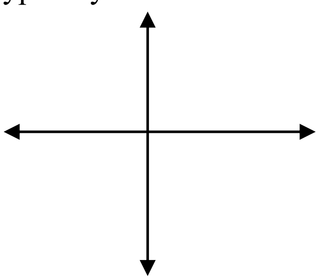
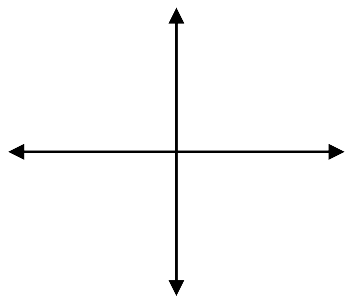
Section 4.3
Trigonometric Functions of Angles

Unit Circle Trigonometry

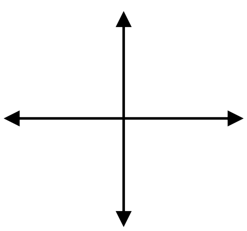
An angle is in standard position if its vertex is at the origin and its initial side is along the positive x axis. Positive angles are measured counterclockwise from the initial side. Negative angles are measured clockwise. We will typically use the θ to denote an angle.



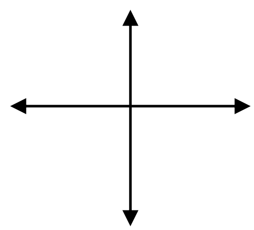


<u>Example 1:</u> Draw each angle in standard position.

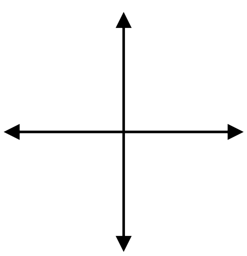
a. $\mathbf{240°}$



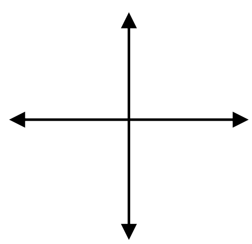
b. $\mathbf{-150°}$



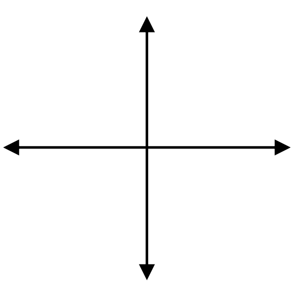
c. $\dfrac{7\pi}{3}$

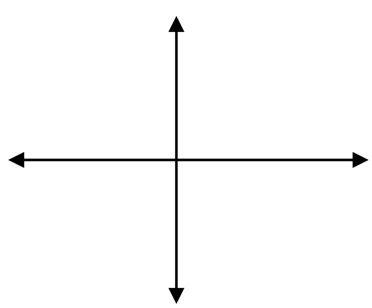


d. $\dfrac{-5\pi}{4}$



Angles that have the same terminal side are called co-terminal angles. Measures of co-terminal angles differ by a multiple of $\mathbf{360°}$ if measured in degrees or by multiple of $\mathbf{2\pi}$ if measured in radians.





Angles in standard position with the same terminal side are called coterminal angles. The measures of two coterminal angles differ by a factor corresponding to an integer number of complete revolutions.
The degree measure of coterminal angles differ by an integer multiple of 360°. For any angle θ measured in degrees, an angle coterminal with θ can be found by the formula $\theta + n * 360^\circ$.

Example 2: Find three angles, two positive and one negative that are co-terminal with each angle.

A. $\theta = 40°$

B. $\theta = -\dfrac{\pi}{6}$

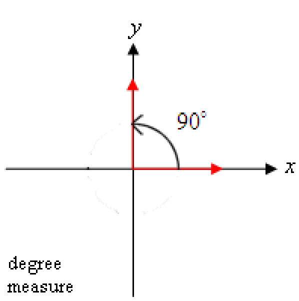
An angle is formed by two rays that have a common endpoint (vertex). One ray is called the **initial side** and the other the **terminal side**. A terminal angle can lie in any quadrant, on the *x*-axis or *y*-axis.

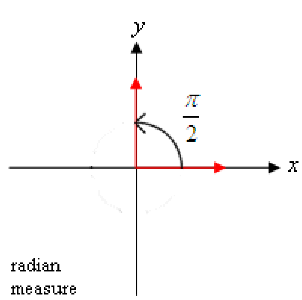
An angle is in **standard position** if the vertex is at the origin of the two-dimensional plane and its initial side lies along the positive *x*-axis.

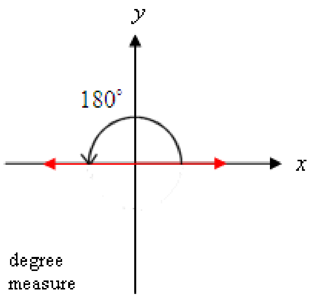
Positive angles are generated by counterclockwise rotation. **Negative angles** are generated by clockwise rotation.

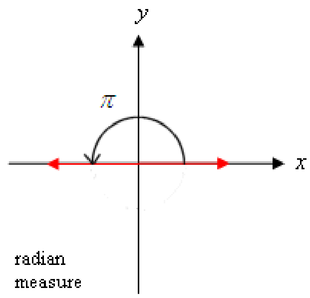
An angle in standard position whose terminal side lies on either the x-axis or the y-axis is called a quadrantal angle.

The figures below show examples of quadrantal angles given in both degree meaure and radian measure.









The Reference Angle or Reference Number

Let θ be an angle in standard position. The **reference angle** associated with θ is the acute angle (with positive measure) formed by the x-axis and the terminal side of the angle θ. When radian measure is used, the reference angle is sometimes referred to as the reference number (because a radian angle measure is a real number).

Example 3: Draw each angle in standard position and specify the reference angle.

a. 405^{o}

Note: Notice that in part a. 405^{0} terminates in the same position as 45^{0}. These type of angles are called **coterminal angles**. Every angle has infinitely many coterminal angles. An angle of x^{o} is coterminal with angles $x^{o} + k\,360^{o} = x + 2\pi k$, where k is an integer.

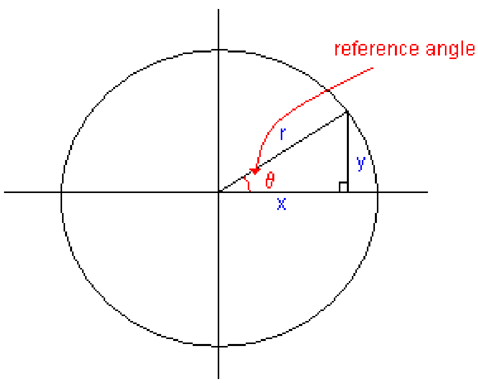
b. $\theta = 330^{o}$

c. $\theta = -\dfrac{2\pi}{3}$

d. $\theta = \dfrac{5\pi}{4}$

We previously defined the six trigonometry functions of an angle as ratios of the length of the sides of a right triangle. Now we will look at them using a circle centered at the origin in the coordinate plane. This circle will have the equation $x^2 + y^2 = r^2$. If we select a point P(x, y) on the circle and draw a ray from the origin though the point, we have created an angle in standard position.

Trigonometric Functions of Angles



If the circle is the unit circle, then r = 1 and we get the following:

$\cos\theta = x \qquad \sec\theta = \dfrac{1}{x} \quad (x \neq 0)$

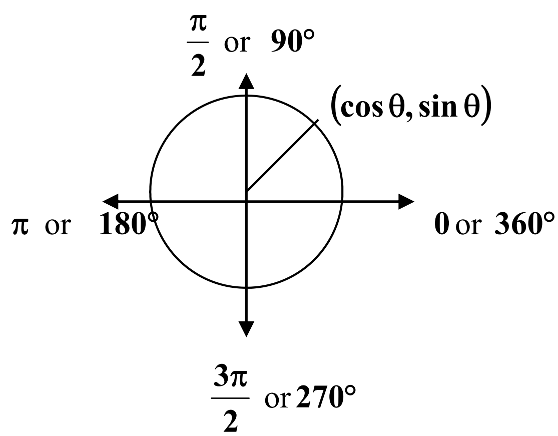
$\sin\theta = y \qquad \csc\theta = \dfrac{1}{y} \quad (y \neq 0)$

$\tan\theta = \dfrac{y}{x} \quad (x \neq 0) \qquad \cot\theta = \dfrac{x}{y} \quad (y \neq 0)$

Example 4: For each quadrantal angle, give the coordinates of the point where the terminal side of the angle interests the unit circle.

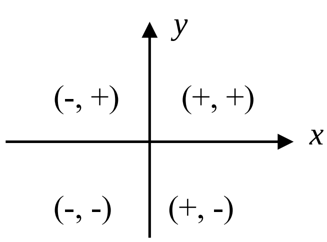
a. -180^o, then find sine and cotangent, if possible, of the angle.

b. $\frac{3\pi}{2}$, then find tangent and cosecant, if possible, of the angle.

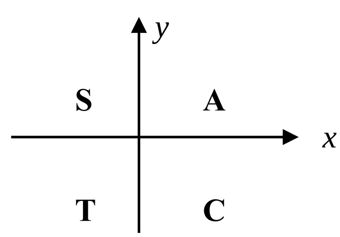


Recall: $(x, y) = (\cos\theta,\ \sin\theta)$

All **S**tudents **T**ake **C**alculus



OR



This should help you to know which trigonometric functions are positive in which quadrant.

Example 5: Name the quadrant in which the given conditions are satisfied.

a. $\sin(\theta) < 0, \cos(\theta) > 0$

b. $\tan(\theta) > 0, \sin(\theta) < 0$

c. $\cos(\theta) < 0, \csc(\theta) > 0$

d. $\sec(\theta) > 0, \cot(\theta) > 0$

Example 6: Rewrite each expression in terms of its reference angle, deciding on the appropriate sign (positive or negative).

a. $\tan\left(\frac{7\pi}{4}\right)$

b. $\sin(140^o)$

c. $\sec\left(-\frac{2\pi}{3}\right)$

Find the trig functions for the angle $\frac{\pi}{6}$ or $\mathbf{30°}$ and $\frac{\pi}{3}$ or $\mathbf{60°}$.

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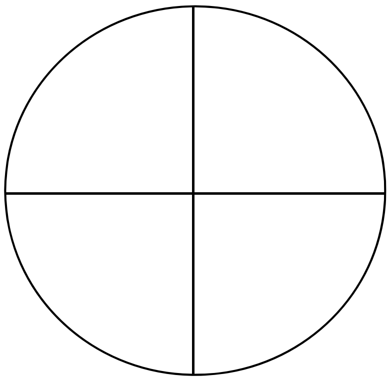
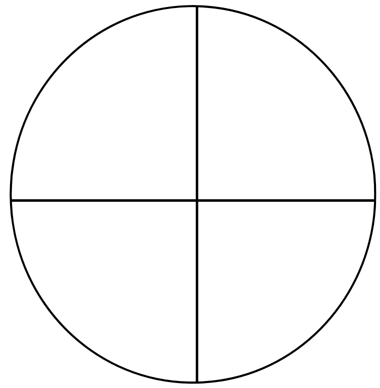




$\sin\frac{\pi}{6} = \frac{1}{2}$

$\csc\frac{\pi}{6} = 2$

$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$\sec\frac{\pi}{6} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\cot\frac{\pi}{6} = \sqrt{3}$

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\csc\frac{\pi}{3} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

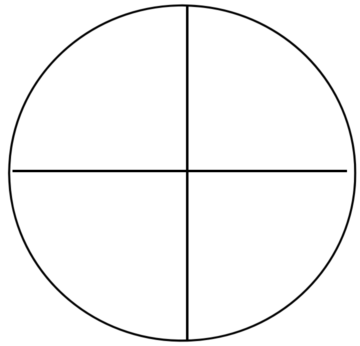
$\cos\frac{\pi}{3} = \frac{1}{2}$

$\sec\frac{\pi}{3} = 2$

$\tan\frac{\pi}{3} = \sqrt{3}$

$\cot\frac{\pi}{3} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

Find the trig functions for the angle $\frac{\pi}{4}$ or $\mathbf{45°}$.



$$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} \qquad \csc\frac{\pi}{4} = \frac{2}{\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

$$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \qquad \sec\frac{\pi}{4} = \frac{2}{\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

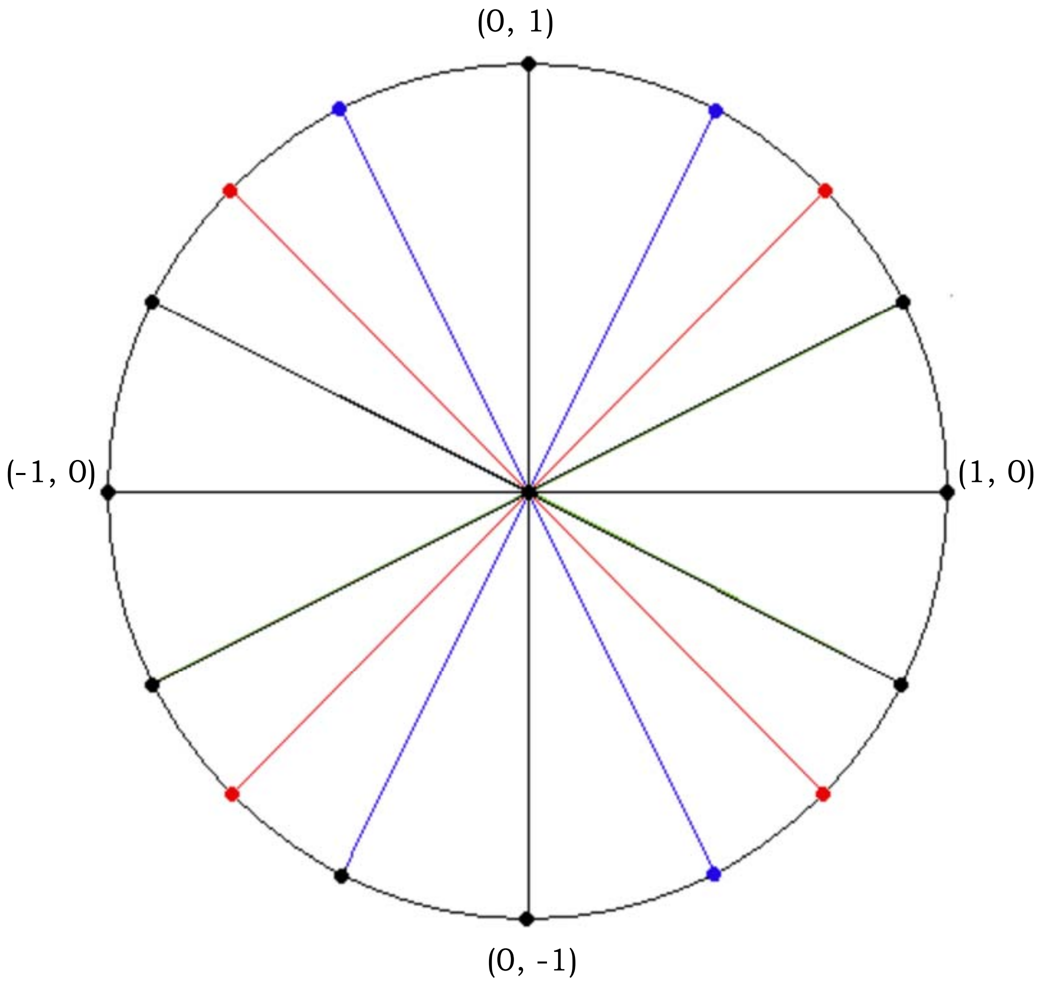
$$\tan\frac{\pi}{4} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1 \qquad \cot\frac{\pi}{4} = 1$$

You will need to find the trigonometric functions of quadrant angles and of angles measuring **30°,45°or60°** without using a calculator.
Here is a simple way to get the first quadrant of trigonometric functions. Under each angle measure, write down the numbers 0 to 4. Next take the square root of the values and simplify if possible. Divide each value by 2. This gives you the sine values of each of the angles you need. To fine the cosine values, write the previous line in reverse order. Now find the tangent values by using the sine and cosine values.

Angle in degrees	0	30	45	60	90
Angle in Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
Sine					
Cosine					
Tangent					

Recall: $\csc\theta = \frac{1}{\sin\theta}$ $\qquad \sec\theta = \frac{1}{\cos\theta}$ $\qquad \cot\theta = \frac{1}{\tan\theta}$

Unit circle



Example 7: Let the point $P(x, y)$ denote the point where the terminal side of angle θ (in standard position) meets the unit circle. P is in Quadrant III and $y = -5/13$. Evaluate the six trig functions of θ.

Example 8: Suppose that $\csc\theta = -\frac{9}{4}$ and that $270^\circ < \theta < 360^\circ$. Find $\cot\theta$ and $\sin\theta$.

Evaluating Trigonometric Functions Using Reference Angles

1. Determine the reference angle associated with the given angle.
2. Evaluate the given trigonometric function of the reference angle.
3. Affix the appropriate sign determined by the quadrant of the terminal side of the angle in standard position.

Example 9: Evaluate the following.

a. $\sin(300^o)$

b. $\cos(-240^o)$

c. $\sec(135^o)$

d. $\cos\left(\frac{3\pi}{4}\right)$

e. $\sin\left(\frac{4\pi}{3}\right)$

f. $\tan\left(-\frac{\pi}{2}\right)$

g. csc(2π)

h. cot(-5π)

i. $\sin\left(-\frac{11\pi}{2}\right)$