Math 1314
Lesson 23
Partial Derivatives

When we are asked to find the derivative of a function of a single variable, $f(x)$, we know exactly what to do. However, when we have a function of two variables, there is some ambiguity. With a function of two variables, we can find the slope of the tangent line at a point P from an infinite number of directions. We will only consider two directions, either parallel to the x axis or parallel to the y axis. When we do this, we fix one of the variables. Then we can find the derivative with respect to the other variable.

So, if we fix y, we can find the derivative of the function with respect to the variable x. And if we fix x, we can find the derivative of the function with respect to the variable y.

These derivatives are called **partial derivatives**.

First-Order Partial Derivatives

We will use two different notations:

$$\frac{\partial f}{\partial x} = f_x \quad \text{In this case, you consider } y \text{ as a constant.}$$

$$\frac{\partial f}{\partial y} = f_y \quad \text{In this case, you consider } x \text{ as a constant.}$$

We can use GGB to determine the first-order partial derivatives. The command is:
derivative[<function>,<variable>]

Example 1: Suppose $f(x, y) = x^2 - 3xy^2 + 4y^2$. *Enter the function into GGB.*

a. Find $\dfrac{\partial f}{\partial x} = f_x$.

Command: Answer:

b. Find $\dfrac{\partial f}{\partial y} = f_y$.

Command: Answer:

We can also evaluate the first partial derivatives at a given point.

Example 2: Given $f(x, y) = 4x^3 y^2 + 2x^2 y^3 - 12x^2 + 3y^2 + 10$, find each of the following:
Enter the function into GGB.

a. $f_x\big|_{(-1,3)}$
Commands: Answers:

b. $f_y\big|_{(-1,3)}$
Commands: Answers:

Second-Order Partial Derivatives

Sometimes we will need to find the second-order partial derivatives. To find a second-order partial derivative, you will take respective partial derivatives of the first partial derivative. There are a total of 4 second-order partial derivatives.

There are two notations, but we will only use one of them.

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \qquad\qquad f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \qquad\qquad f_{yx} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

Example 3: Let $f(x, y) = 3x^2 - x^3 y^3 + 5xy + 6y^3$. Evaluate $f_{xy}\big|_{(1,2)}$ and $f_{yy}\big|_{(1,2)}$.

Enter the function into GGB.

Commands: Answers:

derivative[   ]

derivative[   ]

derivative[   ]

derivative[   ]

A function of the form $f(x, y) = ax^b y^{1-b}$ where a and b are positive constants and $0 < b < 1$ is called a ***Cobb-Douglas production function***. In this function, x represents the amount of money spent for labor, and y represents the amount of money spent on capital expenditures such as factories, equipment, machinery, tools, etc. The function measures the output of finished products.

The first partial with respect to x is called the **marginal productivity of labor**. It measures the change in productivity with respect to the amount of money spent for labor. In finding the first partial with respect to x, the amount of money spent on capital is held at a constant level.

The first partial with respect to y is called the **marginal productivity of capital**. It measures the change in productivity with respect to the amount of money spent on capital expenditures. In finding the first partial with respect to y, the amount of money spent on labor is held at a constant level.

Example 4: A country's production can be modeled by the function $f(x, y) = 50x^{2/3} y^{1/3}$ where x gives the units of labor that are used and y represents the units of capital that were used. Currently the country uses 125 units of labor and 27 units of capital.

a. Use marginal analysis to estimate the change that will result in the addition of 1 more unit of labor if the number of units of capital is unchanged.
Command: Answer:

b. Use marginal analysis to estimate the change that will result in the addition of 1 more unit of capital if the number of units of labor remains the same.
Command: Answer: