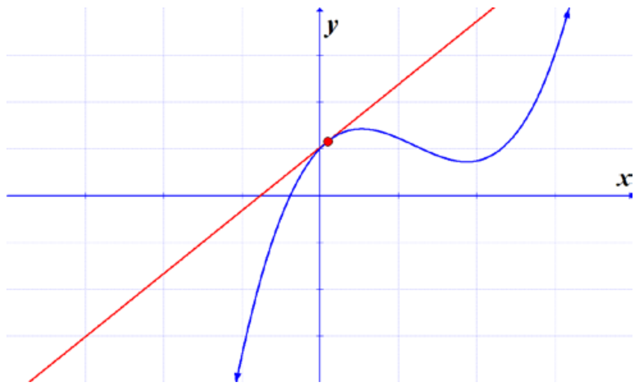
Math 1314
Lesson 6
The Limit Definition of the Derivative; Rules for Finding Derivatives

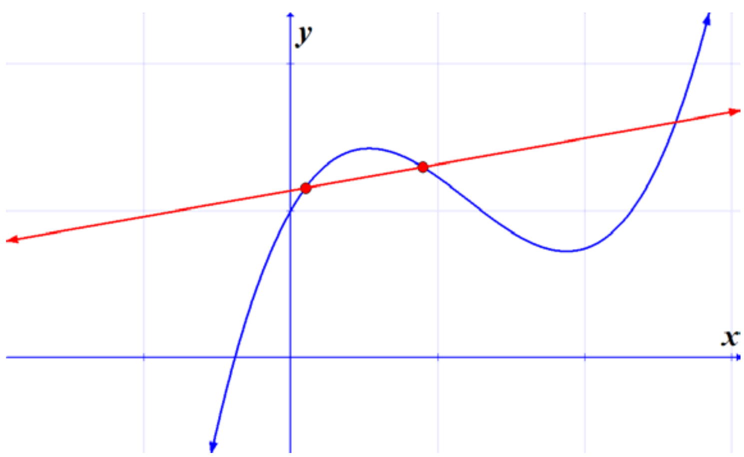
We now address the first of the two questions of calculus, the tangent line question.

We are interested in finding the slope of the tangent line at a specific point.



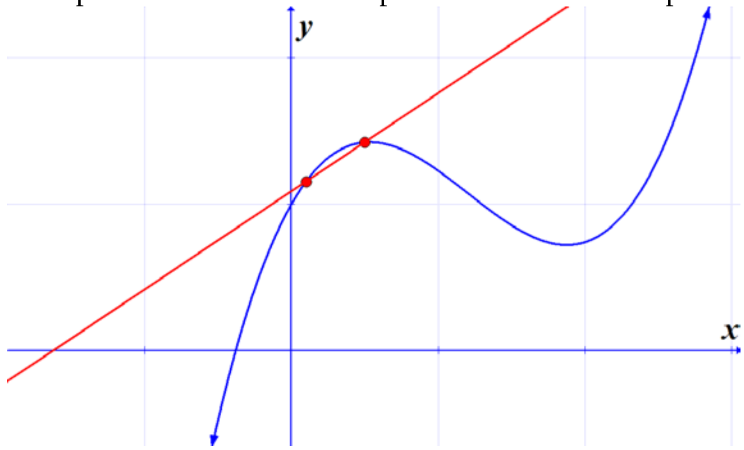
We need a way to find the slope of the tangent line analytically for every problem that will be exact every time.

We can draw a secant line across the curve, then take the coordinates of the two points on the curve, *P* and *Q*, and use the slope formula to approximate the slope of the tangent line.



Now suppose we move point *Q* closer to point *P*. When we do this, we'll get a better approximation of the slope of the tangent line. When we continue to move point *Q* even closer to point *P*, we get an even better approximation. We are letting the distance between *P* and *Q* get smaller and smaller.

Now let's give these two points names. We'll express them as ordered pairs.



Now we'll apply the slope formula to these two points.

$$m = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$

This expression is called a **difference quotient** also called the **average rate of change.**

The last thing that we want to do is to let the distance between *P* and *Q* get arbitrarily small, so we'll take a limit. This gives us the definition of the **slope of the tangent line**.

The slope of the tangent line to the graph of f at the point $P(x, f(x))$ is given by

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists.

We find the **instantaneous rate of change** when we take the limit of the difference quotient.

The **derivative of *f* with respect to *x*** is the function f' (read "*f* prime") defined by

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. The domain of $f'(x)$ is the set of all *x* for which the limit exists.

We can use the derivative of a function to solve many types of problems.   But first we need a method for finding the derivative.

Now we know that the derivative is a formula for finding the slope of a tangent line to any point on the graph of the original function.

Rules for Finding Derivatives

First, a bit of notation: $\frac{d}{dx}[f(x)]$ is a notation that means "the derivative of *f* with respect to *x*, evaluated at *x*."

Rule 1: The Derivative of a Constant

$\frac{d}{dx}[c] = 0,$ where *c* is a constant.

Example 1: Find the derivative of each function.

a. $f(x) = -17$

b. $g(x) = \sqrt{11}$

Rule 2: The Power Rule

$\frac{d}{dx}[x^n] = nx^{n-1}$ for any real number *n*

Example 2: Find the derivative of each function.

a. $f(x) = x^5$

b. $g(x) = x^{-10}$

c. $h(x) = \frac{1}{x^3}$

d. $j(x) = \sqrt{x}$

Rule 3: Derivative of a Constant Multiple of a Function

$\frac{d}{dx}\left[cf(x)\right] = c\frac{d}{dx}\left[f(x)\right]$ where c is any real number

Example 3: Find the derivative of each function.

a. $f(x) = -6x^4$

b. $g(x) = \frac{1}{4}x^{-4}$

c. $h(x) = 5x$

Rule 4: The Sum/Difference Rule

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}\left[f(x)\right] \pm \frac{d}{dx}\left[g(x)\right]$$

Example 4: Find the derivative: $f(x) = 10x^4 + 3x^2 - 6x + 5.$

Example 5: Find the derivative: $f(x) = -2x^7 - 6\sqrt[6]{x} + \frac{1}{2x^4} - 1.$

Sometimes we need to find the derivative of the derivative. Since the derivative is a function, this is something we can readily do. The derivative of the derivative is called the second derivative, and is denoted $f''(x)$. To find the second derivative, we will apply whatever rule is appropriate given the first derivative.

Example 6: Find the second derivative: $f(x) = 4x^5 - x^2 - 7x + 5$

Note, there are many other rules for finding derivatives "by hand." We will not be using those in this course. Instead, we will use GeoGebra for finding more complicated derivatives.