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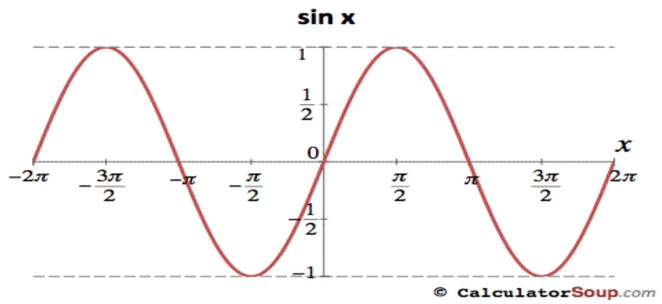
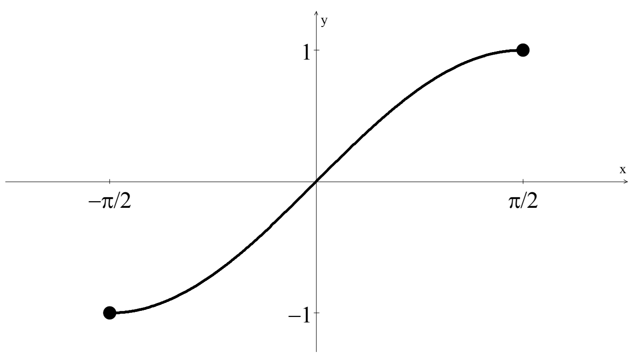
Section 5.4a
Inverse Trigonometric Functions

The function sin(x) is graphed below. Notice that this graph does not pass the horizontal line test; therefore, it does not have an inverse.



Restricted Sine Function and It's Inverse

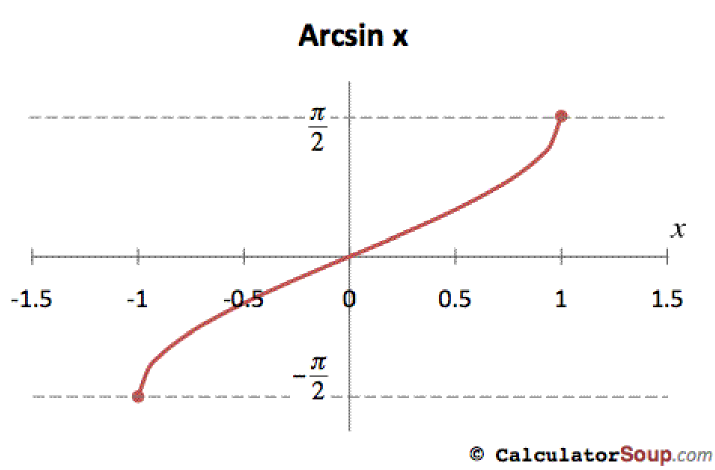
However, if we restrict it from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$ then we have created the **"Restricted" sine function** and it's one-to-one.



Domain: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Range: [-1, 1]

Since the restricted sine function is one-to-one, it has an inverse $f(x) = \sin^{-1}(x) = \arcsin(x)$.

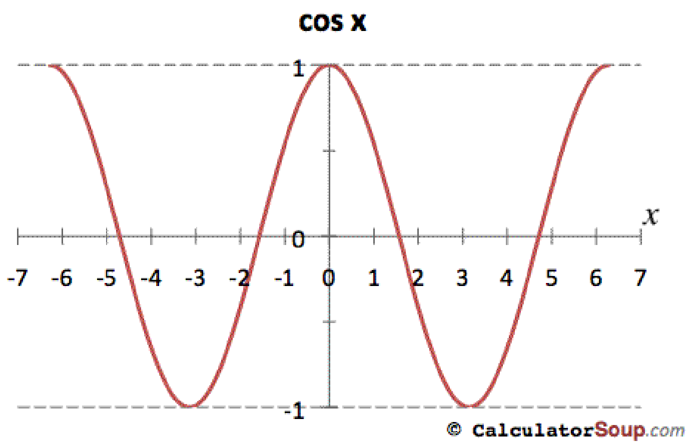


Domain: [-1, 1]

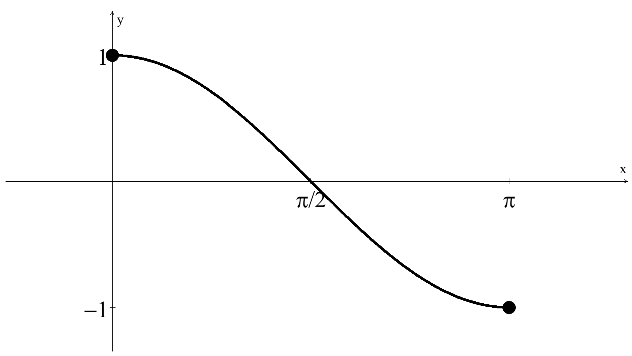
Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Restricted Cosine Function and It's Inverse

The function cos(x) is graphed below. Notice that this graph does not pass the horizontal line test; therefore, it does not have an inverse.



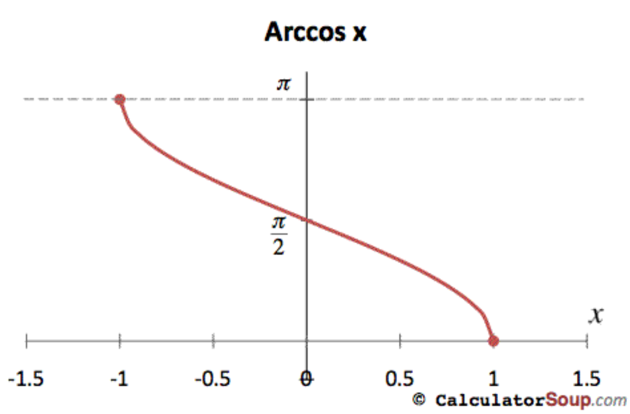
However, if we restrict it from $x = 0$ to $x = \pi$ then we have created the **"Restricted" cosine function** and it's one-to-one.



Domain: $[0, \pi]$

Range: [-1, 1]

Since the restricted cosine function is one-to-one, it has an inverse $f(x) = \cos^{-1}(x) = \arccos(x)$.

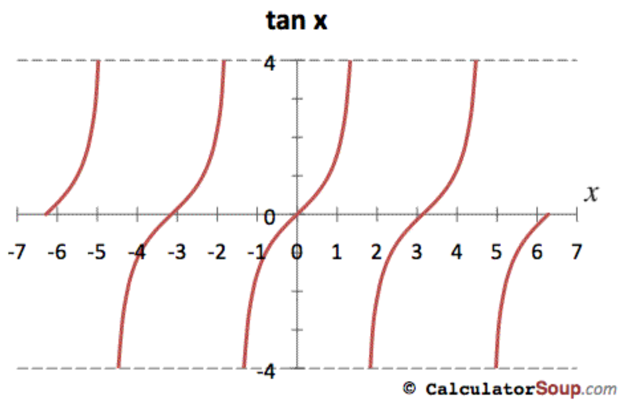


Domain: [-1, 1]

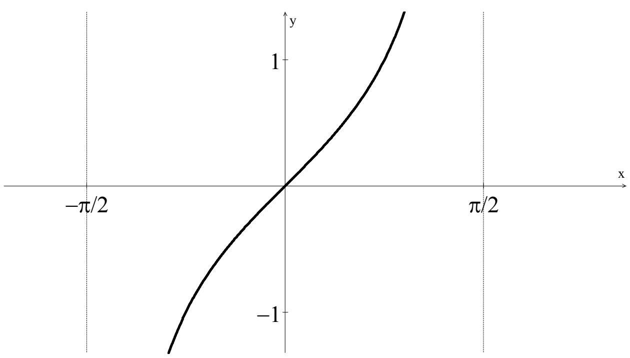
Range: $[0, \pi]$

Restricted Tangent Function and It's Inverse

The function tan(x) is graphed below. Notice that this graph does not pass the horizontal line test; therefore, it does not have an inverse.



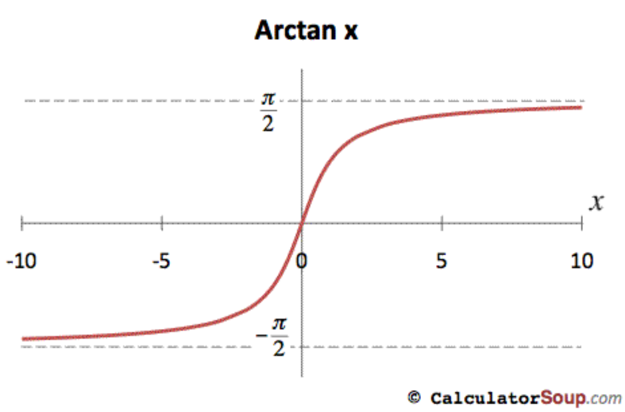
However, if we restrict it from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$ then we have created the **"Restricted" tangent function** and it's one-to-one.



Domain: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Range: $(-\infty, \infty)$

Since the restricted tangent function is one-to-one, it has an inverse $f(x) = \tan^{-1}(x) = \arctan(x)$.



Domain: $(-\infty, \infty)$

Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

The other inverse trigonometric functions, $y = \cot^{-1}(x)$, $y = \csc^{-1}(x)$, and $y = \sec^{-1}(x)$, can be defined in a manner similar to the inverse trigonometric functions shown above, that is, by restricting the domains of the cotangent, cosecant, and secant functions, respectively, so that the resulting functions are one-to-one.

We now want to evaluate inverse trig functions. With these problems, instead of giving you the angle and asking you for the value, you'll be given the value and ask be asked **what angle gives you that value**.

When we covered the unit circle, we saw that there were two angles that had the same value for most of our angles. With inverse trig, we can't have that. We need a unique answer, because of our need for 1-to-1 functions. We'll have one quadrant in which the values are positive and one value where the values are negative. The restricted graphs we looked at can help us know where these values lie. We'll only state the values that lie in these intervals (same as the intervals for our graphs):

QUADRANTS 1 AND 4

Inverse sine: $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

Inverse tangent: $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

Inverse cosecant: $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

QUADRANTS 1 AND 2

Inverse cosine: $[0, \pi]$

Inverse cotangent: $(0, \pi)$

Inverse secant: $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

Example 1: Compute each of the following:

a. $\sin^{-1}\left(\frac{1}{2}\right)$

b. $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$

c. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$.

d. $\arctan(-1)$.

e. $\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)$.

f. $\sec^{-1}(\sqrt{2})$.

g. $\cot^{-1}(0)$

Evaluating composite functions and their inverses:

Here is a summary of properties that maybe helpful when evaluating inverse trigonometric functions:

$\sin(\sin^{-1}(x)) = x$ when $x \in [-1, 1]$
$\cos(\cos^{-1}(x)) = x$ when $x \in [-1, 1]$
$\tan(\tan^{-1}(x)) = x$ when $x \in (-\infty, \infty)$

$\sin^{-1}(\sin(x)) = x$ when $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}(\cos(x)) = x$ when $x \in [0, \pi]$
$\tan^{-1}(\tan(x)) = x$ when $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Notation: $\sin^{-1}(x) = \arcsin(x)$. This is true for all trig functions. The arc in front of the trig function is the older notation but you will see both notations and need to know their meanings.

a. The inverse property $\cos(\cos^{-1}(x)) = x$ applies to every x in $[-1, 1]$. To evaluate $\cos(\cos^{-1}(.06))$, observe that x= .06. This value of x lies in $[-1, 1]$, which is the domain of the inverse cosine function. This means we can use the inverse property$\cos(\cos^{-1}(x)) = x$ so $\cos(\cos^{-1}(.06)) = .06$

b. The inverse property $\sin^{-1}(\sin(x))$= x applies for every x in $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. To evaluate $\sin^{-1}\left(\sin\left(\frac{3\pi}{2}\right)\right)$, there is a problem because $\frac{3\pi}{2} \notin \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. To evaluate this expression you need to evaluate $\sin\left(\frac{3\pi}{2}\right)$ which equals -1. So $\sin^{-1}\left(\sin\left(\frac{3\pi}{2}\right)\right) = \sin^{-1}(-1) = \frac{-\pi}{2}$.

c. The inverse property $\cos(\cos^{-1}(x)) = x$ applies if for every x in $[-1, 1]$. too attempt to evaluate $\cos(\cos^{-1}(2\pi))$, observe that $x = 2\pi$. That value of x does not lie in $[-1, 1]$, which is the domain of the inverse cosine function. $\cos^{-1}(2\pi)$ also cannot be rewritten like the previous example. Thus $\cos(\cos^{-1}(2\pi))$, is not defined because $\cos^{-1}(2\pi)$ is not defined.

Example 2: Find the exact value, if possible

a. Evaluate $\cos(\arccos(-2))$

b. Evaluate $\sin\left(\sin^{-1}\left(\frac{4}{7}\right)\right)$.

c. Evaluate $\cos^{-1}(\cos(\pi))$.

d. Evaluate $\arctan\left(\tan\left(\frac{2\pi}{3}\right)\right)$

e. Evaluate $\sin^{-1}\left(\sin\left(\frac{5\pi}{3}\right)\right)$.