Math 3339

Homework 4 (Chapter 5)

Name:_________________________________ PeopleSoft ID:_______________

Instructions:

- Homework will NOT be accepted through email or in person. Homework must be submitted through CourseWare BEFORE the deadline.
- Print out this file use or software and complete the problems.
- Write in black ink or dark pencil or type your solutions in the space provided. You must show all work for full credit.
- Submit this assignment at http://www.casa.uh.edu under "Assignments" and choose **hw3**.
- Total possible points: **15**
- You can use RStudio for any of these problems unless otherwise indicated.

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1. Section 5.2.3
   a. Problem 1
   b. Problem 2
   c. Find the cumulative distribution for the previous density function.

Answers:

1. For $0 \leq x \leq 1$ let $f(x) = kx(1 - x)$, where k is a constant. Find the value of k such that f is a density function.
2. Find the mean and variance of the distribution in the preceding exercise.

a. $\int_0^1 kx(1-x)dx \ = \ k\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = \ k\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{k}{6} = 1$, which implies that $\underline{k = 6}$

b. $E(X) = \int_0^1 6x^2(1-x)dx = 6\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 = 6\left(\frac{1}{3} - \frac{1}{4}\right) = 0.5$, this is the mean

$E(X^2) = \int_0^1 6x^3(1-x)dx = 6\left(\frac{x^4}{4} - \frac{x^5}{5}\right)\Big|_0^1 = 6\left(\frac{1}{4} - \frac{1}{5}\right) = 0.3$, $\mathrm{Var}(X) = 0.3 - 0.5^2 = \underline{0.05}$

c. $F(X) = \begin{cases} 0, & x < 0 \\ 3x^2 - 2x^3, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$

2. Section 5.2.3
    a. Problem 3
    b. Problem 4
    c. Problem 5

3. For $x \geq 0$, let $f(x) = 2xe^{-x^2}$. Show that f is a density function.
4. Find the cumulative distribution for the density in the preceding exercise.
5. Find the pth quantile of this distribution.

Answers:

a. $\int_0^\infty 2x\, e^{-x^2}\, dx = -e^{-x^2}\Big|_0^\infty = 0 - (-1) = 1$, Thus this is a density function.

b. $\int_0^x 2y\, e^{-y}\, dy = -e^{-y^2}\Big|_0^x = -e^{-x^2} - (-1) = 1 - e^{-x^2}$ for $x \geq 0$

$$F(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-x^2}, & x \geq 0 \end{cases}$$

c. Set p = F(x) and solve for x,

p = 1 - e^{-x^2}

1 – p = e^{-x^2}

ln(1 – p) = $-x^2$

$x = \sqrt{-\ln(1-p)}$

3. Let X denote the amount of time for which a book on 2-hour reserve at a college library is checked out by a randomly selected student and suppose that X has cumulative distribution function, CDF

$$F(X) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{4} & 0 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$

Use this to compute the following:

a. $P(X \le 1)$
b. $P(0.5 \le X \le 1.5)$
c. $P(1.5 < X)$
d. Determine the median checkout duration. That is find x such that $F(x) = 0.5$.
e. Compute $F'(x)$ to obtain the density function $f(x)$.
f. Determine E(X) and Var(X).

Answers:

a. $P(X \le 1) = F(1) = \frac{1}{4} = \underline{0.25}$
b. $P(0.5 \le X \le 1.5) = F(1.5) - F(0.5) = \frac{1.5^2}{4} - \frac{0.5^2}{4} = 0.5$
c. $P(1.5 < X) = P(X > 1.5) = 1 - P(X \le 1.5) = 1 - F(1.5) = 1 - \frac{1.5^2}{4} = \underline{0.4375}$
d. $F'(x) = \frac{x}{2}$, $f(x) = \begin{cases} \frac{x}{2}, & 0 \le x \le 2 \\ 0, & otherwise \end{cases}$
e. $E(X) = \int_0^2 x * \frac{x}{2}\, dx = \frac{x^3}{6}\Big|_0^2 = \frac{8}{6} = 1.3333$

   $E(X^2) = \int_0^2 x^2 * \frac{x}{2}\, dx = \frac{x^4}{8}\Big|_0^2 = \frac{16}{8} = 2$

   $Var(X) = 2 - 1.3333^2 = \underline{0.2223}$

4. Section 5.3.2; problem 5

The shuttle bus from your parking lot and your office building operates on a 15 minute schedule. You arrive at the parking lot at a random time during the bus's cycle, that is, the time you have to wait for the bus is uniformly distributed over the interval from 0 to 15. What is the standard deviation of your waiting time? What is the probability that you will have to wait more than 2 standard deviations?

Answers:

Let X = the time you wait for the bus, X ~ Unif(0,15),

$\sigma_X = SD(X) = \sqrt{\frac{(15-0)^2}{12}} = 4.330127$

$P(X > 2*4.330127) = 1 - P(X \leq 8.660254) = 1 - 0.5773 = 0.4227$

5. Section 5.5.4; problem 1

Let $Z \sim \text{Norm}(0, 1)$. Use the normal table and also R's "pnorm" function to find

(a) $P\,r(Z \leq 1.45)$

(b) $P\,r(Z > -1.28)$

(c) $P\,r(-0.674 \leq Z < 1.036)$

(d) $P\,r(Z > 0.836)$

Answers:

a) Table: $P(Z \leq 1.45) = 0.9265$

R-studio:

```
pnorm(1.45)
[1] 0.9264707
```

b) Table: $P(Z > -1.28) = 1 – P(Z < -1.28) = 1 – 0.1003 = 0.8997$

R-studio:

```
1-pnorm(-1.28)
[1] 0.8997274
```

c) Table: $P(-0.674 \leq Z \leq 1.036) \approx P(Z \leq 1.04) – P(Z \leq -0.67) = 0.8508 – 0.2514 = 0.5994$

R-studio:

```
pnorm(1.036) - pnorm(-0.674)
[1] 0.5997433
```

d) Table: $P(Z > 0.836) \approx 1 – P(Z \leq 0.84) = 1 – 0.7995 = 0.2005$

R-studio:

```
1-pnorm(0.836)
[1] 0.2015775
```

6. Section 5.5.4; problem 2

Use the normal table and also R's "pnorm" function to find

(a) P r(X ≤ 6.13), X ~ Norm(1, 4)

(b) P r(X > −2.35), X ~ Norm(−1, 2)

(c) P r(−0.872 < X ≤ 7.682), X ~ Norm(2.5, 5)

(d) P r(X > 0.698), X ~ Norm(−2, 4)

Answers;

a) Table: $P(X \leq 6.13) = P\left(Z \leq \frac{6.13-1}{4}\right) = P(Z \leq 1.2825) \approx P(Z \leq 1.28) = 0.8997$

R-studio:

```
pnorm(6.13,1,4)
[1] 0.9001663
```

b) Table: $P(X > -2.35) = 1 - P(X \leq -2.35) = 1 - P\left(Z \leq \frac{-2.35-(-1)}{2}\right) = 1 - P(Z \leq -0.675)$

$\approx 1 - 0.2483 = 0.7517$

R-studio:

```
1-pnorm(-2.35,-1,2)
[1] 0.7501621
```

c) Table: $P(-0.872 < X \leq 7.682) = P(X \leq 7.682) - P(X \leq -0.872)$

$$= P\left(Z \leq \frac{7.682-2.5}{5}\right) - P\left(Z \leq \frac{-0.872-2.5}{5}\right) = P(Z \leq 1.0364) - P(Z \leq -0.6744)$$

$$\approx 0.8508 - 0.2154 = 0.5994$$

R-studio:

```
pnorm(7.682,2.5,5) - pnorm(-0.872,2.5,5)
[1] 0.5999637
```

d) Table: $P(X > 0.698) = 1 - P(X \leq 0.698) = 1 - P\left(Z \leq \frac{0.698-(-2)}{4}\right) = P(Z \leq 0.6745) \approx 0.2154$

R-studio:

```
1-pnorm(0.698,-2,4)
[1] 0.2499967
```

7. Section 5.5.4
   a. Problem 3
   b. Problem 4

   Use the normal table and also R's "qnorm" function to find
   (a) The 90th percentile of Norm(0, 5).
   (b) The 15th percentile of Norm(1, 3).
   (c) The interquartile range, i.e., the distance from the first to third quartiles of Norm(0, 1).
   4. Use the result of 3(c) to find the interquartile range of Norm(μ, σ).

Answers:

a) Table: $z \approx 1.28$; $X = 1.28(5) + 0 = 6.4$
   R-studio:
```
qnorm(.9,0,5)
[1] 6.407758
```

b) Table: $z \approx -1.04$; $X = -1.04(3) + 1 = -2.12$
   R-studio:
```
qnorm(.15,1,3)
[1] -2.1093
```

c) When the mean is zero and the standard deviation is 1, this is the standard normal (z).
   Table: For Q1 (25^{th} percentile) $z \approx -0.67$; Q3 (75^{th} percentile) $z \approx 0.67$
   IQR = Q3 – Q1 = 0.67 – (-0.67) = 1.32
   R-studio:
```
> qnorm(.75)-qnorm(.25)
[1] 1.34898
```

d) For any generic X with mean μ and standard deviation σ, notice that $X = z\sigma + \mu$ so from part c of #3;
   $IQR = (0.67\sigma + \mu) - (-0.67\sigma + \mu) = 1.32\sigma$
   If using R-studio it would be $IQR = 1.35\sigma$

8. In each case, determine the value of the constant c that makes the probability statement correct.
   a. $\Phi(c) = 0.9838$
   b. $P(0 \leq Z \leq c) = 0.291$
   c. $P(c \leq Z) = 0.121$
   d. $P(-c \leq Z \leq c) = 0.668$
   e. $P(c \leq |Z|) = 0.016$

   Answers:
   a. c = qnorm(0.9838) = 2.14
   b. c = qnorm(.5+.291) = 0.81
   c. $P(c \leq Z) = P(Z \geq c)$; c = qnorm(1-0.121) = 1.17
   d. c = qnorm(1.668/2) = 0.97
   e. $P(c \leq |Z|) = P(|Z| \geq c) = P(Z \leq -c \text{ or } Z \geq c)$, -c=qnorm(0.016/2) = -2.41 and c = 2.41

9. Suppose the force acting on a column that helps to support a building is a normally distributed random variable X with mean value 15.0 kips and standard deviation 1.25 kips. Compute the following probabilities.
   a. $P(X \leq 15)$
   b. $P(X \leq 17.5)$
   c. $P(X \geq 10)$
   d. $P(14 \leq X \leq 18)$
   e. $P(|X - 15| \leq 3)$

Answers:
All of these answers were using the pnorm function in R-studio

a. $P(X \leq 15) = 0.5$ (since the mean is 15)
b. `pnorm(17.5,15,1.25) = 0.9772499`
c. `1-pnorm(10,15,1.25) = 0.9999683`
d. `pnorm(18,15,1.25)-pnorm(14,15,1.25) = 0.7799471`
e. Notice that $P(|X - 15| \leq 3) = P(12 \leq X \leq 18) =$ `pnorm(18,15,1.25)-pnorm(12,15,1.25)` $= 0.9836$

10. A test was done to look at the maximum speed of mopeds. They found that the maximum speed had a normal distribution with a mean of 29 mph and standard deviation of 1.4 mph. Consider randomly selecting a moped.
    a. What is the probability that the maximum speed is at most 31 mph?
    b. What is the probability that the maximum speed is at least 29.8 mph?
    c. What is the probability that maximum speed differs from the mean value by at most 1.5 standard deviations?
    d. Suppose we select 49 mopeds, what is the probability that the sample mean will be at least 29.5 mph?

Answers:

X = maximum speed of moped, X ~N(29,1.4)

a. $P(X \leq 31)$ = pnorm(31,29,1.4) = 0.9234
b. $P(X \geq 29.8)$ = 1 – pnorm(29.8,29,1.4) = 0.2839
c. $P(29 - 1.5(1.4) < X < 29 + 1.5(1.4)) = P(26.9 < X < 31.1)$ = pnorm(31.1,29,1.4) – pnorm(26.9,29,1.4) = 0.8664
d. n = 49, $P(\bar{X} \geq 29.5)$ = 1 – pnorm(29.5,29,1.4/sqrt(49)) = 0.0062

11. Suppose only 70% of all drivers in a certain state regularly wear a seat belts. A random sample of 500 drivers is selected. What is the probability that
    a. Between 320 and 370 (inclusive) of the drives in the sample regularly wear a seatbelt?
    b. Fewer than 325 of those in the sample regularly wear a seatbelt?
    c. Fewer than 315?
    Hint: Use the binomial approximation to determine these probabilities.

Answers:

p = 0.7, n = 500, $E(X) = np = 500(0.7) = 350$,

$$SD(X) = \sqrt{np(1-p)} = \sqrt{500(0.7)(1-0.7)} = 10.24695$$

Notice, np = 350 and n(1-p) = 150, since both are greater than 10 we can use the Normal distribution as an approximation to find these probabilities.

a. $P(320 \leq X \leq 370) = P(X \leq 370) - P(X \leq 319)$
   = `pnorm(370.5,350,10.24695) - pnorm(319.5,350,10.24695)`
   = 0.9758242

b. $P(X < 325) = P(X \leq 324)$ = pnorm(324.5,350,10.24695) = 0.0064

c. $P(X < 315) = P(X \leq 314)$ = pnorm(314.5,350,10.25695) = 0.0003

Side note: These are close to the probabilities if we were to use pbinom instead:

```
> pbinom(370,500,.7) - pbinom(319,500,.7)
[1] 0.9767255

> pbinom(324,500,.7)
[1] 0.006982576

> pbinom(314,500,.7)
[1] 0.0003339827
```

12. Let X = the time between two successive arrivals at the drive-up window of a local bank. If X has an exponential distribution with $\lambda = 1$, compute the following:
    a. The expected time between two successive arrivals.
    b. The standard deviation of the time between successive arrivals.
    c. $P(X \leq 4)$
    d. $P(2 \leq X \leq 5)$

Answers

a. $E(X) = \frac{1}{\lambda} = 1$, this is because we know it is an exponential distribution
b. SD(X) = 1, the exponential distribution the mean and variance are the same.
c. $P(X \leq 4)$ = pexp(4,1) = 0.9817
d. $P(2 \leq X \leq 5)$ = pexp(5,1) – pexp(2,1) = 0.1286

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13. Suppose that the time spent online to do homework by a randomly selected student has a gamma distribution with mean 20 minutes and variance 80 minutes^2.
    a. What are the values of α and β?
    b. What is the probability that a student spends online at most 24 minutes?
    c. What is the probability that a student spends between 20 and 40 minutes online?

Answers:

a. $\mu = \alpha\beta = 20$, $\sigma^2 = \alpha\beta^2 = 80$, $\alpha = \frac{20}{\beta}$, $80 = \frac{20}{\beta}\beta^2 = 20\beta$, $\beta = 4$, $\alpha = 5$
b. $P(X \leq 24)$ = pgamma(24,5,1/4) = 0.7149
c. $P(20 < X < 40)$ = pgamm(40,5,1/4) – pgamma(20,5,1/4) = 0.4112

14. The inside diameter of a randomly selected piston ring is a random variable with mean value 12 cm and standard deviation 0.04 cm.

   a. If $\bar{X}$ is the sample mean diameter for a random sample of n = 16 rings, where is the sampling distribution of $\bar{X}$ centered, and what is the standard deviation of the $\bar{X}$ distribution?
   b. Answer the questions posted in part (a) for a sample size of n =64 rings.
   c. For which of the two random samples, the one of part (a) or the one of part (b), is $\bar{X}$ more likely to be within 0.01 cm of 12 cm? Explain your reasoning.

   Answers:

   a. $E(\bar{X}) = 12$, $SD(\bar{X}) = \frac{0.04}{\sqrt{16}} = 0.01$
   b. $E(\bar{X}) = 12$, $SD(\bar{X}) = \frac{0.04}{\sqrt{64}} = 0.005$
   c. Assuming Normal distribution
      n = 16, pnorm(12.01,12,0.01) – pnorm(11.99,12,0.01) = 0.6827
      n = 64, pnorm(12.01,12,0.005) – pnorm(11.99,12,0.005) = 0.9544
      It is more likely to be within 0.01 with n=64.