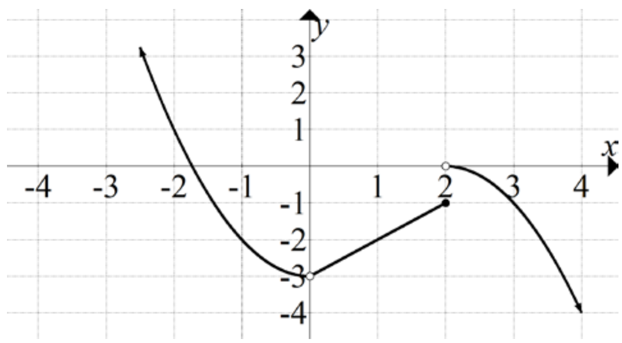
Math 1314
Lesson 5
One-Sided Limits and Continuity

Sometimes we are only interested in the behavior of a function when we look from one side and not from the other. In this case, we are looking at a **one-sided limit**.

We write $\lim_{x \to a^+} f(x)$ for a **right-hand limit**. We write $\lim_{x \to a^-} f(x)$ for a **left-hand limit**.

Theorem: Let *f* be a function that is defined for all values of *x* close to the target number *a*, except perhaps at *a* itself. Then $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$.

Example 1: Given the graph of f below:



Find each of the following limits, if it exist.

a. $\lim_{x \to 0^-} f(x)$ b. $\lim_{x \to 0^+} f(x)$ c. $\lim_{x \to 0} f(x)$

Example 2: Suppose $f(x) = \begin{cases} x^2 - x + 2, & x < 1 \\ x + 1, & 1 \le x < 2 \\ -x^3 - 5, & x \ge 2 \end{cases}$. Find each of the following limits, if it exist.

a. $\lim_{x \to 2^-} f(x)$ b. $\lim_{x \to 2^+} f(x)$ c. $\lim_{x \to 2} f(x)$

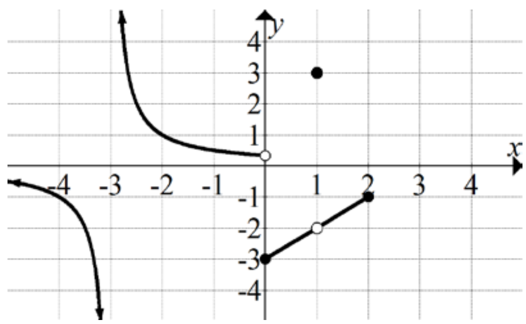
Continuity at a Point

A function is a **continuous** at a point if its graph has no gaps, holes, breaks or jumps at that point. Stated a bit more formally, a function *f* is said to be continuous at the point $x = a$ if the following three conditions are met:

1. $f(a)$ is defined
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

If a function is not continuous at $x = a$, then we say it is discontinuous there.

Example 3: The graph of a function given below is discontinuous at some values of *x*. State the *x*-values of where the function is discontinuous then state why the function is discontinuous at each one of those points.



a. Discontinuous at:
- Is $f(\quad)$ defined?
- Does $\lim_{x \to} f(x)$ exists?
- Does $\lim_{x \to} f(x) = f(\quad)$?

b. Discontinuous at:
- Is $f(\quad)$ defined?
- Does $\lim_{x \to} f(x)$ exists?
- Does $\lim_{x \to} f(x) = f(\quad)$?

c. Discontinuous at:
- Is $f(\quad)$ defined?
- Does $\lim_{x \to} f(x)$ exists?
- Does $\lim_{x \to} f(x) = f(\quad)$?

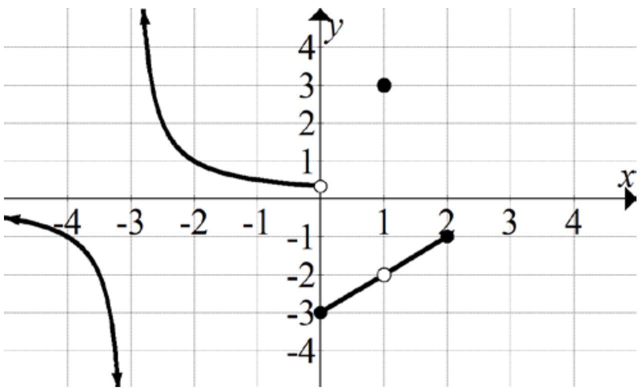
Discontinuities

A function can have a removable discontinuity, a jump discontinuity or an infinite discontinuity.

Let $f(x)$ be discontinuous at $x = a$. Then:

If	Type of Discontinuity
$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$	Jump
$\lim_{x \to a} f(x) \neq f(a)$	Removable
$\lim_{x \to a^-} f(x) \to \pm\infty$ or $\lim_{x \to a^+} f(x) \to \pm\infty$	Infinite

Example 4: Let's revisit the graph from Example 3.



State the type of discontinuity at:

a. x = -3

b. x = 0

c. x = 1

Example 5: Let $f(x)=\begin{cases} x-6, & x\le 0 \\ x^2+5x+6, & x>0 \end{cases}$ is the function continuous at x = 0?

We need to check:
1. Is $f(0)$ defined?

2. Does $\lim\limits_{x\to 0} f(x)$ exist? Must check $\lim\limits_{x\to 0^-} f(x)$ and $\lim\limits_{x\to 0^+} f(x)$.

3. $\lim\limits_{x\to 0} f(x) = f(0)$? *i.e. Compare #1 and #2 above.*

Example 6: Let $f(x)=\begin{cases} x+1, & x\le 2 \\ 4-x, & x>2 \end{cases}$ is the function continuous at x = 2? If it is discontinuous, identify the type of discontinuity.
We need to check:
1. Is $f(2)$ defined?

2. Check to see if $\lim\limits_{x\to 2} f(x)$ exist.
Must check: $\lim\limits_{x\to 2^-} f(x)$ and $\lim\limits_{x\to 2^+} f(x)$

Does $\lim\limits_{x\to 2} f(x)$ exist?

3. $\lim\limits_{x\to 2} f(x) = f(2)$? *i.e. Compare #1 and #2 above.*

Example 7: Is $f(x)=\begin{cases} 2x^2+9, & x<3 \\ 2, & x=3 \\ x^3, & 3<x \end{cases}$ continuous at $x = 3$? If it is discontinuous, identify the type of discontinuity.

We need to check:
1. Is $f(3)$ defined?

2. Check to see if $\lim_{x\to 3} f(x)$ exist.
Must check: $\lim_{x\to 3^-} f(x)$ and $\lim_{x\to 3^+} f(x)$

Does $\lim_{x\to 3} f(x)$ exist?

3. Does $\lim_{x\to 3} f(x) = f(3)$? *i.e. Compare #1 and #2 above.*

Example 8: Let $f(x)=\begin{cases} \dfrac{x^2-25}{5+x}, & x\neq -5 \\ -10, & x=-5 \end{cases}$ is the function continuous at $x=-5$?

We need to check:
1. Is $f(-5)$ defined?

2. Does $\lim_{x\to -5} f(x)$ exist? Must check $\lim_{x\to -5^-} f(x)$ and $\lim_{x\to -5^+} f(x)$.

3. $\lim_{x\to -5} f(x) = f(-5)$? *i.e. Compare #1 and #2 above.*