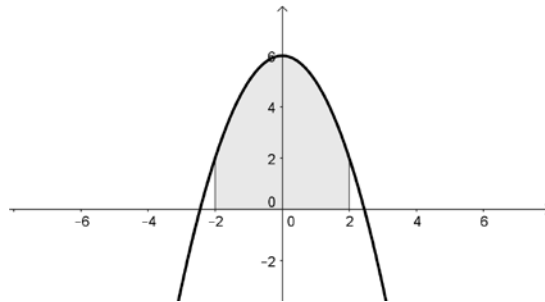


**Math 1314**  
**Lesson 16**  
**Area and Riemann Sums**  
**and**  
**Lesson 17**  
**Riemann Sums Using GeoGebra; Definite Integrals**

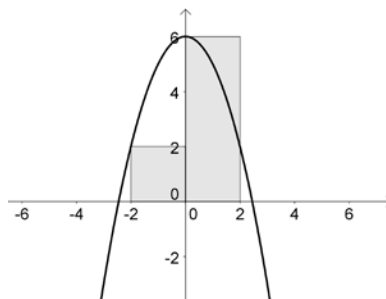
The second question studied in calculus is the area question. If a region conforms to a known formula from geometry, then finding the area is not difficult; simply determine the dimensions and apply the appropriate formula.

Suppose we want to find the area of a region that is not so nicely shaped. For example, consider the function shown below. The area below the curve and above the  $x$  axis cannot be determined by a known formula, so we'll need a method for approximating the area.



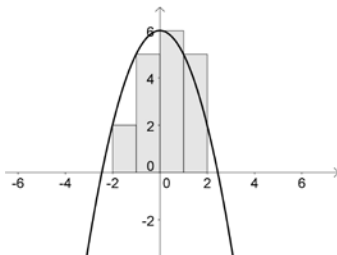
Suppose we want to find the area under the parabola and above the  $x$  axis, between the lines  $x = 2$  and  $x = -2$ .

We can approximate the area under the curve by subdividing the interval  $[-2, 2]$  into smaller intervals and then draw rectangles extending from the  $x$  axis up to the curve. Suppose we divide the region into two parts and draw two rectangles. We can find the area of each rectangle and add them together. That will give us an approximation of the area under the curve. This method is called “finding a Riemann sum.”

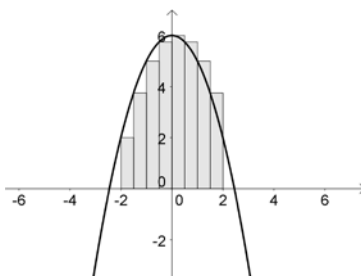


This would not give a very good approximation, as a large region in Quadrant 2 will be left out in the approximation of the area, and a large region in Quadrant 1 will be included and should not be.

Now suppose we increase the number of rectangles that we draw to four. We'll find the area of each of the four rectangles and add them up. Here's the graph for this situation.



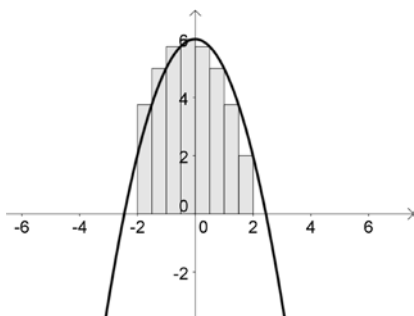
The approximation will be more accurate, but it still isn't perfect. Let's increase the number of rectangles to 8:



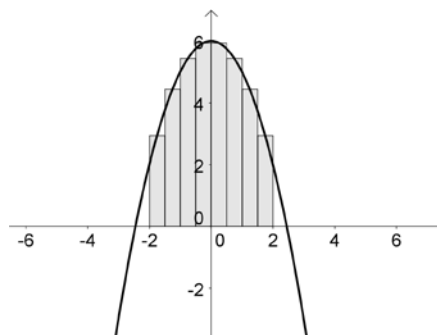
As we add more and more rectangles, the accuracy improves. We're still not to an exact area, but the area we'd find using more rectangles is clearly more accurate than the area we'd find if we just used 2 rectangles.

Suppose we let the number of rectangles increase without bound. If we do this, the width of each rectangle becomes smaller and smaller, as the number of rectangles approaches infinity, there will be no area that is included that shouldn't be and none left out that should be included. This process is beyond the scope of this course, so we will limit the number of rectangles in the problems we work to a finite number.

Using left endpoints is not the only option we have in working these problems. We can also use right endpoints or midpoints. The first graph below shows the region with eight rectangles, using right endpoints. The second graph below shows the region with eight rectangles, using midpoints.



Right Endpoints



Midpoints

Now, how do we approximate the area?

1. **Start by finding the width of each rectangle.** We can compute the width of the rectangles using this formula:  $\Delta x = \frac{b-a}{n}$ . In this formula,  $a$  and  $b$  are the endpoints of the interval  $[a, b]$  and  $n$  is the number of rectangles.  $\Delta x$  stands for “the change in  $x$ .”

2. **Now find the height of the rectangles.** Subdivide the interval into  $n$  subintervals, each of width  $\Delta x$ . Use the appropriate point in each subinterval to compute the value of the function at each of these points (gives the heights of the rectangles).

3. **Find the area of each rectangle and add them up.**  $A \approx [f(x_1) + f(x_2) + \dots + f(x_n)] \Delta x$

To get an exact area, we would need to let the number of rectangles increase without bound:

$$A = \lim_{n \rightarrow \infty} [f(x_1) + f(x_2) + \dots + f(x_n)] \cdot \Delta x$$

This last computation is quite difficult, we will not work problem of this type. Instead, we will use a limited number of rectangles in the problems that we work.

The process we are using to approximate the area under the curve is called “finding a Riemann sum.” These sums are named after the German mathematician who developed them.

Example 1: Let  $f(x) = 2x^2 + 1$ . Approximate the area under the curve, using 4 subdivisions, on the interval  $[0, 2]$  using left endpoints.

Now suppose the function, interval and/or subdivisions we wish to work with are not so nice. You would not want to work this type of problem by hand. We can work Riemann sum problems using GeoGebra. The command is:

RectangleSum[<Function>,<Start x-Value>,<End x-Value>,<Number of rectangles>,<Position for rectangle start>]

“Position of rectangle start”: 0 corresponds to left endpoints, 0.5 corresponds to midpoints and 1 corresponds to right endpoints.

Example 2: Approximate the area between the x axis and the graph of  $f(x) = 0.3x^3 - 0.807x^2 - 3.252x + 6.717$  on  $[-2.82, 1.33]$  with 50 rectangles and midpoints. Enter the function in GGB.

Command:

Answer:

Example 3: Approximate the area between the x axis and the function

$f(x) = \frac{15}{0.083x^2 + 19.17x + 1}$  on  $[0.075, 8.21]$  using 12 rectangles and left endpoints.

Enter the function in GGB.

Command:

Answer:

## Upper and Lower Sums Using GeoGebra

You can also find a related quantity using GeoGebra, the upper sum and/or the lower sum. Rather than always using the left endpoint, the right endpoint or the midpoint of the interval to find the height of the rectangle, the upper sum uses the biggest  $y$  value on each interval as the height of the rectangle and the lower sum uses the smallest  $y$  value on each interval as the height of the rectangle, no matter where on the interval that value occurs.

The command for Upper Sum is:

UpperSum[<Function>,<Start x-Value>,<End x-Value>,<Number of Rectangles>]

The command for Lower Sum is:

LowerSum[<Function>,<Start x-Value>,<End x-Value>,<Number of Rectangles>]

Example 4: Use GeoGebra to find the upper sum and the lower sum for  $f(x) = \frac{1}{2}e^x + 2$  on the interval  $[-3, 5]$  using 35 rectangles. *Enter the function in GGB.*

Command:

Upper Sum Answer:

Command:

Lower Sum Answer:

## The Definite Integral

Let  $f$  be defined on  $[a, b]$ . If  $\lim_{n \rightarrow \infty} ([f(x_1) + f(x_2) + \dots + f(x_n)]\Delta x)$  exists for all choices of representative points in the  $n$  subintervals of  $[a, b]$  of equal width  $\Delta x = \frac{b-a}{n}$ , then this limit is called the definite integral of  $f$  from  $a$  to  $b$ . The definite integral is noted by  $\int_a^b f(x)dx = \lim_{n \rightarrow \infty} ([f(x_1) + f(x_2) + \dots + f(x_n)]\Delta x)$ . The number  $a$  is called the lower limit of integration and the number  $b$  is called the upper limit of integration.