MATH 3307
Lesson 15

Standard Normal Calculations

As suggested in the previous section, all normal distributions share many common properties. In fact, if change the units to σ and center the graph at μ=0, all normal distributions would be exactly the same. This is called **standardizing**. If *x* is an observation from a normal distribution with mean μ and standard deviation σ, the **standardized value** of *x* is called the ***z*-score** and is computed with the formula below.

Z-Score: $$\boxed{z = \frac{x - \mu}{\sigma}}$$

The Z-Score

A z-score tells us how many standard deviations the observed value falls from the mean.

We can use z-scores to “standardize” values that are on different scales to compare them.

Example:

Bon took the ACT and scored 31. Craig took the SAT and scored (CR+M) 1390. If both tests are normally distributed, who did better? The ACT has a mean of 21.1 and a standard deviation of 4.7. The SAT has a mean of 1010 and a standard deviation of 174.5.

ACT (Bon)

$x = 31$

$\mu = 21.1$

$\sigma = 4.7$

$$Z = \frac{(31-21.1)}{4.7}$$

$$\boxed{Z = 2.106}$$

SAT (Craig)

Higher score

$x = 1390$

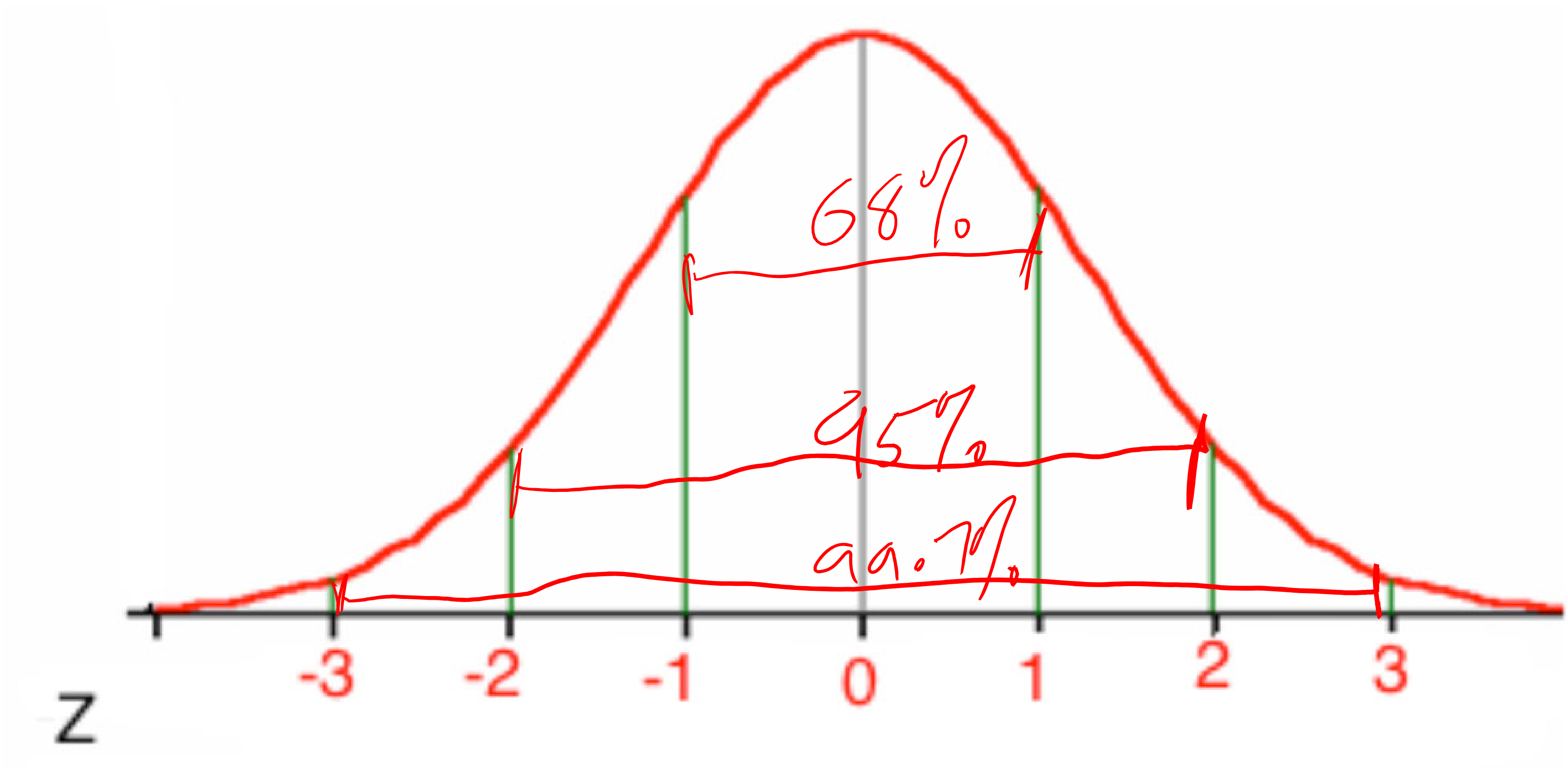
$\mu = 1010$

$\sigma = 174.5$

$$Z = \frac{(1390-1010)}{174.5}$$

$$\boxed{Z = 2.178}$$

The standard normal distribution is the normal distribution with $N(0,1)$:

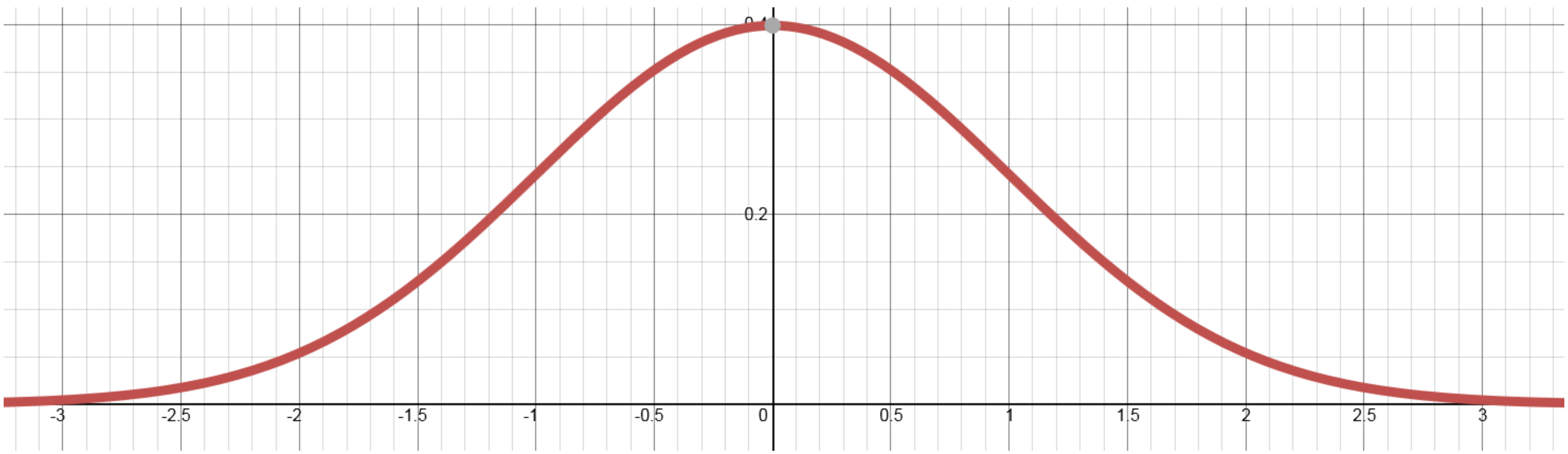


Little known fact:

This is just in case you are ever asked on a game show. You do not need to know this formula

The Normal Distribution Curve has an equation of the following:

$$y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$



Using Tables of Z-Values:

The Table of Z-Values gives areas under the standard normal curve for values of z. The table entry for each value of z gives the area under the curve to the left of z – in other words, it gives p(Z < z) .

http://assessment.casa.uh.edu/Content/Math/2311/Fall2020/T2/Z-table.pdf

$P(Z < 1.25) = .8944$

standard normal (see next slide)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Example: Using Table A, find the following probabilities:

All reference Z (std normal)

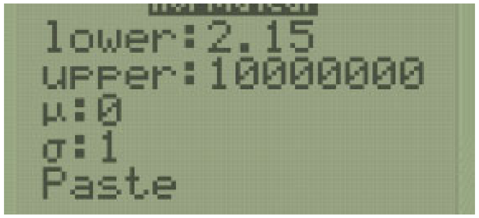
A. $p(Z < -1.06)$

B. $p(Z < 2.15)$

C. $p(Z > 2.15)$

D. $p(-1.06 < Z < 2.15)$

```
> pnorm(-1.06)
[1] 0.1445723
> pnorm(2.15)
[1] 0.9842224
> 1-pnorm(2.15)
[1] 0.01577761
> pnorm(2.15)-pnorm(-1.06)
[1] 0.8396501
```



Now let's repeat with calculator and R-Studio.

If we want to use the table for probabilities and are not given *z*, we must compute the *z*-score using the formula above.

$$z = \frac{x - \mu}{\sigma}$$

Table A only uses z scores

Example: Popper 10

① $z = \frac{(80-100)}{15}$ ② $z = \frac{(105-100)}{15}$

μ σ

If *X* has distribution *N*(100,15), standardize *X* and use Table A to find the following probabilities:

1. Find the z-score corresponding to x = 80. C
2. Find the z-score corresponding to x = 105. A

a. 0.333 b. -0.333 c. -1.333 d. 1.333

3. $p(X < 80)$

   a. 0.9082 b. 0.9999 c. 0.1333 d. 0.0912

4. $p(X > 105)$

   a. 0.6298 b. 0.0004 c. 0.3694 d. 0.9996

5. $p(80 < X < 105)$

   a. 0.6298 b. 0.0000 c. 0.5393 d. 0.0918

1

```
> (80-100)/15
[1] -1.333333
```

2

```
> (105-100)/15
[1] 0.3333333
```

3

```
> pnorm(80,100,15)
[1] 0.09121122
> pnorm(-1.333333)
[1] 0.09121127
```

4

```
> 1-pnorm(105,100,15)
[1] 0.3694413
> 1-pnorm(0.33333333)
[1] 0.3694413
```

5

```
> pnorm(105,100,15)-pnorm(80,100,15)
[1] 0.5393474
> pnorm(0.3333333)-pnorm(-1.3333333)
[1] 0.5393474
>
```

Known Percentile Rank

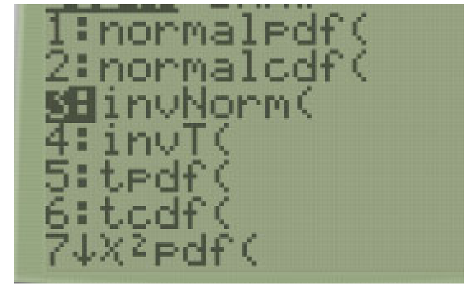
Now, let's suppose we know the percentile rank or the probability and want to find the corresponding *z*-score.

We can use Table A and look up the percentile (remember, it shows the area to the left) or we can use the command invNorm(percent) on the TI or qnorm(percent) in R.

If you know the probability, but you want to find the x or z values that produced it. Use this method.

When using a p-command (greater than), the 1- goes outside, when using a q-command (greater than) the 1- goes inside.

Example: Find the value of *c* so that



unknown p-value is known

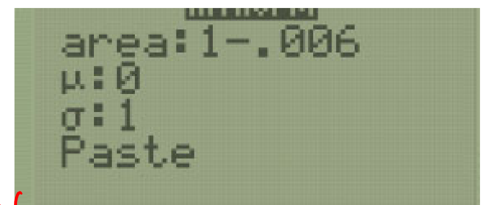
A. $P(Z < c) = 0.7704$

less than

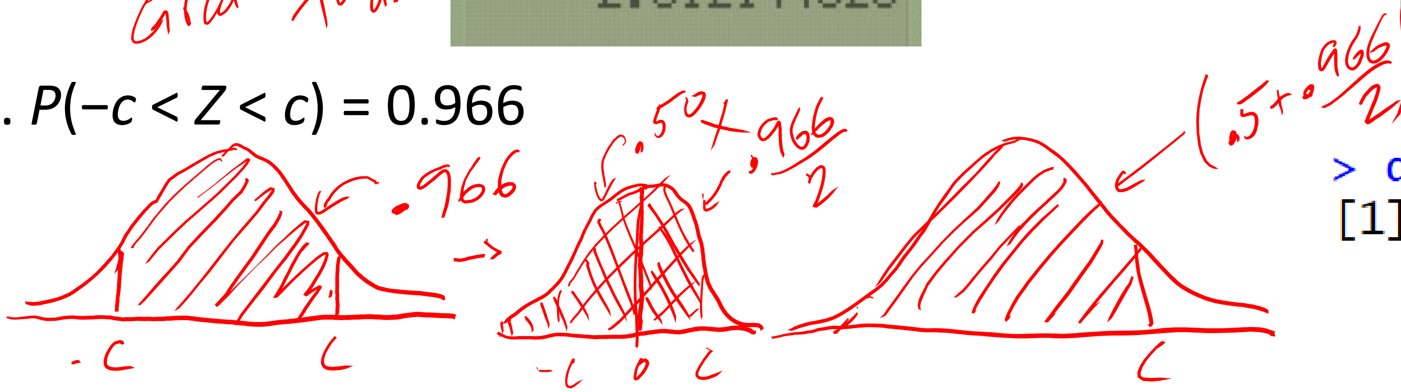
```
> qnorm(0.7704)
[1] 0.7401648
> qnorm(1-0.006)
[1] 2.512144
```

B. $P(Z > c) = 0.006$

Greater Than



C. $P(-c < Z < c) = 0.966$




```
> qnorm(.5+.966/2)
[1] 2.120072
```

Another example:

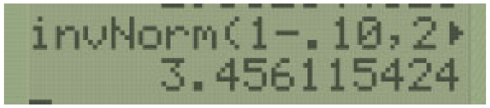
10% above us

$N(2.7, 0.59)$

Suppose you rank in the top 10% of your class. If the mean gpa is 2.7 and the standard deviation is 0.59, what is your gpa? (Assume a normal distribution).

$P(X > c) = .10$


```
> qnorm(1-.10,2.7,0.59)
[1] 3.456115
```



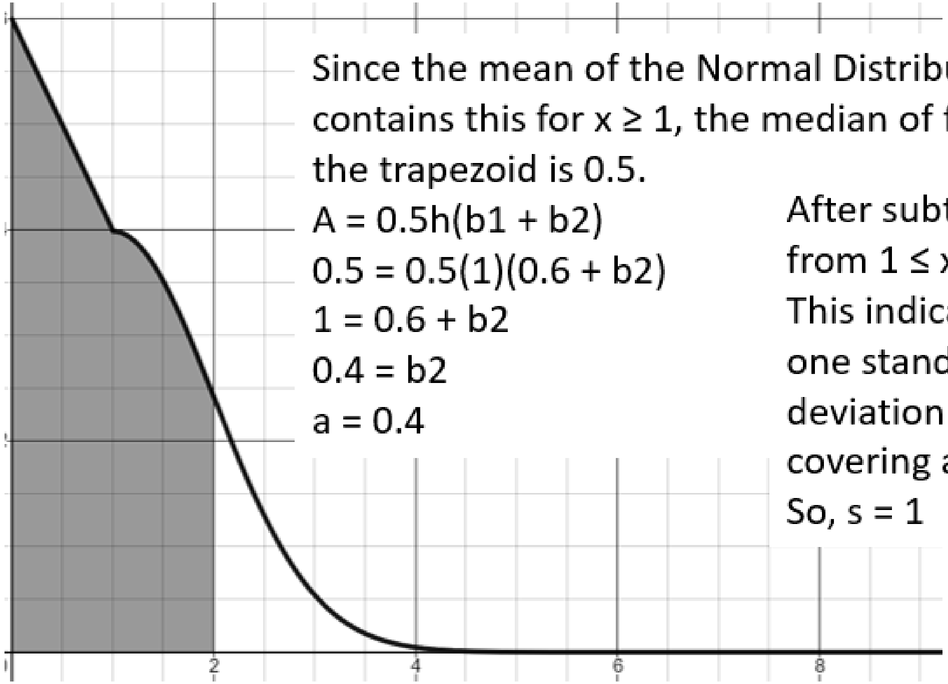
Deciding to use qnorm versus pnorm has NOTHING to do with z or x. It has to do with what you are looking for as an answer. If you are finding a probability, use pnorm. If you finding a c-value, use qnorm.

The following graph illustrates a probability density function, f(x) defined for $[0, \infty)$, comprised of the following:

(1) a segment connecting (0,0.6) to (1, a).

(2) equivalent to the normal distribution N(1, s) for $x \geq 1$.

Assuming the shaded area is equal to 0.841345, determine the values of a and s.



Since the mean of the Normal Distribution is 1, and f(x) contains this for $x \geq 1$, the median of f(x) is 1, so the area of the trapezoid is 0.5.

A = 0.5h(b1 + b2)

0.5 = 0.5(1)(0.6 + b2)

1 = 0.6 + b2

0.4 = b2

a = 0.4

After subtracting 0.5, the area from $1 \leq x \leq 2$ is 0.341345.

This indicates one standard deviation away from the mean, covering a distance of 1 unit.

```
> qnorm(0.341345+0.5)
[1] 1.000001
```

So, s = 1