^x$$

$$(5^2)^{3x-2} = \left(\frac{1}{\sqrt{5^3}}\right)^x$$

$$= \left(\frac{1}{(5^3)^{1/2}}\right)^x$$

$$= \left(\frac{1}{5^{3/2}}\right)^x$$

$$(5^2)^{3x-2} = (5^{-3/2})^x$$

$$(5^2)^{(3x-2)} = \left(5^{-\frac{3}{2}}\right)^x$$

$$5^{6x-4} = 5^{\frac{-3x}{2}}$$

$$2 \cdot \left(6x-4 = \frac{-3x}{2}\right) \cdot 2$$

$$12x - 8 = -3x$$

$$-12x \qquad -12x$$

$$\frac{-8}{-15} = \frac{-15x}{-15}$$

$$8/15 = x$$

$$\boxed{x = 8/15}$$

$$3^{x+5} = 5^x$$

$$\ln 3^{x+5} = \ln 5^x$$

$$(x+5)\ln 3 = x \ln 5$$

$$\cancel{x \ln 3} + 5\ln 3 = x \ln 5$$

[Think of $3x + 5 = 5x$]

$$-\cancel{x \ln 3} \qquad\qquad -x\ln 3$$

$$5 \ln 3 = x \ln 5 - x \ln 3$$

$$5 \ln 3 = x(\ln 5 - \ln 3)$$

$$5 \ln 3 = x(\ln 5 - \ln 3)$$

$$\frac{5 \ln 3}{\ln 5/3} = \frac{x\left(\cancel{\ln(5/3)}\right)}{\cancel{\ln 5/3}}$$

$$\boxed{\frac{5 \ln 3}{\ln(5/3)} = x}$$

$$5 \cdot \frac{\ln 3}{\ln(5/3)} = x \quad \text{(change of base)}$$

$$\boxed{5 \log_{(5/3)} 3 = x}$$

An equation in which the logarithm of the variable occurs is called a logarithmic equation. To solve such an equation, isolate the logarithmic expression on one side of the equation. Then write the equation in exponential form and solve for the variable, or use the law of logarithms to simplify and solve for the variable.

simplify (however possible)

rewrite as exponent.

possibility 1: $\log_a(\text{stuff}) = \log_a(\text{other stuff})$

[same base]

stuff = other stuff

possibility: $\log_a(\text{stuff}) = \text{Number}$

Rewrite in exponential form

$a^{\text{Number}} = \text{stuff}$

* Check for extraneous roots *

check that the x values do not make original logarithms undefined.

Example 5: Solve for x: $\log_3(x-4) = 2$

$$3^2 = x - 4$$

$$9 = x - 4$$

$$\boxed{13 = x}$$

Check: $x = 13$

$$x - 4$$

$$13 - 4$$

$$9 > 0 \checkmark$$

(inside of original logarithms are positive)

Example 6: Solve for x: $\log_6(x) + \log_6(x+1) = \log_6 2$

$$\log_6(x(x+1)) = \log_6 2$$

$$\log_6(x^2+x) = \log_6 2$$

$$x^2 + x = 2$$

$$x^2 + x - 2 = 0$$

$$(x+2)(x-1) = 0$$

$x + 2 = 0$ $\qquad$ $x - 1 = 0$

$\cancel{x = -2}$ $\qquad$ $\boxed{x = 1}$

reject

Check: $x = -2$

$x \to -2 < 0$ reject

$x+1 \to$ Don't Bother

Check: $x = 1$

$x \to 1 > 0$ ✓

$x+1 \to 1+1 = 2 > 0$ ✓

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Example 7: Solve for x: $\log_6 x + \log_6(x+5) = 2$

$\log_6(x(x+5)) = 2 \rightarrow \log_6(x^2+5x) = 2$

9. Simplify the equation:

a. $\log_6(2x + 5) = 2$ b. $\log_6(7x) = 2$ c. $\log_6(x^2 + 5x) = 2$ d. $\log_6(x^2 + 5) = 2$

$6^2 = x^2+5x$

10. Apply Inverse Functions to each side:

a. $x^2 + 5x = 2$ b. $x^2 + 5x = 36$ c. $x^2 + 5x = e^2$ d. $x^2 + 5x = 100$

$x^2+5x-36=0$

$(x+9)(x-4)=0$

$x^2+5x=36$

11. Solve the Resulting Equation for x:

$x+9=0$ $x-4=0$

$x=-9$ $x=4$

a. {-9, 4} b. {-4, 9} c. {-9, -4} d. No Real Solution

12. Check your answers to eliminate Extraneous Roots. The solution is/are:

a. {-9, 4} b. {-9} c. {4} d. No Real Solution

Check: $x=-9$ reject

$x \rightarrow -9$

$x+5 \rightarrow$

Check $x=4$

$x \rightarrow 4>0$ ✓

$x+5 \rightarrow 4+5=9>0$ ✓

Example 8: Solve for x: $\ln(x^2 + 4) = 2$

$$e^2 = x^2 + 4$$

$$e^2 - 4 = x^2$$

$$\boxed{\pm\sqrt{e^2 - 4} = x}$$

check

① is $\sqrt{e^2 - 4}$ defined?

meaning: is $e^2 - 4 \geq 0$

remember $e \approx 2.7$

$$2 < e < 3$$

$$4 < e^2 < 9$$

$$0 < e^2 - 4 < 5$$ is $e^2 - 4 \geq 0$? Yes

② check $x = +\sqrt{e^2 - 4}$

$x^2 + 4 \to (\sqrt{e^2 - 4})^2 + 4 = e^2 - 4 + 4 = e^2 > 0$ ✓

check $x = -\sqrt{e^2 - 4}$

$x^2 + 4 \to (-\sqrt{e^2 - 4})^2 + 4 = e^2 - 4 + 4 = e^2 > 0$ ✓

Solve the following:

$$\log_2(x+5) - \log_2(x) = 1$$

$$\log_2 \frac{x+5}{x} = 1$$

$$2^1 = \frac{x+5}{x}$$

$$\frac{2}{1} = \frac{x+5}{x}$$

$$2x = \cancel{x} + 5$$
$$-x \quad -\cancel{x}$$

$$\boxed{x = 5}$$

Check $x = 5$

$x+5 \to 5+5 = 10 > 0$ ✓

$x \to 5 > 0$ ✓

Solve the following:

$\log_2 x = \log_4(x + 56)$

(2 different bases) apply change of base to one of them.

$$\frac{\log_4 x}{\log_4 2} = \log_4(x+56)$$

$\log_4 2 = n$

$4^n = 2$

$n = 1/2$

$$\frac{\log_4 x}{1/2} = \log_4(x+56)$$

$2\log_4 x = \log_4(x+56)$

$\log_4 x^2 = \log_4(x+56)$

$x^2 = x+56$

$x^2 - x - 56 = 0$

$(x-8)(x+7) = 0$

$x-8=0$ $\quad$ $x+7=0$

$\boxed{x=8}$ $\quad$ $x=-7$ / reject

check: $\quad$ check:

$x \to 8$ $\quad$ $x \to -7$ reject

$x+56 \to 8+56 = 64$

side note:

left: $\log_2 8 = a$ $\quad$ right: $\log_4(64) = b$

$2^a = 8$ $\quad$ $4^b = 64$

$a = 3$ $\quad$ $b = 3$

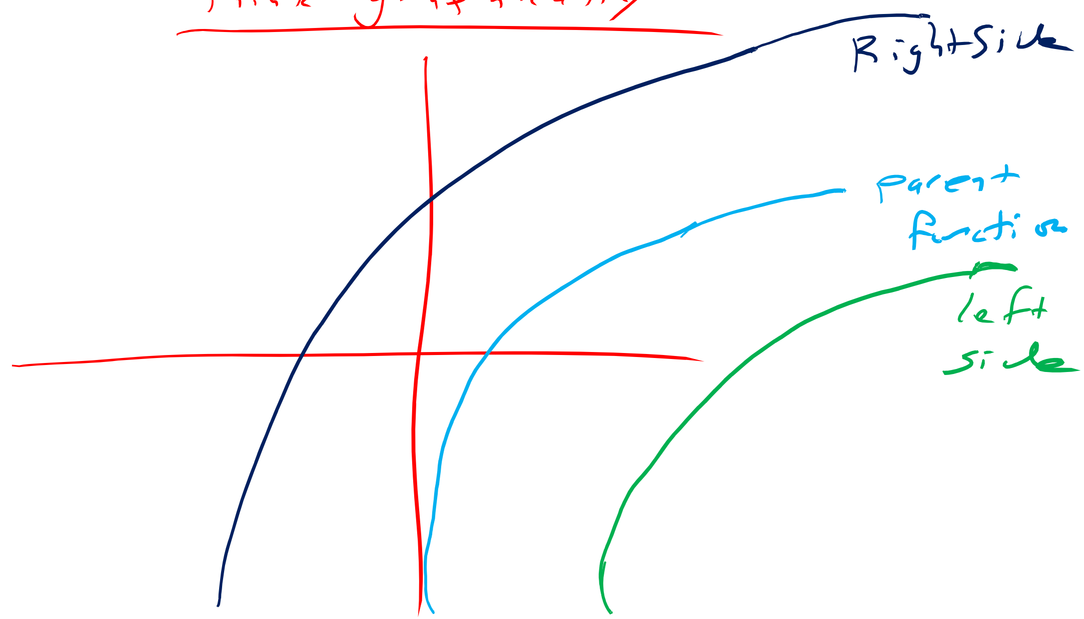
Solve the following:

$$\ln(x - 5) = \ln(x + 5)$$

$$x - 5 = \cancel{x} + 5$$
$$-x \qquad -\cancel{x}$$
$$-5 = 5$$

No Answer

Think graphically:



Graphs will not intersect