ERGODIC THEORY AND THERMODYNAMIC FORMALISM

MATH 7394 – VAUGHN CLIMENHAGA

Abstract. Ergodic theory is a central part of the theory of dynamical systems, studying the asymptotic statistical properties of systems evolving in time that preserve an invariant measure. Systems with chaotic behavior generally possess many invariant measures, and thermodynamic formalism borrows tools from statistical mechanics to select a distinguished measure that is physically relevant. The first part of the class will cover topics in classical ergodic theory, including Birkhoff’s ergodic theorem, entropy, and the classification of Bernoulli automorphisms.

The remainder of the course will discuss thermodynamic formalism, including the description of Sinai-Ruelle-Bowen measure via absolute continuity, the description of Parry measure via a variational principle, and the connection between the two via the general theory of equilibrium states. Some time will be spent describing the different approaches to thermodynamic formalism and SRB measures in uniform hyperbolicity: Ruelle-Perron-Frobenius operators indirectly via symbolic dynamics or directly via anisotropic Banach spaces; specification and expansivity; and the geometric approach via averaged pushforwards. Time permitting, we will discuss connections to dimension theory and geometric measure theory, and will conclude with a discussion of the nonuniformly hyperbolic setting.

This document will evolve as the course progresses, to better reflect the material that is actually covered. The sections below are of varying lengths and will take different numbers of lectures to discuss. Most but not all of the topics we will cover are found in Walters’ textbook [Wal82]. Another useful reference is the textbook by Oliveira and Viana [VO16]. The majority of the references below are to primary sources in the research literatres, and vary widely in accessibility, readability, and how closely the notation and terminology conforms to what we use. Bowen’s monograph [Bow08] deserves special mention as an excellent reference for Markov partitions and the spectral approach. Where relevant, I also give references to some of my blog posts: All references of the form “BLOG:X” are links to https://vaughnclimenhaga.wordpress.com/X.

0.1. Introduction. Stochastic behavior in deterministic systems as a general phenomenon underlying randomness in the real world [Rue89]. Other overall surveys: [Rue80, ER85].

1. Basic examples


2. Classical ergodic theory


Ergodic hierarchy: ergodicity, weak mixing, mixing, multiple mixing, Bernoullicity. Rates of mixing and decay of correlations.

2.3. Measure-theoretic entropy. See [Kat07] for a historical overview.

2.3.1. Kolmogorov, Sinai, and Ornstein. Information function and measure-theoretic entropy. Invariance under isomorphism. Pinsker partition and the K property. \( d \)-bar metric and Ornstein theory [Orn70].


3. Thermodynamic formalism in uniform hyperbolicity


3.1.1. Expanding maps. Construction of ACIPs for expanding circle maps by averaged pushforwards using Ruelle–Perron–Frobenius operator [LY73].


Rectangles. Physical measures [You02].


3.2. Measures of maximal entropy in symbolic dynamics.


3.3. Equilibrium states and ACIPs.

3.3.1. The geometric potential. Margulis–Ruelle inequality in a restricted setting: give details of argument for maps conjugate to doubling. Characterize ACIP as measure maximizing $h_\mu(f) + \int \phi d\mu$ for geometric potential $\phi$ [Wal78, Led81, DKU90].

3.3.2. The general variational principle. Topological pressure, variational principle, and equilibrium states. Upper semi-continuity of entropy as a sufficient condition for existence. Measure-theoretic entropy as Legendre transform of topological pressure.

3.3.3. Log Jacobians and moduli of topological conjugacy. Recovering $\phi$ from its equilibrium state for the doubling map. Conjugacy/isomorphism between doubling map with equilibrium state and non-linear version with Lebesgue.

3.4. Equilibrium states for general hyperbolic systems. When do two potentials have the same equilibrium state? Cohomologous potentials and the Livsic theorem:


3.4.2. Markov partitions and indirect use of transfer operators. Bowen’s monograph [Bow08]. Also mention Sinai (for partitions and for Gibbs measures) and Ruelle.

3.4.3. Anisotropic Banach spaces and direct use of transfer operators. [BKL02, Bal17]

3.4.4. Expansivity and specification. Bowen’s uniqueness argument [Bow75]. Gibbs property implies local product structure — [BLOG:2017/06/16/239] — and equilibrium states have positive entropy — [BLOG:2017/01/26/entropy-bounds-for-equilibrium-states/]

3.5. Dimension theory and geometric measure theory.

3.5.1. Hausdorff dimension and Bowen’s equation. [Bow79]

3.5.2. Entropy and pressure as dimensional quantities. [Bow73, PP84, Cli11]

3.5.3. Construction of general equilibrium states as limits of averaged pushforwards. [CPZ19]

Slowly mixing sets: [BLOG:2014/04/22/slowly-mixing-sets/]
Uniform bounds via Baire: [BLOG:2017/01/24/exponential-decay-of-correlations/]

3.6.1. Exponential decay of correlations via spectral theory.
[BLOG:2013/01/30/spectral-methods-in-dynamics/]
[BLOG:2013/02/08/spectral-methods-in-dynamics-part-2/]
[BLOG:2013/02/13/function-spaces-and-compactness/]

3.6.2. Decay of correlations via coupling.
[BLOG:2013/02/15/markov-chains-and-mixing-times/]
[BLOG:2013/02/21/markov-chains-and-mixing-times-part-2-coupling/]
[BLOG:2013/03/06/markov-chains-and-mixing-times-part-3/]
3.6.3. Decay of correlations via the Hilbert metric.

BLOG:2013/03/30/convex-cones-and-the-hilbert-metric/
BLOG:2013/04/17/the-perron-frobenius-theorem-and-the-hilbert-metric/

3.6.4. Central limit theorem.

BLOG:2013/03/17/spectral-methods-3-central-limit-theorem/
BLOG:2013/05/18/central-limit-theorem-for-dynamical-systems-using-martingales/

3.6.5. Large deviations principles.

4. Nonuniform hyperbolicity

4.1. Tower constructions.

4.1.1. Hofbauer towers for interval maps. β-transformations and others. Sarig’s theory for countable-state TMS.


BLOG:2017/06/07/alpha-beta-shifts/

4.3. Open problems. Hyperbolicity for standard map. SRB for weakly dissipative Hénon.

References


