EQUILIBRIUM MEASURES FOR SOME PARTIALLY HYPERBOLIC SYSTEMS

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Abstract. We study thermodynamic formalism for topologically transitive partially hyperbolic systems in which the center-stable bundle is integrable and nonexpanding, and show that every potential function satisfying the Bowen property has a unique equilibrium measure. Our method is to use tools from geometric measure theory to construct a suitable family of reference measures on unstable leaves as a dynamical analogue of Hausdorff measure, and then show that the averaged pushforwards of these measures converge to a measure that has the Gibbs property and is the unique equilibrium measure.

1. Introduction

Consider a dynamical system $f: X \to X$, where $X$ is a compact metric space and $f$ is continuous. Given a continuous potential function $\varphi: X \to \mathbb{R}$, the topological pressure of $\varphi$ is the supremum of $h_\mu(f) + \int \varphi \, d\mu$ taken over all $f$-invariant Borel probability measures on $X$, where $h_\mu$ denotes the measure-theoretic entropy. A measure achieving this supremum is called an equilibrium measure, and one of the central questions of thermodynamic formalism is to determine when $(X, f, \varphi)$ has a unique equilibrium measure.

When $f: M \to M$ is a diffeomorphism and $X \subset M$ is a topologically transitive locally maximal hyperbolic set, so that the tangent bundle splits as $E^u \oplus E^s$, with $E^u$ uniformly expanded and $E^s$ uniformly contracted by $Df$, it is well-known that every Hölder continuous potential function has a unique equilibrium measure $[6]$. In the specific case when $\varphi = -\log |\det(Df|_{E^u})|$ is the geometric potential, this equilibrium measure is the unique Sinai–Ruelle–Bowen (SRB) measure, which is also characterized by the fact that its conditional measures along unstable leaves are absolutely continuous with respect to leaf volume.

In this paper we study the case when uniform hyperbolicity is replaced by partial hyperbolicity. Here the analogue of SRB measures are the $u$-measures constructed by the second author and Sinai in $[27]$ using a geometric construction based on pushing forward leaf volume and averaging. We follow this approach, replacing leaf volume with a family of reference measures defined using a Carathéodory dimension structure. We give conditions on $f$ and $\varphi$ under which the averaged pushforwards of these reference measures converge to the unique equilibrium measure. See $[2]$ and $[1.1]$ for the precise class of maps and potentials that we consider, $[3]$ for the definition of the reference measures, and $[5]$ for some applications. These applications include time-1 maps of Anosov flows and frame flows, a setting in

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1In $[27]$ these are called “$u$-Gibbs measures”; here we use the terminology from $[28]$. 
which equilibrium measures are well understood for the flows, but the corresponding theory for the time-1 maps is less developed. We also refer to the survey paper [15] for a general overview of how our techniques work in the uniformly hyperbolic setting.

Here we briefly survey some known results on thermodynamic formalism in partial hyperbolicity. The first remark is that whenever $f$ is $C^\infty$, the entropy map $\mu \mapsto h_\mu(f)$ is upper semi-continuous [26] and thus existence of an equilibrium measure is guaranteed by weak*-compactness of the space of $f$-invariant Borel probability measures; however, this nonconstructive approach does not address uniqueness or describe how to produce an equilibrium measure.

Even without the $C^\infty$ assumption, the expansivity property would be enough to guarantee that the entropy map is upper semi-continuous, and thus gives existence; moreover, for expansive systems the construction in the proof of the variational principle [38, Theorem 9.10] actually produces an equilibrium measure. Partially hyperbolic systems are not expansive in general, but when the center direction is one-dimensional, they are entropy-expansive [16, §5.3]; this notion, introduced by Bowen in [4], also suffices to guarantee that the standard construction produces an equilibrium measure. On the other hand, when the center direction is multi-dimensional, Buzzi and Fisher constructed examples with positive tail entropy, so that the system is not even asymptotically entropy-expansive [12].

For results on SRB measures in partial hyperbolicity, we refer to [3, 1, 8, 9, 16]. For the broader class of equilibrium measures, existence and uniqueness questions have been studied for certain classes of partially hyperbolic systems. In general one should not expect uniqueness to hold without further conditions; see [33] for an open set of topologically mixing partially hyperbolic diffeomorphisms in three dimensions with more than one measure of maximal entropy (MME) – that is, multiple equilibrium measures for the potential $\varphi = 0$. Some results on existence and uniqueness of an MME are available when the partially hyperbolic system is semiconjugated to the uniformly hyperbolic one; see [11, 37]. For some partially hyperbolic systems obtained by starting with an Anosov system and making a perturbation that is $C^1$-small except in a small neighborhood where it may be larger and is given by a certain bifurcation, uniqueness results can be extended to a class of nonzero potential functions [13, 14].

The largest set of results is available for the examples known as “partially hyperbolic horseshoes”: existence of equilibrium measures was proved by Leplaideur, Oliveira, and Rios [25]; examples of rich phase transitions were given by Díaz, Gelfert, and Rams [17, 20, 18]; and uniqueness for certain classes of Hölder continuous potentials was proved by Arbieto and Prudente [2] and Ramos and Siqueira [31]. A related class of partially hyperbolic skew-products with non-uniformly expanding base and uniformly contracting fiber was studied by Ramos and Viana [32]. We point out that our results study a class of systems, rather than specific examples, and that we establish uniqueness results, rather than the phase transition results that have been the focus of much prior work.

Another class of partially hyperbolic examples is obtained by considering the time-1 map of an Anosov flow. When the flow is transitive, it is well-known that the flow has a unique MME, and in this case a simple argument communicated to us by F. Rodriguez Hertz (see §5.1) shows that this gives uniqueness for the map as well. For nonzero potentials a new approach is needed, and in §5 we apply our techniques to this setting; the class of potential
functions to which our results apply includes all scalar multiples of the geometric potential, whose equilibrium measures are precisely the $\mu$-measures from [27]. We also study the time-1 map for frame flows in negative curvature, which are partially hyperbolic. In this latter setting, equilibrium measures for the flow (but not the map) were recently studied by Spatzier and Visscher [36].

In §2 we give background definitions and describe the classes of systems we will study. In §3 we recall the general notion of a Carathéodory dimension structure and use it to define a family of reference measures on unstable leaves. In §4 we describe the class of potential functions that we will consider, and formulate our main results. Some applications are given in §5, and the proofs are given in §§6–8. In Appendix A we gather some proofs of background results that we do not claim are new, but which seemed worth proving here for completeness.

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2. Preliminaries

2.1. Partially hyperbolic sets. Let $M$ be a compact smooth connected Riemannian manifold, $U \subset M$ an open set and $f : U \to M$ a diffeomorphism onto its image. Let $\Lambda \subset U$ be a compact $f$-invariant subset on which $f$ is partially hyperbolic in the broad sense: that is

- the tangent bundle over $\Lambda$ splits into two invariant and continuous subbundles $T\Lambda M = E^{cs} \oplus E^{u}$;
- there is a Riemannian metric $\| \cdot \|$ on $M$ and numbers $0 < \nu < \chi$ with $\chi > 1$ such that for every $x \in \Lambda$

\begin{align}
\|dfv\| &\leq \nu \|v\| \text{ for } v \in E^{cs}(x), \\
\|dfv\| &\geq \chi \|v\| \text{ for } v \in E^{u}(x).
\end{align}

Remark 2.1. One could replace the requirement that $\chi > 1$ with the condition that $\nu < 1$; to apply our results in this setting it suffices to replace $f$ with $f^{-1}$, and so we limit our discussion to the case when $E^{u}$ is uniformly expanding.

Our main results in §4 will require the following two conditions:

(C1) $\nu \leq 1$ where $\nu$ is the number in (2.1);
(C2) $f|\Lambda$ is topologically transitive.

We will also assume a dynamical coherence condition; see (A1)–(A4) below.

Since the distributions $E^{cs}$ and $E^{u}$ are continuous on $\Lambda$, there is $\kappa > 0$ such that $\angle(E^{u}(x), E^{cs}(x)) > \kappa$ for all $x \in \Lambda$; we also have $p \geq 1$ such that $\dim E^{u}(x) = p$ and $\dim E^{cs}(x) = \dim M - p$ for all $x \in \Lambda$.

The following well known result describes existence and some properties of local unstable manifolds; see [30, §4] for a proof.

Proposition 2.2. There are numbers $\tau > 0$, $\lambda \in (\chi^{-1}, 1)$, $C_{1} > 0$, $C_{2} > 0$, and for every $x \in \Lambda$ a $C^{1+\alpha}$ local manifold $V_{loc}^{u}(x) \subset M$ such that

1. $T_{x}V_{loc}^{u}(x) = E^{u}(x)$;
The number $\tau$ is the size of the local manifolds and will be fixed at a sufficiently small value to guarantee various estimates.

2.2. Local product structure and rectangles. In what follows we will assume that the distribution $E^{cs}(x)$ can be locally integrated as follows.

(A1) There are numbers $\tau > 0$, $C_3 > 0$, $r_0 > 0$ and for every $x \in \Lambda$ local manifold $V_{loc}^{cs}(x) \subset M$ such that

(a) $T_x V_{loc}^{cs}(x) = E^{cs}(x)$;
(b) $V_{loc}^{cs}(x) = \exp_x \{ v + \psi^{cs}_x(v) : v \in B^{cs}_x(0, \tau) \}$, where $B^{cs}_x(0, \tau)$ is the ball in $E^{cs}(x)$ centered at zero of radius $\tau > 0$ and $\psi^{cs}_x : B^{cs}_x(0, \tau) \to E^{u}(x)$ is a $C^{1+\alpha}$ function.

(A2) For every $y \in V_{loc}^{cs}(x)$ and $n \geq 0$ we have $d(f^n(x), f^n(y)) \leq d(x, y)$.

(A3) For every $y \in B(x, r_0) \cap \Lambda$ we have that $V_{loc}^{u}(x)$ and $V_{loc}^{cs}(y)$ intersect at exactly one point which we denote by $[x, y]$.

We also require $\Lambda$ to have the following product structure, so that the point $[x, y]$ from $\Lambda$ lies in $\Lambda$.

(A4) $\Lambda = \left( \bigcup_{x \in \Lambda} V_{loc}^{u}(x) \right) \cap \left( \bigcup_{x' \in \Lambda} V_{loc}^{cs}(x') \right)$.

Remark 2.3. If the number $\nu$ from (2.1) satisfies $\nu < 1$, so that the set $\Lambda$ is uniformly hyperbolic and $E^{cs}$ is uniformly contracted under $f$, then conditions (A1)-(A4) hold whenever $\Lambda$ is locally maximal for $f$; in particular, our results apply to every topologically transitive locally maximal hyperbolic set.

Given $x \in \Lambda$ and $r \in (0, \tau)$, we write $B_{\Lambda}(x, r) = B(x, r) \cap \Lambda$ for convenience. We also write

$B^{u}(x, r) = B(x, r) \cap V_{loc}^{u}(x), \quad B^{cs}_{\Lambda}(x, r) = B^{cs}(x, r) \cap \Lambda,$

and similarly with $u$ replaced by $cs$.

Definition 2.4. A closed set $R \subset \Lambda$ is called a rectangle if $[x, y] = V_{loc}^{u}(x) \cap V_{loc}^{cs}(y)$ exists and is contained in $R$ for every $x, y \in R$.

One can easily produce rectangles by fixing $x \in \Lambda$, $\delta > 0$ sufficiently small, and putting

\begin{equation}
(2.2) \quad R = R(x, \delta) := \left( \bigcup_{y \in B^{cs}_{\Lambda}(x, \delta)} V_{loc}^{u}(y) \right) \cap \left( \bigcup_{z \in B^{cs}_{\Lambda}(x, \delta)} V_{loc}^{cs}(z) \right).
\end{equation}

Note that the intersection of two rectangles is either empty or is itself a rectangle. The following result is standard: for completeness, we give a proof in Appendix A.

Lemma 2.5. For every $\epsilon > 0$ and every Borel measure $\mu$ on $\Lambda$ (whether invariant or not), there is a finite set of rectangles $R_1, \ldots, R_N \subset \Lambda$ satisfying the following properties:

(1) each $R_i$ is the closure of its (relative) interior;
(2) $\Lambda = \bigcup_{i=1}^{N} R_i$, and the (relative) interiors of the $R_i$ are disjoint;
\( \mu(\partial R_i) = 0 \) for all \( i \);
\( \text{diam } R_i < \epsilon \) for all \( i \).

We refer to \( \mathcal{R} = \{ R_1, \ldots, R_N \} \) as a partition by rectangles. Note that even though the rectangles may overlap, the fact that the boundaries are \( \mu \)-null implies that there is a full \( \mu \)-measure subset of \( \Lambda \) on which \( \mathcal{R} \) is a genuine partition. In particular, when \( \mu \) is \( f \)-invariant, we can use a partition by rectangles for the computation of measure-theoretic entropy.

We end this section with one more definition.

**Definition 2.6.** Given a rectangle \( R \) and points \( y, z \in R \), the holonomy map \( \pi_{yz} : V_u^R(y) \to V_u^R(z) \) is defined by

\[
\pi_{yz}(x) = V_{\text{loc}}^u(z) \cap V_{\text{loc}}^s(y) = [z, y].
\]

The holonomy map is a homeomorphism between \( V_u^R(y) \) and \( V_u^R(z) \). One can define a holonomy map between \( V_{cs}^R(y) \) and \( V_{cs}^R(z) \) in the analogous way, sliding along unstable leaves; if need be, we will denote this map by \( \pi_{yz}^u \) and the holonomy map from Definition 2.6 by \( \pi_{yz}^{cs} \). The notation \( \pi_{yz} \) without a superscript will always refer to \( \pi_{yz}^{cs} \).

2.3. Measures with local product structure. We recall some facts about measurable partitions and conditional measures; see [34] or [21, §5.3] for proofs and further details.

Given a measure space \( (X, \mu) \), a partition \( \xi \) of \( X \), and \( x \in X \), write \( \xi(x) \) for the partition element containing \( x \). The partition is said to be measurable if it can be written as the limit of a refining sequence of finite partitions. In this case there exists a system of conditional measures \( \{ \mu_{\xi x} \}_{x \in X} \) such that:

\( \text{1) Each } \mu_{\xi x} \text{ is a probability measure on } \xi(x); \)
\( \text{2) If } \xi(x) = \xi(y), \text{ then } \mu_{\xi x} = \mu_{\xi y}; \)
\( \text{3) For every } \psi \in L^1(X, \mu), \text{ we have} \)

\[
\int_X \psi \, d\mu = \int_X \int_{\xi(x)} \psi \, d\mu_{\xi x} \, d\mu(x).
\]

Moreover, the system of conditional measures is unique mod zero: if \( \mu_{\xi x}^\xi \) is any other system of measures satisfying the conditions above, then \( \mu_{\xi x}^\xi = \nu_{\xi x}^\xi \) for \( \mu \)-a.e. \( x \).

We will be most interested in the following example: Given a rectangle \( R \subset \Lambda \) and a point \( x \in R \), we consider the measurable partition \( \xi \) of \( R \) by unstable sets of the form \( V_u^R(x) = V_{\text{loc}}^u(x) \cap R \).

Let \( \{ \mu_{x}^u \}_{x \in R} \) denote the corresponding system of conditional measures; given a partition element \( V = V_R^u(x) \), we may also write \( \mu_V = \mu_{x}^u \). Let \( \tilde{\mu} \) denote the factor-measure on \( R/\xi \) defined by \( \tilde{\mu}(E) = \mu(\bigcup_{V \in E} V) \). Then for every \( \psi \in L^1(R, \mu) \), we have

\[
\int_{R} \psi \, d\mu = \int_{R/\xi} \int_V \psi(z) \, d\mu_V(z) \, d\tilde{\mu}(V).
\]

Note that the factor space \( R/\xi \) can be identified with the set \( V_R^{cs}(x) = V_{\text{loc}}^{cs}(x) \cap R \) for any \( x \in R \), so that the measure \( \tilde{\mu} \) can be viewed as a measure on this set, in which case (2.5).
becomes
\begin{equation}
\int_R \psi \, d\mu = \int_{V_R^u(x)} \int_{V_R^u(y)} \psi(z) \, d\mu^u_y(z) \, d\mu(y).
\end{equation}

Note that the conditional measures \( \mu^u_x \) depend on the choice of rectangle \( R \), although this is not reflected in the notation. In fact the ambiguity only consists of a normalizing constant, as the following lemma shows.

**Lemma 2.7.** Let \( \mu \) be a Borel measure on \( \Lambda \), and let \( R_1, R_2 \subset \Lambda \) be rectangles. Let \( \{ \mu^1_x \}_{x \in R_1} \) and \( \{ \mu^2_x \}_{x \in R_2} \) be the corresponding systems of conditional measures on the unstable sets \( V_R^u(x) \). Then for \( \mu \)-a.e. \( x \in R_1 \cap R_2 \), the measures \( \mu^1_x \) and \( \mu^2_x \) are scalar multiples of each other when restricted to \( V_R^u(x) \cap V_R^u(x) \).

See Appendix A for a proof of Lemma 2.7 and the following corollary.

**Lemma 2.8.** If \( \mu \) is an \( f \)-invariant Borel measure on \( \Lambda \), then for \( \mu \)-a.e. \( x \in \Lambda \) and any choice of two rectangles containing \( x \) and \( f(x) \), the corresponding systems of conditional measures are such that \( f_* \mu^u_x \) is a scalar multiple of \( \mu^u_{f(x)} \) on the intersection of the corresponding unstable sets.

In particular, Lemma 2.8 demonstrates that statements of the form “such-and-such a property holds for \( \mu^u_x \)-a.e. \( y \in V^u_{\text{loc}}(x) \)” do not depend on the choice of rectangle used to define \( \mu^u_x \).

For the next definition, we recall that two measures \( \nu, \mu \) are said to be equivalent if \( \nu \ll \mu \) and \( \mu \ll \nu \); in this case we write \( \nu \sim \mu \). Also, given a rectangle \( R \), a point \( p \in R \), and measures \( \nu^u_p, \nu^{cs}_p \) on \( V^u_R(p), V^{cs}_R(p) \) respectively, we can define a measure \( \nu = \nu^u_p \otimes \nu^{cs}_p \) on \( R \) by \( \nu([A, B]) = \nu^u_p(A) \nu^{cs}_p(B) \) for \( A \subset V^u_R(p) \) and \( B \subset V^{cs}_R(p) \). The following lemma is proved in Appendix A.

**Lemma 2.9.** Let \( R \) be a rectangle and \( \mu \) a measure with \( \mu(R) > 0 \). Then the following are equivalent.

1. \( (\pi^u_y)_z, \mu^u_y \ll \mu^u_z \) for \( \mu \)-a.e. \( y, z \in R \).
2. \( (\pi^u_z)_y, \mu^u_y \sim \mu^u_z \) for \( \mu \)-a.e. \( y, z \in R \).
3. \( (\pi^{cs}_y)_z, \mu^{cs}_y \ll \mu^{cs}_z \) for \( \mu \)-a.e. \( y, z \in R \).
4. \( (\pi^{cs}_z)_y, \mu^{cs}_y \sim \mu^{cs}_z \) for \( \mu \)-a.e. \( y, z \in R \).
5. there exist \( p \in R \) and measures \( \tilde{\mu}_p^u, \tilde{\mu}_p^{cs} \) on \( V^u_R(p), V^{cs}_R(p) \) such that \( \mu|_R \sim \tilde{\mu}_p^u \otimes \tilde{\mu}_p^{cs} \).
6. \( \mu|_R \sim \mu^u_y \otimes \mu^{cs}_y \) for \( \mu \)-a.e. \( y \in R \).

**Definition 2.10.** A measure \( \mu \) on \( \Lambda \) has local product structure if for any rectangle \( R \subset \Lambda \) with \( \mu(R) > 0 \), one (and hence all) of the conditions in Lemma 2.9 holds.

### 2.4. Equilibrium measures

Let \( \varphi : \Lambda \to \mathbb{R} \) be a continuous function, which we call a potential function. Given an integer \( n \geq 0 \), the dynamical metric of order \( n \) is
\begin{equation}
d_n(x, y) = \max \{ d(f^k x, f^k y) : 0 \leq k < n \}
\end{equation}
and for each \( r > 0 \), the associated Bowen balls are given by
\begin{equation}
B_n(x, r) = \{ y : d_n(x, y) < r \}.
\end{equation}
A set $E \subset \Lambda$ is said to be $(n, r)$-separated if $d_n(x, y) \geq r$ for all $x \neq y \in E$, and $(n, r)$-spanning for $X \subset M$ if $X \subset \bigcup_{x \in E} B_n(x, r)$.

Let $S_n\varphi(x) = \sum_{k=0}^{n-1} \varphi(f^k x)$ denote the $n$th Birkhoff sum along the orbit of $x$. The partition sums of $\varphi$ on a set $X \subset M$ are the following quantities:

$$Z_n^\text{span}(X, \varphi, r) := \inf \left\{ \sum_{x \in E} e^{S_n \varphi(x)} : E \subset X \text{ is } (n, r)\text{-spanning for } X \right\},$$

$$Z_n^\text{sep}(X, \varphi, r) := \sup \left\{ \sum_{x \in E} e^{S_n \varphi(x)} : E \subset X \text{ is } (n, r)\text{-separated} \right\}.$$  

The topological pressure of $\varphi$ on $X \subset \Lambda$ is given by

$$P(\varphi) = \lim_{r \to 0} \lim_{n \to \infty} \frac{1}{n} \log Z_n^\text{span}(\Lambda, \varphi, r) - \lim_{r \to 0} \lim_{n \to \infty} \frac{1}{n} \log Z_n^\text{sep}(\Lambda, \varphi, r),$$

where equality of the limits follows from [38, Theorem 9.4].

Denote by $\mathcal{M}(f)$ the set of $f$-invariant Borel probability measures on $\Lambda$. The variational principle [38, Theorem 9.10] establishes that

$$P(\varphi) = \sup_{\mu \in \mathcal{M}(f)} \left\{ h_{\mu}(f) + \int \varphi \, d\mu \right\}.$$  

We call a measure $\mu \in \mathcal{M}(f)$ an equilibrium measure for $\varphi$ if it achieves the supremum in (2.11).

We say that a measure $\mu \in \mathcal{M}(f)$ is a Gibbs measure (or that $\mu$ has the Gibbs property) with respect to $\varphi$ if for every small $r > 0$ there is $Q = Q(r) > 0$ such that for every $x \in \Lambda$ and $n \in \mathbb{N}$, we have

$$Q^{-1} \leq \frac{\mu(B_n(x, r))}{\exp(-P(\varphi)n + S_n \varphi(x))} \leq Q.$$  

A straightforward computation with partition sums shows that every Gibbs measure for $\varphi$ is an equilibrium measure for $\varphi$; however, the converse is not true in general, and there are examples of systems and potentials with equilibrium measures that do not satisfy the Gibbs property.

3. Carathéodory dimension structure

We recall the Carathéodory dimension construction described in [29, §10], which generalizes the definition of Hausdorff dimension and measure.

3.1. Carathéodory dimension and measure. A Carathéodory dimension structure, or C-structure, on a set $X$ is given by the following data.

1. An indexed collection of subsets of $X$, denoted $\mathcal{F} = \{ U_s : s \in \mathcal{S} \}$.
2. Functions $\xi, \eta, \psi : \mathcal{S} \to [0, \infty)$ satisfying the following conditions:
   (H1) if $U_s = \emptyset$, then $\eta(s) = \psi(s) = 0$; if $U_s \neq \emptyset$, then $\eta(s) > 0$ and $\psi(s) > \varepsilon$;
   (H2) for any $\delta > 0$ one can find $\varepsilon > 0$ such that $\eta(s) \leq \delta$ for any $s \in \mathcal{S}$ with $\psi(s) \leq \varepsilon$;

\[\text{In [29, Condition [H1]] includes the requirement that there is } s_0 \in \mathcal{S} \text{ such that } U_{s_0} = \emptyset, \text{ but this can safely be omitted, since we can always formally enlarge our collection by adding the empty set, without changing any of the definitions below.}\]
(H3) for any $\epsilon > 0$ there exists a finite or countable subcollection $\mathcal{G} \subset \mathcal{S}$ that covers $X$ (meaning that $\bigcup_{s \in \mathcal{G}} U_s \supset X$) and has $\psi(\mathcal{G}) := \sup\{\psi(s) : s \in \mathcal{S}\} \leq \epsilon$.

Note that no conditions are placed on $\xi$.

The $C$-structure $(\mathcal{S}, \mathcal{F}, \xi, \eta, \psi)$ determines a one-parameter family of outer measures on $X$ as follows. Fix a nonempty set $Z \subset X$ and consider some $\mathcal{G} \subset \mathcal{S}$ that covers $Z$ as in (H3). Interpreting $\psi(\mathcal{G})$ as the largest size of sets in the cover, we can define for each $\alpha \in \mathbb{R}$ an outer measure on $X$ by

$$m_C(Z, \alpha) := \liminf_{\varepsilon \to 0} \inf_{\mathcal{G} \in \mathcal{S}} \sum_{s \in \mathcal{G}} \xi(s)\eta(s)^\alpha,$$

where the infimum is taken over all finite or countable $\mathcal{G} \subset \mathcal{S}$ covering $Z$ with $\psi(\mathcal{G}) \leq \epsilon$.

Defining $m_C(\emptyset, \alpha) := 0$, this gives an outer measure by [29, Proposition 1.1]. The measure induced by $m_C(\cdot, \alpha)$ on the $\sigma$-algebra of measurable sets is the $\alpha$-Carathéodory measure; it need not be $\sigma$-finite or non-trivial.

**Proposition 3.1** ([29 Proposition 1.2]). For any set $Z \subset X$ there exists a critical value $\alpha_C \in \mathbb{R}$ such that $m_C(Z, \alpha) = \infty$ for $\alpha < \alpha_C$ and $m_C(Z, \alpha) = 0$ for $\alpha > \alpha_C$.

We call $\dim_C Z = \alpha_C$ the Carathéodory dimension of the set $Z$ associated to the $C$-structure $(\mathcal{S}, \mathcal{F}, \xi, \eta, \psi)$. By Proposition 3.1 $\alpha = \dim_C X$ is the only value of $\alpha$ for which (3.1) can possibly produce a non-zero finite measure on $X$, though it is still possible that $m_C(X, \dim_C X)$ is equal to 0 or $\infty$.

### 3.2. A C-structure on local unstable leaves

Given a potential $\varphi$, a number $r > 0$, and a point $x \in \Lambda$, we define a $C$-structure on $X = V_{loc}^u(x) \cap \Lambda$ in the following way. For our index set we put $\mathcal{S} = X \times \mathbb{N}$, and to each $s = (x, n) \in X \times \mathbb{N}$, we associate the $u$-Bowen ball

$$U_s = B_n^u(x, r) = B_n(x, r) \cap V_{loc}^u(x);$$

then $\mathcal{F}$ is the collection of all such balls. Set

$$\xi(x, n) = e^{S_n\varphi(x)}, \quad \eta(x, n) = e^{-n}, \quad \psi(x, n) = \frac{1}{n}.$$ 

It is easy to see that $(\mathcal{S}, \mathcal{F}, \xi, \eta, \psi)$ satisfies (H1)-(H3) and defines a $C$-structure, whose associated outer measure is given by

$$m_C(Z, \alpha) = \lim_{N \to \infty} \inf_{\mathcal{G}} \sum_{(x, n) \in \mathcal{G}} e^{S_n\varphi(x)}e^{-n\alpha},$$

where the infimum is over all $\mathcal{G} \subset \mathcal{S}$ such that $\bigcup_{(x, n) \in \mathcal{G}} B_n^u(x, r) \supset Z$ and $n \geq N$ for all $(x, n) \in \mathcal{G}$.

Given $x \in \Lambda$ and the corresponding $X = V_{loc}^u(x) \cap \Lambda$, we are interested in computing

1. the Carathéodory dimension of $X$, as determined by this $C$-structure;
2. the (outer) measure on $X$ defined by (3.4) at $\alpha = \dim_C X$.

We settle the first problem in Theorem 4.2 below in which we prove (among other things) that for small $r$, under the assumptions (C1)-(C2) on the map $f$ and some regularity assumptions on the potential function $\varphi$ (see Section 4.1), the $C$-structure defined...
on $X = V^u_{\text{loc}}(x) \cap \Lambda$ as above satisfies $\dim_C X = P(\varphi)$ for every $x \in \Lambda$. In particular, $P_X(\varphi) = P(\varphi)$. This allows us to consider the outer measure on $X$ given by

$$m^c_x(Z) := m_C(Z, P(\varphi)) = \lim_{N \to \infty} \inf \sum_i e^{-n_i P(\varphi)} e^{S_{n_i} \varphi(x_i)},$$

where the infimum is taken over all collections $\{B^u_{n_i}(x_i, r_i)\}$ of $u$-Bowen balls with $x_i \in V^u_{\text{loc}}(x) \cap \Lambda$, $n_i \geq N$, which cover $Z$; for convenience we write $C = (\varphi, r)$ to keep track of the data on which the reference measure depends.

One must do some work to show that this outer measure is finite and nonzero; we do this in §§4.1.6.1–6.5. We also show that this measure is Borel.

4. Main results

4.1. Assumptions on the map and the potential. As in [2] let $f: U \to M$ be a diffeomorphism onto its image, where $M$ is a compact smooth Riemannian manifold and $U \subset M$ is open, and suppose that $\Lambda \subset U$ is a compact $f$-invariant set on which $f$ is partially hyperbolic in the broad sense, with $T \Lambda M = E^c \oplus E^s$. Suppose moreover that [C1] and [C2] are satisfied, so that $\|df|E^c\| \leq 1$ and $f|\Lambda$ is topologically transitive. Finally, suppose that the local product structure conditions [A1]–[A4] are satisfied.

A potential function $\varphi: \Lambda \to \mathbb{R}$ is said to have the $u$-Bowen property if there exists $Q_u > 0$ such that for every $x \in \Lambda$, $n \geq 0$, and $y \in B^u_n(x, r) \cap \Lambda$, we have $|S_n \varphi(x) - S_n \varphi(y)| \leq Q_u$. Similarly, we say that $\varphi$ has the $cs$-Bowen property if there exists $Q_{cs} > 0$ such that for every $x \in \Lambda$, $n \geq 0$, and $y \in V^c_{\text{loc}}(x)$, we have $|S_n \varphi(x) - S_n \varphi(y)| \leq Q_{cs}$. Let $C_B(\Lambda)$ be the set of all functions $\varphi: \Lambda \to \mathbb{R}$ that satisfy both the $u$- and $cs$-Bowen properties.

Remark 4.1. As mentioned in Remark 2.3, all of the conditions in the first paragraph above are satisfied if $\Lambda$ is a transitive locally maximal hyperbolic set for $f$. Moreover, in this case it follows from [15, Lemma 6.6] that $C_B(\Lambda)$ contains every Hölder continuous potential function.

4.2. Statements of main results. From now on we fix $f$, and $\varphi \in C_B(\Lambda)$ as described above. Our first result, which we prove in [6] shows that the measure $m^c_x$ defined in (3.3) is finite and nonzero.

Theorem 4.2. Fix $0 < r < \tau/3$. There is $K > 0$ such that for every $x \in \Lambda$, the following are true.

1. For the $C$-structure defined on $X = V^u_{\text{loc}}(x) \cap \Lambda$ by $u$-Bowen balls $B^u_n(x, r)$ and (3.3), we have $\dim_C X = P(\varphi)$ for every $x \in \Lambda$. In particular, $P_X(\varphi) = P(\varphi)$.
2. $m^c_x$ is a Borel measure on $X := V^u_{\text{loc}}(x) \cap \Lambda$.
3. $m^c_x(V^u_{\text{loc}}(x) \cap \Lambda) \in [K^{-1}, K]$.
4. If $V^u_{\text{loc}}(x) \cap V^u_{\text{loc}}(y) \cap \Lambda \neq \emptyset$, then $m^c_x$ and $m^c_y$ agree on the intersection.

Definition 4.3. Consider a family of measures $\{\mu_x : x \in \Lambda\}$ such that $\mu_x$ is supported on $V^u_{\text{loc}}(x)$. We say that this family has the $u$-Gibbs property\footnote{Note that this is a different notion than the idea of $u$-Gibbs state from [27].} with respect to the potential
function $\varphi: \Lambda \to \mathbb{R}$ if there is $Q_0 = Q_0(r) > 0$ such that for all $x \in \Lambda$ and $n \in \mathbb{N}$, we have
\begin{equation}
Q_0^{-1} \leq \frac{\mu_x(B^n(x, r))}{e^{-nP(\varphi)} + \delta_n(x)} \leq Q_0.
\end{equation}

The following two results are proved in (4.1) the first establishes the scaling properties of the measures $m^C_x$ under iteration by $f$, which then leads to the $u$-Gibbs property.

**Theorem 4.4.** For every $x \in \Lambda$, we have $f^*m^C_{f(x)} := m^C_{f(x)} \circ f \ll m^C_x$, with Radon–Nikodym derivative $e^{P(\varphi) - \varphi}$.

**Corollary 4.5.** The family of measures $\{m^C_x\}_{x \in \Lambda}$ has the $u$-Gibbs property. In particular, for every relatively open $U \subset V^u_{\text{loc}}(x) \cap \Lambda$, we have $m^C_U(U) > 0$.

Given a rectangle $R$ and points $y, z \in R$, let $\pi_{yz}: V^u_R(y) \to V^u_R(z)$ be the holonomy map from Definition 2.6. We say that $\pi_{yz}$ is absolutely continuous with respect to the system of measures $m^C_y$ if the pullback measure $\pi_{yz}^*m^C_z$ is equivalent to the measure $m^C_y$ for every $y, z \in \Lambda$.

In this case the Jacobian of $\pi_{yz}$ is the function $\text{Jac} \pi_{yz}: V^u_R(y) \to (0, \infty)$ defined by the following Radon–Nikodym derivative:
\[ \text{Jac} \pi_{yz} = \frac{d\pi_{yz}^*m^C_z}{dm^C_y}. \]

**Theorem 4.6.** The holonomy map is absolutely continuous with respect to the system of measures $m^C_y$. Moreover, there is $C > 0$ such that for every rectangle $R$ and every $y, z \in R$, the Jacobian of $\pi_{zy}$ satisfies $C^{-1} \leq \text{Jac} \pi_{zy}(x) \leq C$ for $m^C_y$-a.e. $x$.

The measure $m^C_x$ can be extended to a measure on $\Lambda$ by taking $m^C_x(A) := m^C_x(A \cap V^u_{\text{loc}}(x))$ for any Borel set $A \subset \Lambda$. We consider the evolution of (the normalization of) this measure by the dynamics; that is the sequence of measures
\begin{equation}
\mu_n := \frac{1}{n} \sum_{k=0}^{n-1} f^km^C_x(V^u_{\text{loc}}(x)).
\end{equation}

Our main result is the following, which we prove in (8).

**Theorem 4.7.** Under the conditions in (4.1), the following are true.

1. For every $x \in \Lambda$, the sequence of measures from (4.2) is weak* convergent as $n \to \infty$ to a limiting probability measure $\mu_\varphi$, which is independent of $x$.

2. The measure $\mu_\varphi$ is ergodic, gives positive weight to every open set in $\Lambda$, has the Gibbs property (2.12) and is the unique equilibrium measure for $(\Lambda, f, \varphi)$.

3. For every rectangle $R \subset \Lambda$ with $\mu_\varphi(R) > 0$, the conditional measures $\mu^u_{y}$ generated by $\mu_\varphi$ on unstable sets $V^u_R(y)$ are equivalent for $\mu_\varphi$-almost every $y \in R$ to the reference measures $m^C_y|_{V^u_R(y)}$. Moreover, there exists $C_0 > 0$, independent of $R$ and $y$, such that for $\mu_\varphi$-almost every $y \in R$ we have
\begin{equation}
C_0^{-1} \leq \frac{d\mu^u_y}{dm^C_y}(z)m^C_y(R) \leq C_0 \text{ for } \mu^u_y\text{-a.e. } z \in V^u_R(y).
\end{equation}

This looks similar to the notion of local product structure in Lemma 2.9 but the difference here is that the system of measures $m^C_y$ are not assumed to arise as conditional measures for some Borel measure on $\Lambda$. 

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This text snippet has been formatted for readability and clarity. It contains mathematical expressions and theorems that are central to the study of dynamical systems and measures. Theorems and corollaries are clearly delineated, with references to earlier results and definitions. The text is self-contained, providing a logical flow of ideas and proofs, which is crucial for understanding the developments in the field of dynamical systems.
(4) The measure $\mu_\varphi$ has local product structure as in Definition 2.10.

**Remark 4.8.** In particular, the zero potential always satisfies the $u$- and cs-Bowen properties, and thus Theorem 4.7 demonstrates that every partially hyperbolic set satisfying (C1) (C2) and (A1) (A4) has a unique measure of maximal entropy, which can be obtained via the “pushforward and average” procedure applied to the appropriate reference measure on any unstable leaf, and which has the Gibbs property and local product structure.

5. Applications

5.1. Time-1 map of an Anosov flow.

**Definition 5.1.** A $C^1$ flow $f^t: M \to M$ on a smooth compact manifold $M$ is called an Anosov flow if there exists a Riemannian metric and a number $0 < \lambda < 1$ such that the tangent bundle splits into three subbundles $TM = E^s \oplus E^c \oplus E^u$, each invariant under the flow such that

1. $\frac{d}{dt} f^t(x)\big|_{t=0} \in E^c(x) \setminus \{0\}$ and $\dim E^c(x) = 1$;
2. $\|Df^t\| E^s \leq \lambda^t$ and $\|Df^{-t}\| E^u \leq \lambda^t$ for all $t \geq 0$.

It is well known that if an Anosov flow $f^t$ is of class $C^r$, $r \geq 1$, then for each $x \in M$ there are a pair of embedded $C^r$-discs $W^s(x)$ and $W^u(x)$ called local strong stable and unstable manifolds, and a number $C > 0$ such that

1. $T_x W^s(x) = E^s(x)$ and $T_x W^u(x) = E^u(x)$;
2. if $y \in W^u(x)$, then $d(f^{-t}(x), f^{-t}(y)) \leq C \lambda^t d(x, y)$ for all $t \geq 0$;
3. if $y \in W^s(x)$, then $d(f^t(x), f^t(y)) \leq C \lambda^t d(x, y)$ for all $t \geq 0$.

We define weak-unstable and weak-stable manifolds through $x$ by

$$W^{uc} = \bigcup_{t \in (-r,r)} W^u(f^t(x)), \quad W^{sc} = \bigcup_{t \in (-r,r)} W^s(f^t(x)).$$

Given an Anosov flow $f^t: M \to M$ one can define a diffeomorphism $f: M \to M$ to be the time-1 map of the flow. That is, $f(x) := f^1(x)$. Observe that such an $f$ is partially hyperbolic in the broad sense with $\Lambda = M$ and $E^{cs} = E^c \oplus E^s$ and satisfies Assumption (C1) from (2) moreover, conditions (A1) (A4) on integrability of $E^{cs}$ and local product structure are satisfied by considering the weak-stable manifolds defined above. We stress that even when the flow is known to have a unique equilibrium measures for a certain potential function, this does not automatically imply uniqueness for the time-1 map.

Given a Hölder continuous potential $\psi: M \to \mathbb{R}$, consider the corresponding potential $\varphi(x) := \int_0^1 \psi(f^t(x)) dt$. We have the following.

**Theorem 5.2.** Let $f^t: M \to M$ be an Anosov flow on a smooth compact manifold $M$. Let $f = f^1$ be the time-1 map of $f^t$, and let $m^C_x$ be the reference measures on $V^u_{loc}(x)$ associated to the potential function $\varphi = \int_0^1 \psi \circ f^t \, dt$. If $f$ is topologically transitive, then the following statements hold:

1. For every $x \in M$, the sequence of measures $\mu_n := \frac{1}{n} \sum_{k=0}^{n-1} f^k m^C_x / m^C_x(V^u_{loc}(x))$ is weak$^*$ convergent as $n \to \infty$ to a measure $\mu_\varphi$, which is independent of $x$.
2. The measure $\mu_\varphi$ is ergodic, gives positive weight to every open set in $M$, has the Gibbs property (2.12) and is the unique equilibrium state for $(M, f, \varphi)$. 

Remark 5.3. Theorem 5.2 applies to the geometric potential \( q\varphi \) to all its scalar multiples. This completes the proof of the theorem. □

Using the local product structure of the flow, this shows that \( \varphi \) is the unique measure of maximal entropy for the flow has a unique measure of maximal entropy.

(4) The measure \( \mu \) has local product structure.

Proof. It is enough to check that \( \varphi \in C_B(M) \) and then apply Theorem 4.7. Since the function \( \psi \) is Hölder continuous, so is \( \varphi \). In particular, \( \varphi \) satisfies Bowen’s property along strong stable and unstable leaves. Moreover, \( \varphi \) satisfies Bowen’s property along the flow direction: indeed, consider two points \( x = x(0) \in M \) and \( y = x(\epsilon) \) for some \( \epsilon > 0 \). We have that

\[
|S_n\varphi(x) - S_n\varphi(y)| = \left| \int_0^n \psi(f^\tau(x))d\tau - \int_0^n \psi(f^\tau(y))d\tau \right|
\]

\[
= \left| \int_0^n \psi(f^\tau(x))d\tau - \int_0^{n+\epsilon} \psi(f^\tau(x))d\tau \right|
\]

\[
= \left| \int_0^{\epsilon} \psi(f^\tau(x))d\tau - \int_{n}^{n+\epsilon} \psi(f^\tau(x))d\tau \right| \leq 2\epsilon\|\psi\|_{\infty}.
\]

Using the local product structure of the flow, this shows that \( \varphi \in C_B(M) \), and thus completes the proof of the theorem. □

Remark 5.3. Theorem 5.2 applies to the geometric potential \( \varphi(x) = -\log \det Df|_{E^u(x)} \) and to all its scalar multiples \( q\varphi \) for \( q \in \mathbb{R} \); indeed, taking \( \psi(x) = \lim_{t\to 0} -\frac{1}{t} \log \det Df^t|_{E^u(x)} \), we have \( \varphi = \int_0^1 \psi \circ f^\tau d\tau \), and \( \psi \) is Hölder continuous because the distribution \( E^u \) is Hölder continuous. When \( q = 0 \) the measure produced in Theorem 5.2 is the unique measure of maximal entropy; when \( q = 1 \) it is the unique u-measure.

We mention two alternate approaches to existence and/or uniqueness of equilibrium states in the setting of Theorem 5.2. (Neither of these, however, addresses the structure of \( \mu \) as described in the last two statements of the theorem.) First, the time-1 map of an Anosov flow has the entropy expansivity property [19], which implies existence of an equilibrium measure for \( f \) with respect to any continuous potential function \( \varphi \). However, this approach does not say anything about uniqueness.

A second approach, which gives uniqueness in the case when the flow is topologically mixing and \( \varphi = 0 \), deduces uniqueness of the measure of maximal entropy for \( f \) from the uniqueness of the measure of maximal entropy for the flow \( f^t \). It follows from [7] that the flow has a unique measure of maximal entropy \( \mu \), and that \( (f^t, \mu) \) is mixing. We claim that \( \mu \) is the unique measure of maximal entropy for \( f = f^1 \). First note that since \( (f^t, \mu) \) is mixing, then \( (f, \mu) \) is ergodic. Now if \( \nu \) is any measure of maximal entropy for \( f = f^1 \), then the measure \( \int_0^1 f^t\nu dt \) is invariant under the flow and is the measure of maximal entropy, since for every \( t \) the measure \( f^t\nu \) is \( f \)-invariant and has the same (maximal) entropy as

\[^5\]We would like to thank F. Rodriguez Hertz for providing us with this argument.
\[5.2\text{. Time-1 map of the frame flow.}

Let \( M \) be a closed oriented \( n \)-dimensional manifold of negative sectional curvature. Consider the unit tangent bundle \( SM = \{(x,v) : x \in M, \; v \in T_x M, \; \|v\| = 1\} \), and the frame bundle

\[ FM = \{(x,v_0,v_1,\ldots,v_{n-1}) : x \in M, \; v_i \in T_x M, \]
\[ \text{and } \{v_0,\ldots,v_{n-1}\} \text{ is a positively oriented orthonormal frame at } x \}. \]

The geodesic flow \( g^t : SM \to SM \) is defined by

\[ g^t(x,v) = (\gamma(x,v)(t), \dot{\gamma}(x,v)(t)), \]

where \( \gamma(x,v)(t) \) is the unique geodesic determined by the vector \((x,v)\), and the frame flow \( f^t : FM \to FM \) is given by

\[ f^t(x,v_0,v_1,\ldots,v_{n-1}) = (g^t(x,v_0), \Gamma^t_\gamma(v_1),\ldots,\Gamma^t_\gamma(v_{n-1})), \]

where \( \Gamma^t_\gamma \) is the parallel transport along the geodesic \( \gamma(x,v_0) \). The flow \( f^t \) is partially hyperbolic with splitting \( TFM = E^s \oplus E^c \oplus E^u \), where \( \dim E^c = \dim SO(n-1) \) and \( f^t \) acts isometrically on the center bundle.

Let \( f = f^1 \) be the time-1 map of \( f^t \). As in the previous section, existence of a unique equilibrium measure for the flow does not a priori imply uniqueness for the time-1 map.

Consider a Hölder continuous potential \( \psi : FM \to \mathbb{R} \) which is constant on each \( N_x \), where \( N_x \) is the space of positively oriented orthonormal \( n \)-frames in \( T_x M \). For each such \( \psi \) we consider the corresponding potential \( \varphi(v) := \int_0^1 \psi(f^t(v)) \, d\tau \), where we write \( v = (x,v_0,v_1,\ldots,v_{n-1}) \) for an element of \( FM \). Repeating the argument in the proof of Theorem 5.2, one can show that \( \varphi \in C_B(FM) \).

The results in our paper apply to \( f \) and \( \varphi \) if \( f \) is topologically transitive. To this end, note that the frame flow \( f^t \) preserves a smooth measure that is locally the product of the Liouville measure with normalized Haar measure on \( SO(n-1) \). There are several cases in which the time-1 map \( f \) is known to be ergodic with respect to this measure, and hence topologically transitive by [24, Proposition 4.1.18].

**Proposition 5.4 ([10, Theorem 0.2]).** Let \( f^t \) be the frame flow on an \( n \)-dimensional compact smooth Riemannian manifold with sectional curvature between \(-\Lambda^2\) and \(-\lambda^2\) for some \( \Lambda, \lambda > 0 \). Then in each of the following cases the flow and its time-1 map are ergodic:

- if the curvature is constant,
- for a set of metrics of negative curvature which is open and dense in the \( C^3 \) topology,
- if \( n \) is odd and \( n \neq 7 \),
- if \( n \) is even, \( n \neq 8 \), and \( \lambda/\Lambda > 0.93 \),
- if \( n = 7 \) or \( 8 \) and \( \lambda/\Lambda > 0.99023 \ldots \).

We therefore have the following.

**Theorem 5.5.** Under the hypotheses of Proposition 5.4, the following statements hold:
For every \( v \in FM \), the sequence of measures \( \mu_n := \frac{1}{n} \sum_{k=0}^{n-1} f^k m^c_v / m^c_v (V^u_{loc} (v)) \) from (4.2) is weak* convergent as \( n \to \infty \) to a measure \( \mu_\varphi \), which is independent of \( v \).

(2) The measure \( \mu_\varphi \) is ergodic, gives positive weight to every open set in \( FM \), has the Gibbs property (2.12) and is the unique equilibrium state for \( (FM, f, \varphi) \).

(3) For every rectangle \( R \subset FM \) with \( \mu_\varphi (R) > 0 \), the conditional measures \( \mu_\varphi^v \) generated by \( \mu_\varphi \) on unstable sets \( V^u_R(v) \) are equivalent for \( \mu_\varphi \)-a.e. \( v \in R \) to the reference measures \( m^c_v V^u_R(v) \). Moreover, there exists \( C_0 > 0 \), independent of \( R \) and \( v \), such that for \( \mu_\varphi \)-a.e. \( v \in R \) we have

\[
C_0^{-1} \leq \frac{d\mu_\varphi^v}{dm^c_v} (w) m^c_v (R) \leq C_0 \text{ for } \mu_\varphi^v \text{-a.e. } w \in V^u_R(v).
\]

(4) The measure \( \mu_\varphi \) has local product structure.

**Remark 5.6.** Existence and uniqueness of equilibrium measures for the frame flow and a Hölder continuous potential which is constant on fibers \( N_x \) was shown in [36]; in particular, the flow has a unique measure of maximal entropy. Without a mixing result for this MME, though, the argument from the previous section does not immediately imply uniqueness of an MME for the time-1 map.

### 5.3. Compact center leaves.

Let \( M \) be a compact smooth manifold and \( U \subset M \) an open set. Let \( f: U \to M \) be a diffeomorphism onto its image and \( \Lambda \subset U \) a compact invariant set on which \( f \) is topologically transitive and which has a partially hyperbolic splitting \( T_\Lambda M = E^u \oplus E^c \oplus E^s \), where \( E^u \) and \( E^s \) are uniformly expanding and contracting, respectively. Suppose moreover that the center distribution \( E^c \) is integrable, that all global center leaves are compact, and that \( \|Df|_{E^c}\| \leq 1 \).

**Theorem 5.7.** Let \( f, \Lambda \) be as above and let \( \varphi: \Lambda \to \mathbb{R} \) be a Hölder continuous function that is constant on each center leaf. Then there is \( C > 0 \) such that for every \( x \in \Lambda \), the measures \( m^c_x = m^c_x (\cdot, P(\varphi)) \) on \( X = V^u_{loc} (x) \cap \Lambda \) satisfies

1. \( C^{-1} < m^c_x (V^u_{loc} (x) \cap \Lambda) < C \);
2. \( C^{-1} < \text{Jac} \pi_{x,y} (z) < C \) for all rectangles \( R \) and \( x, y \in R \), \( z \in V^u_R (x) \);
3. the sequence (4.2) converges to a unique equilibrium measure \( \mu = \mu_\varphi \) for \( \varphi \);
4. for \( \mu \)-almost every \( y \in M \) we have that the conditional measure \( \mu_y^v \) is equivalent to \( m^c_y \), with uniform bounds as in (4.3);
5. \( \mu \) has a local product structure.

In particular, Theorem 5.7 applies to topologically transitive skew products with isometric fiber actions over uniformly hyperbolic sets, for all Hölder continuous potential functions that are constant along fibers.

**Remark 5.8.** In the skew product setting one can give an alternate proof of existence and uniqueness by taking \( \Phi (x) = \varphi (x, \cdot) \) to be the obvious potential on the base; however, the other properties of the measure \( \mu_\varphi \) stated in Theorem 5.7 are new.

### 6. Basic properties of reference measures

Now we begin to prove the results from §4 starting with Theorem 4.2 in this section, and the remaining results in §§7–8.
Proof. Smale bracket \([x, y]\) such that \(f \in (6.2)\). Uniform transitivity of local unstable leaves.

That \(E \in E, F\) any there is \(\Delta\) to prove Statement (2) it suffices to show that \(\text{Lemma 6.2.}\) For every \(\forall \), \(\text{Proposition 6.1.}\) Proposition 6.1. For every \(\forall \) in \(\forall \) in \(\forall \) in \(\forall \) in \(\text{many of the techniques used in the proof of Proposition 6.1 are adapted from Bowen’s paper \([5]\); the underlying principle is that if } Z_n \text{ is a ‘nearly multiplicative’ sequence of numbers satisfying } Z_{n+k} = Q^{\pm 1} Z_n Z_k \text{ for some } Q \text{ independent of } n, k, \text{ then } P = \lim_{n \to \infty} \frac{1}{n} \log Z_n \text{ exists and } Z_n = Q^{\pm 1} e^{nP} \text{ (see \([15, \text{lemmas } 6.2–6.4]\) for a proof of this elementary fact). The proofs here are more involved than those in \([5]\) because the partition sums in (6.1) actually depend on } x, r, \text{ so } \text{we must control how they vary when these parameters are changed.}

6.1. Reference measures are Borel. An outer measure \(m\) on a metric space \((X, d)\) is said to be a \textit{metric outer measure} if \(m(E \cup F) = m(E) + m(F)\) whenever \(d(E, F) := \inf\{d(x, y) : x \in E, y \in F\} > 0\). By \([22, 2.3.2(9)]\), every metric outer measure is Borel, so to prove Statement (2) it suffices to show that \(m^c\) is metric. To this end, note that given \(x \in \Lambda\) and \(y \in X = V^c_{\text{loc}}(x) \cap \Lambda\), we have \(d_B^n(y, r) \leq r\lambda^n \to 0\) as \(n \to \infty\), and thus for any \(E, F \subset X\) with \(d(E, F) > 0\), there is \(N \in \mathbb{N}\) such that \(B_n(y, r) \cap B_k(z, r) = \emptyset\) whenever \(y \in E, z \in F,\) and \(k, n \geq N\). In particular, for this (and larger) \(N\), every \(\mathcal{G}\) used in (3.4) splits into two disjoint subsets, one that covers \(E\) and one that covers \(F\), which implies that \(m^c(E \cup F) = m^c(E) + m^c(F)\), so \(m^c\) is a metric outer measure.

6.2. Uniform transitivity of local unstable leaves. For convenience, given \(x \in \Lambda\) and \(\delta \in (0, \tau)\) we will write

\[
B_{\Lambda}(x, \delta) = B(x, \delta) \cap \Lambda \quad \text{and} \quad B^u_{\Lambda}(x, \delta) := B^u(x, \delta) \cap \Lambda = B(x, \delta) \cap V^u_{\text{loc}}(x) \cap \Lambda.
\]

Later on we will need the following consequence of topological transitivity.

Lemma 6.2. For every \(\delta > 0\) there is \(n \in \mathbb{N}\) such that for every \(x, y \in \Lambda\), there is \(0 \leq k \leq n\) such that \(f^k(B^u_{\Lambda}(x, \delta)) \cap B^s(y, \delta) \neq \emptyset\).

Proof. Let \(\Delta_\epsilon = \{(x, y) \in \Lambda \times \Lambda : d(x, y) \leq \epsilon\}\), where \(\epsilon > 0\) is small enough so that the Smale bracket \([x, y] = V^s_{\text{loc}}(x) \cap V^u_{\text{loc}}(y)\) defines a continuous map \(\Delta_\epsilon \to \Lambda\). The function \(G(x, y) = \max\{|d([x, y], x), d([x, y], y)|\}\) is continuous on \(\Delta_\epsilon\) and vanishes on the diagonal \(\Delta_0\). Thus there is \(\delta_1 \in (0, \delta/2)\) such that \(d(x, y) < \delta_1\) implies \(G(x, y) < \delta/2\), and similarly there is \(\delta_2 \in (0, \delta_1/2)\) such that \(d(x, y) < \delta_2\) implies \(G(x, y) < \delta_1/2\).
Now fix \( x \in \Lambda \) and let \( U = B_\Lambda(x, \delta_2) \). Given \( n \in \mathbb{N} \), let
\[
\gamma_n := \sup \left\{ \gamma > 0 : \text{there exists } y \in \Lambda \text{ such that } B(y, \gamma) \cap \bigcup_{k=0}^{n} f^k(U) = \emptyset \right\}.
\]

If \( \gamma_n \not\to 0 \), then there are \( y_n \in \Lambda \) and \( \gamma > 0 \) such that \( B_\Lambda(y_n, \gamma) \subset \Lambda \setminus \bigcup_{k=0}^{n} f^k(U) \) for all \( n \), and thus any limit point \( y = \lim_{j \to \infty} y_{n_j} \) has \( B(y, \gamma) \cap f^k(U) = \emptyset \) for all \( k \in \mathbb{N} \), contradicting topological transitivity of \( f|_\Lambda \). Thus \( \gamma_n \to 0 \), and in particular there is \( n \in \mathbb{N} \) such that \( \gamma_n < \delta_2 \), so for every \( y \in \Lambda \), we have \( B(y, \delta_2) \cap \bigcup_{k=0}^{n} f^k(U) \neq \emptyset \).

Now given \( x, y \in \Lambda \), this argument gives \( k \in [0, n] \) and \( p \in f^k(U) \cap B(y, \delta_2) \); see Figure 6.1. Let \( q = [y, p] \), so \( q \in B^{cs}(y, \delta_1/2) \cap B^{u}(p, \delta_1/2) \) by our choice of \( \delta_2 \). It follows that
\[
f^{-k}(q) \in B^{u}(f^{-k}(p), \delta_1/2) \subset B(x, \delta_1/2 + \delta_2) \subset B(x, \delta_1).
\]

Now by our choice of \( \delta_1 \), we have \( z := [f^{-k}(q), x] \in B^{cs}(f^{-k}(q), \delta/2) \cap B^{u}(x, \delta/2) \), so \( f^k(z) \in B^{cs}(q, \delta/2) \subset B^{cs}(y, \delta) \), which proves the lemma.

6.3. Preliminary partition sum estimates. Now we need to compare the partition sums \( Z^\text{span}_n(B_\Lambda^u(x, r_1), \varphi, r_2) \) and \( Z^\text{sep}_n(B_\Lambda^u(x, r_1), \varphi, r_2) \) from (2.9) for various \( x \in \Lambda \) and \( 0 < r_1, r_2 \leq \tau \). It will be useful to note that given \( x \in \Lambda \) and \( y \in B_\Lambda^u(x, r_2) \), we have
\[
d_n(x, y) = \max_{0 \leq k \leq n} d(f^k(x), f^k(y)) = d(f^n(x), f^n(y)),
\]
so that in particular, \( B_\Lambda^u(x, r_2) = f^{-n}(B_\Lambda^u(f^n x, r_2)) \).

6.3.1. Comparing spanning and separated sets.

**Lemma 6.3.** For every \( x \in \Lambda \), \( n \in \mathbb{N} \), and \( r_1, r_2 \in (0, \tau] \) we have
\[
Z^\text{span}_n(B_\Lambda^u(x, r_1), \varphi, r_2) \leq Z^\text{sep}_n(B_\Lambda^u(x, r_1), \varphi, r_2) \leq e^{Q_n} Z^\text{span}_n(B_\Lambda^u(x, r_1), \varphi, r_2/2).
\]
Proof. If $E \subset B^u(x, r_1)$ is a maximal $(n, r_2)$-separated set, then it must be an $(n, r_2)$-spanning set as well, otherwise we could add another point to it while remaining $(n, r_2)$-separated. Thus

$$Z_{n}^{\text{span}}(B^u_\Lambda(x, r_1), \varphi, r_2) \leq \sum_{z \in E} e^{S_n \varphi(z)} \leq Z_{n}^{\text{sep}}(B^u_\Lambda(x, r_1), \varphi, r_2),$$

which proves the first inequality. Now let $F \subset B^u_\Lambda(x, r_1)$ be any $(n, r_2/2)$-spanning set. Given any $(n, r_2)$-separated set $E \subset B^u_\Lambda(x, r_1)$, every $z \in E$ has a point $y(z) \in F \cap B^u_\Lambda(z, r_2/2)$, and the map $z \mapsto y(z)$ is injective, so

$$Z_{n}^{\text{sep}}(B^u_\Lambda(x, r_1), \varphi, r_2) \leq \sum_{z \in E} e^{S_n \varphi(z)} \leq \sum_{z \in E} e^{S_n \varphi(y(z))} + Q_u \leq e^{Q_u} \sum_{y \in F} e^{S_n \varphi(y)}.$$ 

Taking an infimum over all such $F$ gives the second inequality. \hfill \Box

6.3.2. Changing leaves. Let $\epsilon > 0$ be such that $[y, z]$ exists whenever $d(y, z) < \epsilon$. Without loss of generality we assume that $\epsilon \leq \tau/3$. The following two statements allow us to compare partition sums along different leaves.

**Lemma 6.4.** Given any $r_1 \in (0, \epsilon)$ and $r_2 \in (0, \tau/3]$, there are $n_1 = n_1(r_1) \in \mathbb{N}$ and $Q_2 = Q_2(r_1, r_2) > 0$ such that given any $x, y \in \Lambda$ and $n \geq n_1$ we have

$$(6.2) \quad Z_{n-n_1}^{\text{span}}(B^u_\Lambda(y, r_1), \varphi, 3r_2) \leq Q_2 Z_{n}^{\text{span}}(B^u_\Lambda(x, r_1), \varphi, r_2).$$

**Proof.** Choose $\delta > 0$ small enough that if $x \in \Lambda$, $y \in B^u_\Lambda(x, r_2)$, and $z \in B^c_\Lambda(x, \delta)$, then $V^c_{loc}(y) \cap B^u(z, 2r_2) \neq \emptyset$. By Lemma 6.2, there is $n_1 \in \mathbb{N}$ such that for every $x, y \in \Lambda$ there is $k = k(x, y) \in [0, n_1]$ with $f^k(B^u_\Lambda(x, r_1)) \cap B^c_\Lambda(y, \delta) \neq \emptyset$.

Now given $x, y \in \Lambda$, $n \geq n_1$, and any $(n, r_2)$-spanning set $E \subset B^u_\Lambda(x, r_1)$, we will produce an $(n - n_1, 3r_2)$-spanning set $E' \subset B^u_\Lambda(y, r_1)$. To this end, let $U = \bigcup_{z \in B^u_\Lambda(y, r_1 + 3r_2)} V^u_{loc}(z)$, and let $\pi: U \to B^u_\Lambda(y, r_1 + 2r_2)$ be projection along stable leaves. We first claim that

$$E_1 := \bigcup_{k=0}^{n_1} \pi(f^k(E) \cap U) \subset B^u_\Lambda(y, r_1 + 2r_2)$$

has the property that

$$(6.3) \quad \bigcup_{z \in E_1} B^u_{n-n_1}(z, 2r_2) \supset B^u_\Lambda(y, r_1).$$

Indeed, given $z \in B^u_\Lambda(y, r_1)$, by the choice of $n_1$ there are $k \in [0, n_1]$ and $p \in B^u_\Lambda(x, r_1)$ such that $f^k(p) \in B^c_\Lambda(z, \delta)$, and since $E$ is an $(n, r_2)$-spanning set in $B^u_\Lambda(x, r_1)$, we can choose a point $q \in E \cap B^u_n(p, r_2)$. Then

$$f^k(q) \in B^u_{n-k}(f^k(p), r_2) \subset B^u_{n-n_1}(f^k(p), r_2);$$

our choice of $\delta$ then gives $\pi(f^k(q)) \in B^u_{n-n_1}(z, r_2)$, since $\pi(f^k(p)) = z$. This proves (6.3).

To produce $E'$, consider the sets

$$E_2 := E_1 \cap B^u_\Lambda(y, r_1), \quad E_3 := \{z \in E_1 \setminus E_2 : B^u_{n-n_1}(z, r_2) \cap B^u_\Lambda(y, r_1) \neq \emptyset\}.$$ 

Define a map $T: E_3 \to B^u_\Lambda(y, r_1)$ by choosing for each $z \in E_3$ some $T(z) \in B^u_{n-n_1}(z, r_2)$. Then $E' = E_2 \cup T(E_3)$ is an $(n - n_1, 3r_2)$-spanning set in $B^u_\Lambda(y, r_1)$. 

Given $k \in [0, n_1]$ and $j \in \{2, 3\}$, let $E_j^k = \{p \in E : \pi(f^k(p)) \in E_j\}$, so

\[(6.4)\]
\[E' = \bigcup_{k=0}^{n_1} \pi f^k(E_j^k) \cup T(\pi f^k(E_j^k))\]

By the $cs$-Bowen property, for each $p \in E$ we have

\[(6.5)\]
\[|S_{n-1} \varphi(\pi(f^k(p))) - S_{n-1} \varphi(f^k(p))| \leq Q_{cs}\]

Since $T(z) \in B_{u,n_1}(z, r_2)$, the $u$-Bowen property gives

\[(6.6)\]
\[|S_{n-1} \varphi(T(z)) - S_{n-1} \varphi(z)| \leq Q_u.

Using the $(n - n_1, 3r_2)$-spanning property of $E'$ together with (6.4)–(6.6), we obtain

\[Z_{n-1}^{\text{span}}(B_{\Lambda}^u(y, r_1), \varphi, 3r_2) \leq \sum_{z \in E'} e^{S_{n-1} \varphi(z)}\]
\[\leq \sum_{k=0}^{n_1} \left( \sum_{p \in E_j^k} e^{S_{n-1} \varphi(\pi(f^k(p)))} + \sum_{p \in E_j^k} e^{S_{n-1} \varphi(T(\pi f^k(p)))} \right)\]
\[\leq \sum_{k=0}^{n_1} \left( \sum_{p \in E_j^k} e^{S_{n-1} \varphi(f^k(p)) + Q_{cs}} + \sum_{p \in E_j^k} e^{S_{n-1} \varphi(f^k(p) + Q_{cs} + Q_u)} \right)\]
\[\leq (n_1 + 1) \sum_{p \in E} e^{S_{n} \varphi(p) + Q_{cs} + Q_u + n_1\|\varphi\|}.

Putting $Q_2 := (n_1 + 1)e^{Q_{cs} + Q_u + n_1\|\varphi\|}$ and taking an infimum over all $E$ proves (6.2). \qed

6.3.3. Changing scales.

Lemma 6.5. For every $r_2, r_3 \in (0, \tau]$, there is $n_0 \in \mathbb{N}$ such that for every $x \in \Lambda$ and $r_1 \in (0, \tau]$, we have

\[Z_n^{\text{sep}}(B_{\Lambda}^u(x, r_1), \varphi, r_3) \leq e^{n_0\|\varphi\|} Z_n^{\text{sep}}(B_{\Lambda}^u(x, r_1), \varphi, r_2)\]

Proof. Choose $n_0 \in \mathbb{N}$ such that $r_2 \lambda^{n_0} < r_3$, where $\lambda < 1$ is as in Proposition 2.2. Then if $x \in \Lambda$ and $y, z \in V_{\text{loc}}(x)$ are such that $d_n(y, z) \geq r_3$, we must have $d_{n+n_0}(y, z) \geq r_2$. This shows that any $(n, r_3)$-separated subset $E \subset B_{\Lambda}^u(x, r_1)$ is $(n + n_0, r_2)$-separated. Moreover, we have

\[\sum_{y \in E} e^{S_n \varphi(y)} \leq e^{n_0\|\varphi\|} \sum_{y \in E} e^{S_{n+n_0} \varphi(y)},\]

and taking a supremum over all such $E$ completes the proof. \qed

6.3.4. Correct growth rate. At this point we have enough machinery developed to prove that the leafwise partition sums have the same growth rate as the overall partition sums so that we can use the former to compute the topological pressure in (2.10). This is not yet quite enough to conclude Proposition 6.1 but is an important step along the way.

Lemma 6.6. For every $x \in \Lambda$, $r_1 \in (0, \epsilon)$, and $r_2 \in (0, \tau/3]$, we have

\[(6.7)\]
\[P(\varphi) = \lim_{n \to \infty} \frac{1}{n} \log Z_n^{\text{span}}(B_{\Lambda}^u(x, r_1), \varphi, r_2) = \lim_{n \to \infty} \frac{1}{n} \log Z_n^{\text{sep}}(B_{\Lambda}^u(x, r_1), \varphi, r_2)\]
Proof. Given \( Y \subseteq M \), write
\[
\mathcal{P}_Y^{\text{span}}(r_2) := \lim_{n \to \infty} \frac{1}{n} \log Z_n^{\text{span}}(Y \cap \Lambda, r_2);
\]
define \( \mathcal{P}_Y^{\text{sep}}(r_2), \mathcal{P}_Y^{\text{span}}(r_2) \), and \( \mathcal{P}_Y^{\text{sep}}(r_2) \) similarly. (Since \( \varphi \) is fixed throughout we omit it from the notation.) It follows directly from Lemma 6.3 that
\[
\mathcal{P}_{B_2^u(x, r_1)}^{\text{sep}}(2r_2) \leq \mathcal{P}_{B_2^u(x, r_1)}^{\text{span}}(r_2) \leq \mathcal{P}_{B_2^u(x, r_1)}^{\text{sep}}(r_2) \leq \mathcal{P}_{B_2^u(x, r_1)}^{\text{sep}}(r_2),
\]
and Lemma 6.5 gives \( \mathcal{P}_{B_2^u(x, r_1)}^{\text{sep}}(r_2) \leq \mathcal{P}_{B_2^u(x, r_1)}^{\text{sep}}(2r_2) \). It follows that both limits in (6.7) exist and are independent of \( r_2 \). We denote their common value by \( \mathcal{P}_{B_2^u(x, r_1)}(\varphi) \). To complete the proof we must show that this is equal to \( P(\varphi) \).

One direction is easy: any \((n, r_2)\)-separated subset of \( B_2^u(x, r_1) \) is also an \((n, r_2)\)-separated subset of \( \Lambda \), so we immediately get \( P(\varphi) \geq \mathcal{P}_{B_2^u(x, r_1)}(\varphi) \). For the other direction, consider for each \( y \in \Lambda \) the (relatively) open set \( U_y := \bigcup_{z \in B_2^u(y, r_1)} B_2^u(z, r_2) \). Since \( \Lambda \) is compact, we have \( \Lambda \subseteq \bigcup_{i=1}^N U_{y_i} \) for some \( \{y_1, \ldots, y_N\} \). Now for any \( x \in \Lambda \), Lemma 6.4 gives \((n - n_1, 3r_2)\)-spanning sets \( E_i \) for \( B_2^u(y_i, r_1) \) such that
\[
\sum_{z \in E_i} e^{S_{n_i} \varphi(z)} \leq Q_2 \mathcal{P}_{B_2^u(x, r_1)}^{\text{span}}(B_2^u(x, r_1), \varphi, r_2).
\]
We claim that \( E_i \) is an \((n - n_1, 4r_2)\)-spanning set for \( U_{y_i} \). Indeed, for every \( z \in U_{y_i} \), we have \( [z, y_i] \in B_2^u(y_i, r_1) \) and hence there is \( p \in E_i \) such that \( d_n(p, [z, y_i]) < 3r_2 \). Moreover, \( d_n(z, [z, y_i]) < r_2 \) since \( z \in B_2^u(z, r_2) \), and the triangle inequality proves the claim.

Now writing \( E' = \bigcup_{i=1}^N E_i \), we see that \( E' \) is an \((n - n_1, 4r_2)\)-spanning set for \( \Lambda \), and hence,
\[
\mathcal{P}_{B_2^u(x, r_1)}^{\text{span}}(\Lambda, \varphi, 4r_2) \leq Q_2 N \mathcal{P}_{B_2^u(x, r_1)}^{\text{span}}(B_2^u(x, r_1), \varphi, r_2).
\]
Taking logs, dividing by \( n \), and sending \( n \to \infty \), \( r_2 \to 0 \) gives \( P(\varphi) \leq \mathcal{P}_{B_2^u(x, r_1)}(\varphi) \), which proves Lemma 6.6. \( \square \)

6.4. Uniform control of partition sums. Now we are nearly ready to use the estimates from the preceding sections to prove Proposition 6.1. We need two more lemmas.

Given \( n \in \mathbb{N} \) and \( r_1, r_2 \in (0, \tau] \), consider the quantity
\[
Z_n^u(\varphi, r_1, r_2) := \sup_{x \in \Lambda} Z_n^{\text{sep}}(B_2^u(x, r_1), \varphi, r_2).
\]
We have the following submultiplicativity result.

Lemma 6.7. For every \( x \in \Lambda \), \( r_1, r_2 \in (0, \tau] \), and \( k, \ell \in \mathbb{N} \), we have
\[
Z_k^{\text{sep}}(B_2^u(x, r_1), \varphi, r_2) \leq e^{Q_k} Z_{k+\ell}^{\text{sep}}(B_2^u(x, r_1), \varphi, r_2) Z_{\ell}^{\text{sep}}(\varphi, r_1, r_2).
\]
Proof. Given \( x \in \Lambda\) and \( k, \ell \in \mathbb{N} \), let \( E \subseteq B_2^u(x, r_1) \) be a \((k + \ell, r_2)\)-separated set. Let \( E' \subseteq E \) be a maximal \((k, r_2)\)-separated set, and given \( y \in E' \) let \( E_y = E \cap B_2^u(y, r_1) \). Then \( f^k(E_y) \) is an \((\ell, r_2)\)-separated subset of \( f^k(B_2^u(y, r_1) \cap \Lambda) = B_2^u(f^k(y), r_1) \), and we conclude
that
\[ \sum_{y \in E} e^{S_k + \varphi(y)} = \sum_{y \in E'} \sum_{z \in E_y} e^{S_k + \varphi(z)} = \sum_{y \in E'} \sum_{z \in E_y} e^{S_k \varphi(z)} e^{S_k \varphi(f^k(z))} \]
\[ \leq \sum_{y \in E'} e^{S_k \varphi(y)} + Q_u \sum_{z \in E_y} e^{S_k \varphi(f^k(z))} \]
\[ \leq e^{Q_u} Z_k^{sep}(B^u_\Lambda(x, r_1), \varphi, r_2) \max_{y \in E'} Z_{k}^{sep}(B^u_\Lambda(f^k(y), r_1), \varphi, r_2), \]
where the first inequality uses the \( u \)-Bowen property. Taking a supremum over all choices of \( E \) completes the proof. \( \square \)

Now we can assemble Lemmas 6.3, 6.4, 6.5, and 6.7 into the following result that lets us change parameters \( x \) and \( r_2 \) in partition sums more or less at will. \(^6\)

**Lemma 6.8.** For every \( r_1 \in (0, \epsilon) \) and \( r_2, r_2' \in (0, \tau/3] \) there is \( Q_3 \) such that for every \( x, y \in \Lambda \) and \( n \in \mathbb{N} \), we have
\[ Z_n^{sep}(B^u_\Lambda(x, r_1), \varphi, r_2) \leq Q_3 Z_n^{sep}(B^u_\Lambda(y, r_1), \varphi, r_2'). \]

**Proof.** Let \( n_1 \) be as in Lemma 6.4 (note that it only depends on \( r_1 \), not on \( r_2 \)) and let \( n_0 \) be as in Lemma 6.5 with \( r_3 = r_2'/6 \). Then for all \( x, y \in \Lambda \), we have
\[ Z_n^{sep}(B^u_\Lambda(x, r_1), \varphi, r_2') \leq e^{Q_u} Z_n^{sep}(B^u_\Lambda(x, r_1), \varphi, r_2)/2 \quad \text{(Lemma 6.3)} \]
\[ \leq Q_2 e^{Q_u} Z_n^{sep}(B^u_\Lambda(x, r_1), \varphi, r_2'/6) \quad \text{(Lemma 6.4)} \]
\[ \leq Q_2 e^{Q_u} Z_{n+n_1}^{sep}(B^u_\Lambda(x, r_1), \varphi, r_2'/6) \quad \text{(Lemma 6.3)} \]
\[ \leq Q_2 e^{Q_u+n_0||\varphi||} Z_{n+n_0+n_1}^{sep}(B^u_\Lambda(x, r_1), \varphi, r_2). \quad \text{(Lemma 6.5)} \]
By Lemma 6.7, this gives
\[ Z_n^{sep}(B^u(y, r_1), \varphi, r_2') \leq Q_2 e^{Q_u+n_0||\varphi||} Z_n^{sep}(B^u(x, r_1), \varphi, r_2) Z_n^{u}(\varphi, r_1, r_2). \]
Putting \( Q_3 = Q_2 e^{Q_u+n_0||\varphi||} Z_n^{u}(\varphi, r_1, r_2) \) completes the proof. \( \square \)

**Proof of Proposition 6.1.** For the lower bound, we apply Lemma 6.7 iteratively to get
\[ Z_{nk}^{sep}(B^u_\Lambda(x, r_1), \varphi, r_2) \leq e^{(n-1)Q_u} Z_k^{u}(\varphi, r_1, r_2)^n. \]
Taking logs, dividing by \( nk \), and sending \( n \to \infty \) gives
\[ \frac{1}{k} \log Z_k^{u}(\varphi, r_1, r_2) \geq -\frac{Q_u}{k} + P(\varphi) \]
by Lemma 6.6. Thus for every \( x \in \Lambda \) and \( k \in \mathbb{N} \), Lemma 6.8 gives
\[ Z_k^{sep}(B^u(x, r_1), \varphi, r_2) \geq Q_3^{-1} Z_k^{u}(\varphi, r_1, r_2) \geq Q_3^{-1} e^{-Q_u} e^{kP(\varphi)}, \]
which proves the lower bound in (6.1) by taking \( Q_1 \geq Q_3 e^{Q_u} \).

For the upper bound in (6.1), start by letting \( n_2 \in \mathbb{N} \) be such that \( r_2 \lambda^{-n_2} \geq r_2 + 2r_1 \), where once again \( \lambda < 1 \) is as in Proposition 2.2(3). Now fix \( x \in \Lambda \) and \( n \in \mathbb{N} \), and let

\(^6\)With a little more work we could vary \( r_1 \) as well, but we will not need this.
$E_0 \subset B^u_\Lambda(x, r_1)$ be any $(n, r_2)$-separated set. By Lemma 6.8 for every $z \in \Lambda$ there is a $(n, r_2)$-separated set $G(z) \subset B^u_\Lambda(z, r_1)$ with

$$\sum_{y \in G(z)} e^{S_n \varphi(y)} \geq Q_3^{-1} \sum_{y \in E_0} e^{S_n \varphi(y)}. \tag{6.10}$$

Given $k = 1, 2, \ldots$, construct $E_k \subset B^u_\Lambda(x, r_1)$ iteratively by

$$E_k = \bigcup_{y \in E_{k-1}} f^{-k(n+n_2)}(G(f^{k(n+n_2)}(y))). \tag{6.11}$$

We prove by induction that $E_k$ is a $((k + 1)n + kn_2, r_2)$-separated set. The case $k = 0$ is true by our assumption on $E_0$. For $k \geq 1$, suppose that the set $E_{k-1}$ is $(kn + (k-1)n_2, r_2)$-separated. Then given any $p_1, p_2 \in E_k$ we have one of the following two cases.

1. There is $z \in E_{k-1}$ with $p_1, p_2 \in f^{-k(n+n_2)}(G(f^{k(n+n_2)}(z)))$. By the definition of $G(f^{k(n+n_2)}(z))$, this gives

$$d((k+1)n+kn_2)(p_1, p_2) \geq d_n(f^{k(n+n_2)}(p_1), f^{k(n+n_2)}(p_2)) \geq r_2.$$

2. There are $z_1 \neq z_2 \in E_{k-1}$ such that $p_i \in f^{-k(n+n_2)}(G(f^{k(n+n_2)}(z_i)))$ for $i = 1, 2$. Then $f^i(p_i) \in B^u(f^i(z_i), r_1)$ for all $0 \leq \ell < k(n + n_2)$, and there is $0 \leq j < kn + (k-1)n_2$ such that $f^j(z_2) \in V_{f^j(z_1)}$ and $d(f^j(z_1), f^j(z_2)) > r_2$. By our choice of $n_2$, we have

$$d((f^{j+n_2}(z_1), f^{j+n_2}(z_2)) > r_2 + 2r_1 \geq r_2 + \sum_{i=1}^2 d(f^{j+n_2}(z_i), f^{j+n_2}(p_i)),$$

and the triangle inequality gives $d(f^{j+n_2}(p_1), f^{j+n_2}(p_2)) \geq r_2$.

This completes the induction, and gives the following estimate:

$$Z^{sep}_{(k+1)n+kn_2}(B^u_\Lambda(x, r_1), \varphi, r_2) \geq \sum_{y \in E_k} e^{S_{(k+1)n+kn_2} \varphi(y)}. \tag{6.12}$$

Write $\tilde{Z}_k$ for the sum on the right-hand side of (6.12). The definition of $E_k$ in (6.11) gives

$$\tilde{Z}_k = \sum_{y \in E_{k-1}} \sum_{z \in f^{-k(n+n_2)}(G(f^{k(n+n_2)}(y)))} e^{S_{(k+1)n+kn_2} \varphi(z)} \geq \sum_{y \in E_{k-1}} e^{S_{kn+(k-1)n_2} \varphi(z)-Q_u e^{-n_2} \|\varphi\|} \sum_{p \in G(f^{k(n+n_2)}(y))} e^{S_n \varphi(p)} \geq Q_3^{-1} e^{-Q_u e^{-n_2} \|\varphi\|} \tilde{Z}_{k-1} \sum_{y \in E_0} e^{S_n \varphi(y)}, \tag{6.13}$$

where the last inequality uses (6.10). Writing $Q_4 := Q_3 e^{Q_u e^{-n_2} \|\varphi\|}$ and applying (6.13) $k$ times yields

$$\tilde{Z}_k \geq Q_4^{-k} \left( \sum_{y \in E_0} e^{S_n \varphi(y)} \right)^k.$$
Then (6.12) gives

\[ Z_{(k+1)n+kn_2}^{\text{sep}}(B^u_\Lambda(x,r_1), \varphi, r_2) \geq Q^k_4 \left( \sum_{y \in E_0} e^{S_n \varphi(y)} \right)^k. \]

Taking logs and dividing by \( k \) gives

\[ \frac{n + n_2}{(k + 1)n + kn_2} \log Z_{(k+1)n+kn_2}^{\text{sep}}(B^u_\Lambda(x,r_1), \varphi, r_2) \geq \log Q_4 + \log \sum_{y \in E_0} e^{S_n \varphi(y)}. \]

Sending \( n \to \infty \) and taking a supremum over all choices of \( E_0 \), Lemma 6.6 yields

\[ (n + n_2)P(\varphi) \geq -\log Q_4 + \log Z_n^{\text{sep}}(B^u_\Lambda(x,r_1), \varphi, r_2), \]

and so \( Z_n^{\text{sep}}(B^u_\Lambda(x,r_1), \varphi, r_2) \leq Q_4 e^{(n+n_2)P(\varphi)} \). Choosing \( Q_1 \geq Q_4 e^{n_2P(\varphi)} \) completes the proof of Proposition 6.1.

6.5. **Proof of Theorem 4.2** Fix \( x \in \Lambda \) and set \( X := V^u_{\text{loc}}(x) \cap \Lambda \). We showed in 3.2 that \( m^C_x \) defines a metric outer measure on \( X \), and hence gives a Borel measure. Note that the final claim in Theorem 4.2 about agreement on intersections is immediate from the definition. Thus it remains to prove that \( m^C_x(X) \in [K^{-1}, K] \), where \( K \) is independent of \( x \); this will complete the proof of Theorem 4.2.

We start with the following basic fact about local unstable leaves, which follows easily from Proposition 2.3(1).

**Lemma 6.9.** For all \( r_1, r_2 \in (0, \tau] \) there is \( Q_5 > 0 \) such that for all \( y \in \Lambda \), there are \( k \leq Q_5 \) and \( z_1, \ldots, z_k \in B^u_\Lambda(y, r_2) \) such that \( \bigcup_{i=1}^k B^u_\Lambda(z_i, r_1) \supset B^u_\Lambda(y, r_2) \).

Together with Proposition 6.1 this leads to the following.

**Lemma 6.10.** For every \( r \in (0, \tau/3] \), there is a constant \( Q_6 > 0 \) such that for every \( y \in \Lambda \), \( n \in \mathbb{N} \), and \( N \geq n \), we have

\[ Z_N^{\text{sep}}(B^u_\Lambda(y, r) \cap \Lambda, \varphi, r) \leq Q_6 e^{(N-n)P(\varphi)} e^{S_n \varphi(y)}. \]

**Proof.** Let \( \ell \in \mathbb{N} \) be such that \( \lambda^\ell < \frac{1}{2} \), where \( \lambda < 1 \) is as in Proposition 2.2(3). Given any \( (N, r) \)-separated set \( E \subset B^u_\Lambda(y, r) \cap \Lambda \), we have \( d(f^n(z_1), f^n(z_2)) < 2r \) for every \( z_1, z_2 \in E \), and thus \( d(f^{n-\ell}(z_1), f^{n-\ell}(z_2)) < r \), so \( f^{n-\ell}(E) \) is an \( (N - n + \ell, r) \)-separated subset of \( f^{n-\ell}(B^u_\Lambda(y, r) \cap \Lambda) \subset B^u_\Lambda(f^{n-\ell}(y), r) \).

Applying Proposition 6.1 with \( r_1 = \epsilon/2 \) and \( r_2 = r \), gives \( Q_1 = Q_1(r) \) such that

\[ Z_{N-n+\ell}^{\text{sep}}(B^u_\Lambda(f^{n-\ell}(y), \epsilon/2), \varphi, r) \leq Q_1 e^{(N-n+\ell)P(\varphi)}, \]

and so, applying Lemma 6.9 with \( r_1 = \epsilon/2 \) and \( r_2 = 4 \), gives \( Q_5 = Q_5(r) \) such that

\[ \sum_{z \in f^{n-\ell}(E)} e^{S_n \varphi(z)} \leq Z_{N-n+\ell}^{\text{sep}}(B^u_\Lambda(f^{n-\ell}(y), r), \varphi, r) \leq Q_5 Q_1 e^{(N-n+\ell)P(\varphi)}. \]
Thus we get
\[ \sum_{z \in E} e^{SN \varphi(z)} = \sum_{z \in E} e^{SN-T \varphi(z)} e^{SN-N \varphi(f^n \cdot (z))} \]
\[ \leq e^{Q_0} e^{SN-T \varphi(y)} \sum_{z \in f^n \cdot (E)} e^{SN-N \varphi(z)} \]
\[ \leq e^{Q_0} e^{\|\varphi\|} e^{SN \varphi(y)} Q_0 Q_1 e^{(N-n)P(\varphi)} e^{NP(\varphi)}, \]
and so putting \( Q_6 = e^{Q_0} e^{\|\varphi\|+P(\varphi)} Q_0 Q_1 \) proves the result. \( \square \)

Now we can complete the proof of Theorem 4.2. Fix \( r_1 \in (0, \tau) \), and note that Proposition 6.1 and Lemmas 6.3 and 6.9 apply with \( r_2 = r \). We will find \( Q_7 > 0 \) such that
\[ (6.14) \quad Q_7^{-1} \leq m^C_x(B^u_\Lambda(x, r_1)) = m^C_x(B^u_\Lambda(x, r_1)) \leq Q_7 \]
for every \( x \in \Lambda \); then Lemma 6.9 will complete the proof of the theorem by taking \( K = Q_7 Q_5 \).

For the upper bound in (6.14), let \( Q_1 \) be given by Proposition 6.1 with \( r_2 = r \). By Lemma 6.3 and Proposition 6.1, for every \( N \in \mathbb{N} \) there is an \((N, r)\)-spanning set \( E_N \subset B^u_\Lambda(x, r_1) \) with \( \sum_{y \in E_N} e^{SN \varphi(y)} \leq Q_1 e^{NP(\varphi)} \). Then (3.5) gives
\[ (6.15) \quad m^C_x(B^u_\Lambda(x, r_1)) \leq \lim_{N \to \infty} \sum_{y \in E_N} e^{-NP(\varphi)} e^{SN \varphi(y)} \leq Q_1. \]

For the lower bound in (6.14), let \( \{(y_i, n_i)\}_{i=1}^k \subset (V^{\text{loc}}_x(\Lambda) \times \mathbb{N}) \) be any finite or countable set such that \( B^u_\Lambda(x, r_1) \subset \bigcup_{i=1}^k B^u_{n_i}(y_i, r) \). By compactness, there is \( k \in \mathbb{N} \) such that \( B^u_\Lambda(x, r_1/2) \subset \bigcup_{i=1}^k B^u_{n_i}(y_i, r) \). Fix \( N \geq \max\{n_1, \ldots, n_k\} \) and for each \( 1 \leq i \leq k \), let \( E_i \subset B^u_{n_i}(y_i, r) \cap \Lambda \) be a maximal \((N, r)\)-separated set. Then \( \bigcup_{i=1}^k E_i \) is an \((N, r)\)-spanning set for \( B^u_\Lambda(x, r_1/2) \), and we conclude that
\[ (6.16) \quad \sum_{i=1}^k \sum_{z \in E_i} e^{SN \varphi(z)} \geq Z^\text{span}_N(B^u_\Lambda(x, r_1/2), \varphi, r) \]
\[ \geq e^{-Q_0} Z^\text{sep}_N(B^u_\Lambda(x, r_1/2), \varphi, 2r) \geq Q_1^{-1} e^{-Q_0} e^{NP(\varphi)}, \]
where the second inequality uses Lemma 6.3, and the third uses Proposition 6.1 with \( Q_1 = Q_1(r_1/2, 2r) \).

Now we can use Lemma 6.10 to get the bound
\[ \sum_{z \in E_i} e^{SN \varphi(z)} \leq Z^\text{sep}_N(B^u_{n_i}(y_i, r) \cap \Lambda, \varphi, r) \leq Q_6 e^{(N-n_i)P(\varphi)} e^{SN \varphi(y_i)} \]
for each \( 1 \leq i \leq k \). Summing over \( i \) and using (6.16) gives
\[ Q_1^{-1} e^{-Q_0} e^{NP(\varphi)} \leq \sum_{i=1}^k \sum_{z \in E_i} e^{SN \varphi(z)} \leq \sum_{i=1}^k Q_6 e^{(N-n_i)P(\varphi)} e^{SN \varphi(y_i)}, \]
and dividing both sides by \( e^{NP(\varphi)} \) yields

\[
Q_1^{-1} e^{-Q u} \leq Q_6 \sum_{i=1}^{k} e^{-n_i P(\varphi)} e^{S_{n_i} \varphi(y_i)}.
\]

Taking an infimum over all choices of \( \{(y_i, n_i)\}_i \) and setting \( N \to \infty \) gives

\[
m_C^e(B^u_\Lambda(x, r_1)) \geq Q_1^{-1} e^{-Q u} Q_6^{-1}.
\]

Thus we can prove (6.14) and complete the proof of Theorem 4.2 by putting \( Q_7 = \max\{Q_1, Q_6 e^{Q u} Q_6^{-1}\} \).

7. Behavior of reference measures under iteration and holonomy

7.1. Proof of Theorem 4.4. We will prove that

\[
m_C^e(f(x))(A) = \int_{f^{-1}(A)} e^{P(\varphi) - \varphi(y)} \, dm_C^e(y)
\]

for every \( A \subset V^u_{\text{loc}}(f(x)) \cap \Lambda \), which shows that \( f_*^{-1} m_C^e(f(x)) \ll m_C^e \) and that the Radon–Nikodym derivative is \( g := e^{P(\varphi) - \varphi} \). Given such an \( A \), we approximate the integrand on the right-hand side of (7.1) by simple functions; for every \( T \in \mathbb{N} \) there are real numbers

\[
\inf_{y \in f^{-1}(A)} g(y) = a_1^T < a_2^T < \cdots < a_T^T = \sup_{y \in f^{-1}(A)} (g(y) + 1)
\]

and disjoint sets

\[
E_i^T := \{y \in f^{-1}(A) : a_i^T \leq g(y) < a_{i+1}^T\} \text{ for } 1 \leq i < T
\]

such that \( f^{-1}(A) = \bigcup_{i=1}^{T-1} E_i^T \); since the union is disjoint we have

\[
\int_{f^{-1}(A)} g(y) \, dm_C^e(y) = \lim_{T \to \infty} \sum_{i=1}^{T-1} a_i^T \, m_C^e(E_i^T) = \lim_{T \to \infty} \sum_{i=1}^{T-1} a_i^T \, m_C^e(E_i^T).
\]

To prove (7.1), start by using the first equality in (7.2) and the definition of \( m_C^e \) in (3.5) to write

\[
\int_{f^{-1}(A)} g \, dm_C^e = \lim_{T \to \infty} \sum_{i=1}^{T-1} a_i^T \lim_{N \to \infty} \inf_j e^{-n_j P(\varphi)} e^{S_{n_j} \varphi(z_j)},
\]

where the infimum is taken over all collections \( \{B^u_{n_j}(z_j, r)\} \) of \( u \)-Bowen balls with \( z_j \in V^u_{\text{loc}}(x) \cap \Lambda, n_j \geq N \) that cover \( E_i^T \). Without loss of generality we can assume that

\[
B^u_{n_j}(z_j, r) \cap E_i^T \neq \emptyset \text{ for all } j.
\]

Consider the quantity

\[
R_N := \sup\{|\varphi(y) - \varphi(z)| : y \in \Lambda, z \in B^u_N(y, \tau/3)\},
\]

and note that \( R_N \to 0 \) as \( N \to \infty \) using uniform continuity of \( \varphi \) together with the fact that \( \text{diam} B^u_N(y, \tau/3) \leq \tau \lambda^N \to 0 \). Now by (7.4) we have

\[
a_i^T \geq g(z_j) e^{-R_N} = e^{-R_N} e^{P(\varphi)} e^{-\varphi(z_j)}
\]
for all $j$, and thus (7.3) gives
\[
\int_{f^{-1}(A)} g \, dm^C_x \geq \lim_{T \to \infty} \sum_{i=1}^{T-1} \lim_{N \to \infty} \inf_j \sum_e^{-R_N} e^{-R_N} e^{-(n_j-1)\|\varphi\|} e^{S_{n_j-1}\varphi(f(z_j))},
\]
where again the infimum is taken over all collections $\{B_{n_j}^u(z_j, r)\}$ of $u$-Bowen balls with $z_j \in V_{loc}^u(x) \cap \Lambda$, $n_j \geq N$ that cover $E_i^T$. Observe that to each such collection there is associated a cover of $f(E_i^T)$ by the $u$-Bowen balls $f(B_{n_j}^u(z_j, r)) = B_{n_j-1}^u(f(z_j), r)$, and vice versa, so we get
\[
\int_{f^{-1}(A)} g(y) \, dm^C_x(y) \geq \lim_{T \to \infty} \sum_{i=1}^{T-1} m^C_{f(x)}(f(E_i^T)) = m^C_{f(x)}(A).
\]
The reverse inequality is proved similarly by replacing $a_i^T$ with $a_{i+1}^T$ in (7.3) and using the second equality in (7.2). This completes the proof of (7.1), and hence of Theorem 4.4.

### 7.2. Proof of Corollary 4.5

Iterating (7.1), we obtain
\[
(7.6) \quad m^C_{f^n(x)}(A) = \int_{f^{-n}(A)} e^{nP(\varphi)-S_n\varphi(y)} \, dm^C_x(y)
\]
for all $A \subset V_{loc}^u(f^n(x))$. Fix $\delta \in (0, \tau)$. Putting $A = B^u(f^n(x), \delta)$ and observing that $f^{-n}(B^u(f^n(x), \delta)) = B^u_n(x, \delta)$, we can use Theorem 4.2 and the $u$-Bowen property to get
\[
K \geq m^C_{f^n(x)}(B^u(f^n(x), \delta)) = \int_{B^u_n(x, \delta)} e^{nP(\varphi)-S_n\varphi(y)} \, dm^C_x(y) \geq e^{-Q_u} e^{nP(\varphi)-S_n\varphi(x)} m^C_x(B^u_n(x, \delta)).
\]
This gives $m^C(B^u_n(x, \delta)) \leq e^{Q_u} K e^{nP(\varphi)+S_n\varphi(x)}$, proving the upper bound in (4.1). For the lower bound, let $k \in \mathbb{N}$ be such that $\delta \lambda^{-k} > \tau$; then $B^u_n(y, \delta) \supseteq f^{-k}(V_{loc}^u(f^k(y)))$ for all $y \in \Lambda$, and again Theorem 4.2 and the $u$-Bowen property give
\[
K^{-1} \leq m^C_{f^{n+k}(x)}(V_{loc}^u(f^{n+k}(x))) \leq \int_{B^u_n(x, \delta)} e^{(n+k)P(\varphi)-S_{n+k}\varphi(y)} \, dm^C_x(y) \leq e^{Q_u} e^{k(P(\varphi)+\|\varphi\|)} e^{nP(\varphi)-S_n\varphi(x)} m^C_x(B^u_n(x, \delta)).
\]
This proves Corollary 4.5.

### 7.3. Proof of Theorem 4.6

Let $k \in \mathbb{N}$ be such that for every $y \in \Lambda$ there are $z_1, \ldots, z_k \in B^u_N(y, 2r)$ such that $B^u_N(y, 2r) \subset \bigcup_{i=1}^k B^u_N(z_i, r)$. Let $\delta > 0$ be such that if $x, y \in \Lambda$ are such that $d(x, y) < \delta$, and $a, b \in V_{loc}^u(x) \cap \Lambda$, $p, q \in V_{loc}^u(y) \cap \Lambda$ are such that $p \in V_{\Lambda}^{cs}(a)$, $q \in V_{\Lambda}^{cs}(b)$, then $d(p, q) \leq d(a, b) + r$. Let $N \in \mathbb{N}$ be such that $\tau \lambda^{-N} < \delta$.

Now let $R \subset \Lambda$ be any rectangle. Given $y, z \in R$ and $E \subset V_{R}^u(y)$, we must compare $m^C_y(E)$ and $m^C_z(\pi_{yz}(E))$. For any cover $\{B_{n_i}^u(x_i, r)\}$ of $E$ with $x_i \in V_{R}^u(y)$ and $n_i \geq N$, we note that $d(f^{n_i}(x_i), f^{n_i}(\pi_{yz}(x_i))) < \tau \lambda^N < \delta$ by our choice of $N$, and thus by our choice of
\[ \pi_{yz}(B_n^u(x_i, r) \cap \Lambda) = \pi_{yz}(f^{-n_i}(B_n^u(f^{n_i}(x_i), r))) = f^{-n_i}(\pi_{f^{n_i}(y), f^{n_i}(z)}(B_n^u(f^{n_i}(x_i), r))) \subset f^{-n_i}(B_\Lambda(f^{n_i}(\pi_{yz}(x_i)), 2r)). \]

By our choice of \( k \), for each \( i \) there are points \( x_i^1, \ldots, x_i^k \subset B_\Lambda(f^{n_i}(\pi_{yz}(x_i)), 2r) \) with \( B_n^u(f^{n_i}(\pi_{yz}(x_i)), 2r) \subset \bigcup_{j=1}^{k} B_\Lambda(x_i^j, r) \). Thus \( \{B_n^u(f^{-n_i}(x_i^j), r)\}_{i,j} \) is a cover of \( \pi_{yz}(E) \), and moreover for each \( i, j \), the \( u \)- and \( cs \)-Bowen properties give

\[
|S_{n_i} \varphi(x_i) - S_{n_i} \varphi(x_i^j)| \\
\leq |S_{n_i} \varphi(x_i) - S_{n_i} \varphi(\pi_{yz}(x_i))| + |S_{n_i} \varphi(\pi_{yz}(x_i)) - S_{n_i} \varphi(x_i^j)| \leq Q_u + Q_{cs}.
\]

Now we have

\[
\sum_{i,j} e^{-n_i P(\varphi)} e^{S_{n_i} \varphi(x_i^j)} \leq ke^{Q_u + Q_{cs}} \sum_{i} e^{-n_i P(\varphi)} e^{S_{n_i} \varphi(x_i)},
\]

and taking an infimum over all such covers \( \{B_n^u(x_i, r)\} \) of \( E \) gives

\[
m_c^c(\pi_{yz}(E)) \leq ke^{Q_u + Q_{cs}} m_g^c(E).
\]

By symmetry, this implies the reverse inequality and completes the proof of Theorem 4.6.

8. Proof of Theorem 4.7

To prove items (1)(4) from Theorem 4.7, first observe that each measure \( \mu_n \) has \( \mu_n(\Lambda) = 1 \), and thus by weak*-compactness, every subsequence \( \mu_{n_k} \) has a subsequence \( \mu_{n_{kj}} \) that converges to an \( f \)-invariant limiting probability measure.

The first step in the proof is to show that every limit measure \( \mu \) has conditional measures satisfying Statement (3) which we do in 8.1. By Theorem 4.6, this implies that \( \mu \) has local product structure, so it satisfies Statement (4) as well.

The second step is to use (1,3) to show that every limit measure \( \mu \) satisfies the Gibbs property and gives positive weight to every open set; this is relatively straightforward and is done in 8.2.

The third step is to use the local product structure together with a variant of the Hopf argument to show that every limit measure \( \mu \) is ergodic; see 8.3.

For the fourth step, we recall that in the setting of an expansive homeomorphism, an ergodic Gibbs measure was shown by Bowen to be the unique equilibrium measure; see 5. Lemma 8. In our setting, \( f \) may not be expansive, but we can adapt Bowen’s argument (as presented in 24 Theorem 20.3.7) so that it only requires expansivity along the unstable direction, which still holds; see 8.4.

Once this is done, it follows that \((\Lambda, f, \varphi)\) has a unique equilibrium measure \( \mu_\varphi \), and that every limit measure of \( \{\mu_n\} \) is equal to \( \mu_\varphi \). In particular, every subsequence \( \mu_{n_k} \) has a subsequence that converges to \( \mu_\varphi \), and thus \( \mu_n \) converges to this measure as well, which establishes Statement (1) and completes the proof of Theorem 4.7.
8.1. **Conditional measures of limit measures.** To produce the equilibrium state $\mu_\varphi$ using the reference measure $m_x^c$, we start by writing the measures $f_n^m m_x^c$ in terms of standard pairs $(V_{\text{loc}}^u(y), \rho)$, where $y \in f^n(V_{\text{loc}}^u(x))$ and $\rho: V_{\text{loc}}^u(y) \to [0, \infty)$ is a $m_y^c$-integrable density function; each such pair determines a measure $\rho dm_y^c$ on $V_{\text{loc}}^u(y)$. By controlling the density functions that appear in the standard pairs representing $f_n^m m_x^c$, we can guarantee that every limit measure $\mu$ of the sequence of measures $\mu_n = \frac{1}{n} \sum_{k=0}^{n-1} f_k^m m_x^c$ has conditional measures that satisfy part [3] of Theorem 4.7. This in turn will let us show that $\mu$ is ergodic and has the Gibbs property [2.12]; then Proposition 8.8 will guarantee that $\mu$ is the unique equilibrium state, proving part [2]. Finally, since every limit point of the sequence $\mu_n$ can be characterized as above, it will follow that the sequence converges, proving part [1].

Fix $x \in \Lambda$ and $n \in \mathbb{N}$; let $W = V_{\text{loc}}^u(x)$ and $W_n = f^n(W \cap \Lambda)$. Then the iterate $f_n^m m_x^c$ is supported on $W_n$, and $W_n$ can be covered by finitely many local leaves $V_{\text{loc}}^u(y)$. Iterating the formula for the Radon–Nikodym derivative in Theorem 4.4, we obtain for every $y \in W_n$ and $z \in W_n \cap V_{\text{loc}}^u(y)$ that

$$
\frac{d(f_n^m m_x^c)}{dm_y^c}(z) = e^{-nP(\varphi)+S_n \varphi(f^{-n}z) =: g_n(z)}.
$$

Write $\rho_n^y(z) := g_n(z)/g_n(y)$; then the $u$-Bowen property gives

$$
\rho_n^y(z) = e^{S_n \varphi(f^{-n}z) - S_n \varphi(f^{-n}y)} \in [e^{-Q_u}, e^{Q_u}].
$$

Now suppose $y_1, \ldots, y_s \in W_n$ are such that the local leaves $V_{\text{loc}}^u(y_i)$ are disjoint. Then for every Borel set $E \subset \bigcup_{i=1}^s V_{\text{loc}}^u(y_i)$, we have

$$
f_n^m m_x^c(E) = \sum_{i=1}^s \int_E g_n(z) dm_y^c(z) = \sum_{i=1}^s g_n(y_i) \int_E \rho_n^y(z) dm_y^c(z).
$$

In other words, one can write $f_n^m m_x^c$ on $\bigcup_{i=1}^s V_{\text{loc}}^u(y_i)$ as a linear combination of the measures $\rho_n^y dm_y^c$ associated to the standard pairs $(V_{\text{loc}}^u(y_i), \rho_n^y)$, with coefficients given by $g_n(y_i)$. The crucial properties that we will use are the following.

1. The uniform bounds given by (8.2) on the density functions $\rho_n^y$ allow us to control the limiting behavior of $f_n^m m_x^c$.
2. When $\bigcup_{i=1}^s V_{\text{loc}}^u(y_i)$ covers “enough” of $W_n$, the sum of the weights $\sum_{i=1}^s g_n(y_i)$ can be bounded away from 0 and $\infty$.

Given a rectangle $R$, the intersection $W_n \cap R$ is contained in a disjoint union of local leaves, so that $f_n^m m_x^c|_R$ is given by (8.3). We will use this to prove the following result.

**Lemma 8.1.** If $\mu$ is any limit point of the sequence $\mu_n$ from [4.2], then $\mu$ satisfies Statement [3] of Theorem 4.7: given any rectangle $R$ with $\mu(R) > 0$, the conditional measures of $\mu$ are equivalent to the reference measures $m_y^c$ and satisfy the bound

$$
C_0^{-1} \leq \frac{d\mu_y^u}{dm_y^c}(z)m_y^c(R) \leq C_0 \text{ for } \mu_y^u - a.e. \ z \in V_R^u(y).
$$

Note that although (8.3) gives good control of the conditional measures of $\mu_n$, it is not in general true that the conditionals of a limit are the limits of the conditionals; that is, one does not automatically have $(\mu_n)_y^c \to \mu_y^c$ whenever $\mu_n \to \mu$. In order to establish the
desired properties for the conditional measures of \( \mu \), we will need to use the fact that the conditionals of \( \mu_n \) are represented by density functions for which we have uniform bounds as in (8.2). We will also need the following characterization of the conditional measures, which is an immediate consequence of [21 Corollary 5.21].

**Proposition 8.2.** Let \( \mu \) be a finite Borel measure on \( \Lambda \) and let \( R \subset \Lambda \) be a rectangle with \( \mu(R) > 0 \). Let \( \{ \xi_\ell \}_{\ell \in \mathbb{N}} \) be a refining sequence of finite partitions of \( R \) that converge to the partition \( \xi \) into local unstable sets \( V^u_R(x) = V^u_{\text{loc}}(x) \cap R \). Then there is a set \( R' \subset R \) with \( \mu(R') = \mu(R) \) such that for every \( y \in R' \) and every continuous \( \psi: R \to \mathbb{R} \), we have

\[
\int_{V^u_R(y)} \psi(z) \, d\mu^{\xi}_{V^u_R(y)}(z) = \frac{1}{\mu(\xi(y))} \int_{\xi(y)} \psi(z) \, d\mu(z),
\]

where \( \xi_n(y) \) denotes the element of the partition \( \xi_\ell \) that contains \( y \).

**Proof of Lemma 8.1.** Given a rectangle \( R \subset \Lambda \) with \( \mu(R) > 0 \), let \( \xi_\ell \) be a refining sequence of finite partitions of \( R \) such that for every \( y \in R \) and \( \ell \in \mathbb{N} \), the set \( \xi_\ell(y) \) is a rectangle, and \( \bigcup_{\ell \in \mathbb{N}} \xi_\ell(y) = V^u_R(y) \).

Let \( R' \subset R \) be the set given by Proposition 8.2. We prove that (8.4) holds for each \( y \in R' \). Recall that \( W_k = f^k(V^u_{\text{loc}}(x) \cap \Lambda) \), so that \( f^k m^C_x \) is supported on \( W_k \). Given \( y \in R' \) and \( k, \ell \in \mathbb{N} \), the set \( W_k \cap \xi_\ell(y) \) is contained in \( \bigcup_{i=1}^s V^u_R(z^{(i)}_{k,\ell}) \) for some \( s \in \mathbb{N} \) and \( z^{(1)}_{k,\ell}, \ldots, z^{(s)}_{k,\ell} \in W_k \cap \xi_\ell(y) \). Without loss of generality we assume that the sets \( V^u_R(z^{(i)}_{k,\ell}) \) are disjoint. Following (8.3), we want to write \( f^k m^C_x \) as a linear combination of measures supported on these sets; the only problem is that some of these sets may not be completely contained in \( W_k \).

To address this, let \( I(k) = \{ i \in \{1, \ldots, s\} : V^u_R(z^{(i)}_{k,\ell}) \subset W_k \} \), and let \( \nu_{k,\ell} \) be the restriction of \( m^C_x \) to the set \( \bigcup_{i \in I(k)} f^{-k} V^u_R(z^{(i)}_{k,\ell}) \subset V^u_{\text{loc}}(x) \). Then we have

\[
(f^k m^C_x - f^k \nu_{k,\ell})|_{\xi_\ell(y)} = (f^k m^C_x)|_{Z_k} \quad \text{for} \quad Z_k := \bigcup_{i \in I(k)} W_k \cap \Lambda \cap V^u_{\text{loc}}(z^{(i)}_{k,\ell}),
\]

and since \( V^u_R(z^{(i)}_{k,\ell}) \subset B^u_{\Lambda}(z^{(i)}_{k,\ell}, \tau) \), we obtain that

\[
Z_k \subset \{ z \in W_k : B^u_{\Lambda}(z, 2\tau) \not\subset W_k \}.
\]

Taking the preimage gives

\[
Y_k := f^{-k} Z_k \subset \{ y \in V^u_{\text{loc}}(x) \cap \Lambda : B^u_{\Lambda}(y, 2\tau \lambda^k) \not\subset V^u_{\text{loc}}(x) \},
\]

so \( \bigcap_{k=1}^\infty Y_k = \emptyset \); we conclude that \( f^k m^C_x(Y_k) = m^C_{\xi(y)}(Y_k) \to 0 \) as \( k \to \infty \), so (8.6) gives

\[
\lim_{k \to \infty} \left\| (f^k m^C_x - f^k \nu_{k,\ell})|_{\xi_\ell(y)} \right\| = 0.
\]

It follows that \( \frac{1}{n_j} \sum_{k=0}^{n_j-1} f^k \nu_{k,\ell} \) converges to \( \mu|_{\xi_\ell(y)} \) in the weak* topology, and thus for every continuous \( \psi: \xi(y) \to \mathbb{R} \), (8.3) gives

\[
\int_{\xi(y)} \psi \, d\mu = \lim_{j \to \infty} \frac{1}{n_j} \sum_{k=0}^{n_j-1} \sum_{i \in I(k)} g_k(z^{(i)}_{k,\ell}) \int_{V^u_R(z^{(i)}_{k,\ell})} \psi(z) \rho(z) \, d(m^C_{\xi(y)})(z).
\]
Given $p, q \in R$ and a continuous function $\psi: R \to \mathbb{R}$, (8.2) and Theorem 4.6 give
\[ \int_{V_R^n(p)} \psi(z) \rho_k^p(z) \, dm^C_p(z) = e^{\pm Q_u} \int_{V_R^n(q)} \psi(z) \, dm^C_p(z) = e^{\pm Q_u} C^{-1} \int_{V_R^n(q)} \psi(z') \, dm^C_q(z'). \]
Now assume that $\psi > 0$; then when the leaves $V_R^n(p)$ and $V_R^n(q)$ are sufficiently close, we have $\psi(\pi_{pq}z') = 2\pm 1 \psi(z')$, and thus for all sufficiently large $\ell$, (8.7) gives
\[ (8.8) \quad \int_{\xi_\ell(y)} \psi \, d\mu = (2e^{Q_u} C)^{\pm 1} \left( \lim_{j \to \infty} \frac{1}{n_j} \sum_{k=0}^{n_j-1} \sum_{i \in I(k)} g_k(z^{(i)}_{k,\ell}) \right) \int_{V_R^n(y)} \psi \, dm^C_z. \]
When $\psi \equiv 1$ this gives
\[ \mu(\xi_\ell(y)) = (2e^{Q_u} C)^{\pm 1} \left( \lim_{j \to \infty} \frac{1}{n_j} \sum_{k=0}^{n_j-1} \sum_{i \in I(k)} g_k(z^{(i)}_{k,\ell}) \right) m^C_y(V_R^n(y)), \]
and so (8.5) yields
\[ \int_{V_R^n(y)} \psi \, d\mu_y(V_R^n(y)) = (2e^{Q_u} C)^{\pm 2} \frac{1}{m^C_y(V_R^n(y))} \int_{V_R^n(y)} \psi \, dm^C_z. \]
Since $\psi > 0$ was arbitrary, this proves (8.4) and completes the proof of Lemma 8.1. □

8.2. Local product structure, Gibbs property, and full support. The fact that $\mu$ has local product structure, is fully supported, and has the Gibbs property follows by exactly the same argument as in [15] §6.3.2; we stress that the proofs in that section do not require $E^s$ to be contracting, but rather only require that $\|DF\|_{E^s} \leq 1$, so that they work in our setting using $E^{cs}$ in place of $E^s$. In addition to the results claimed here, that section also includes a proof of ergodicity using the standard Hopf argument, which does require uniform contraction in the stable direction; since this does not apply here, we prove ergodicity using a modified Hopf argument in §8.3. The condition $\|DF\|_{E^s} \leq 1$ is sufficient for the remaining results in [15] §6.3.2.

The first step is to prove local product structure, which follows from Theorem 4.6 and (4.3): given a rectangle $R$ with $\mu(R) > 0$, for $\mu$-a.e. $y, z \in R$ and every $A \subset V_R^n(z)$, we have
\[ (8.9) \quad \mu^\mu_y(\pi_{yz}A) = C_0^{\pm 1} m^C_y(\pi_{yz}A)/m^C_y(R) = C_0^{\pm 1} C^{\pm 2} m^C_z(A)/m^C_z(R) = (C_0 C)^{\pm 2} \mu^\mu_z(A). \]
Thus the properties in Lemma 2.9 hold. The proof of full support and the Gibbs property in [15] use the following rectangles:
\[ R_n(x, \delta) := \left[ B_n^u(x, \delta) \cap \Lambda, B_n^{cs}(x, \delta) \right] = \{ [y, z] : y \in B_n^u(x, \delta) \cap \Lambda, z \in B_n^{cs}(x, \delta) \}. \]
Note that when $n = 0$ we get
\[ R_0(x, \delta) = R(x, \delta) = \left[ B_1^u(x, \delta), B_1^{cs}(x, \delta) \right] = \{ [y, z] : y \in B_1^u(x, \delta), z \in B_1^{cs}(x, \delta) \} \]
as in (2.2). The rectangles $R_n(x, \delta)$ are related to the Bowen balls $B_n(x, \delta)$ as follows.

Lemma 8.3. For every sufficiently small $\delta > 0$, $x \in \Lambda$, and $n \geq 0$, we have $R_n(x, \delta) = \bigcap_{k=0}^{n-1} f^{-k} R(f^k x, \delta)$. Moreover, for every small $\delta > 0$, there is $\delta' > 0$ such that
\[ (8.10) \quad R_n(x, \delta/3) \subset B_n(x, \delta) \cap \Lambda \subset R_n(x, \delta') \]
for every $x \in \Lambda$ and $n \in \mathbb{N}$. Moreover, $\delta' \to 0$ as $\delta \to 0$. 
Proof. We start with the first claim. Given \( p \in R_n(x, \delta) \), there are \( y \in \overline{B^u_n(x, \delta)} \cap \Lambda \) and \( z \in \overline{B^s(\Lambda)}(x, \delta) \) such that \( p = [y, z] \), and then for all \( 0 \leq k < n \) we observe that \( f^k(y) \in B^u_\Lambda(f^kx, \delta) \) and \( f^k(z) \in B^s_\Lambda(f^kx, \delta) \), so
\[
f^k(p) = f^k([y, z]) = [f^k(y), f^k(z)] \in R(x, \delta).
\]
Conversely, if \( p \in \bigcap_{k=0}^{n-1} f^{-k}R(f^kx, \delta) \), then there are \( y_k \in B^u_\Lambda(f^kx, \delta) \) and \( z_k \in B^s_\Lambda(f^kx, \delta) \) such that \( f^k(p) = [y_k, z_k] \) for each \( 0 \leq k < n \). In this case we must have \( y_k = [f^k(p), f^k(x)] = f_k([p, x]) = f_k(y_0) \) for each \( k \), and thus \( y_0 \in \overline{B^u_n(x, \delta)} \), so \( p \in R_n(x, \delta) \).

It remains to prove (8.10). For all \( p \in R_n(x, \delta/3) \) we have \( f^k(p) \in R(x, \delta/3) \subset B(x, \delta) \) for all \( 0 \leq k < n \), so \( p \in \overline{B_n(x, \delta)} \). For the other bound, since \( \nu^u (x) \) and \( \nu^s (x) \) depend continuously on \( x \), for every sufficiently small \( \delta > 0 \) there is \( \delta' > 0 \) such that \( B(x, \delta) \cap \Lambda \subset R(x, \delta') \) for all \( x \in \Lambda \), and \( \delta' \to 0 \) as \( \delta \to 0 \). It follows that
\[
\overline{B_n(x, \delta)} = \bigcap_{k=0}^{n-1} f^{-k}(B(f^kx, \delta)) \subset \bigcap_{k=0}^{n-1} f^{-k}(R(f^kx, \delta')) = R_n(x, \delta_2).
\]

Using the fact that conditional measures of \( \mu \) are equivalent to the reference measures \( m^x \), which scale according to (7.1), [15, Lemma 6.8] shows that given \( \delta > 0 \), there is \( Q_8 > 0 \) such that for every \( x, \delta, n \) as above, we have
\[
\mu(R_n(x, \delta)) = Q_8 e^{-nP(\varphi) + S_n \varphi(x)} \mu(R(x, \delta)).
\]
Thus to prove full support and the Gibbs property, it is enough to show that \( \inf_{x \in \Lambda} R(x, \delta) > 0 \) for every \( \delta > 0 \). For this we need the following.

**Lemma 8.4.** For every sufficiently small \( \delta > 0 \), there is \( \delta' > 0 \) such that for every \( z \in \Lambda \) and \( x \in R(z, \delta') \), we have \( R(z, \delta') \subset R(x, \delta) \).

**Proof.** As in Lemma 8.3, given \( \delta > 0 \) small, there is \( \delta' > 0 \) such that \( B(x, 6\delta') \subset R(x, \delta) \) for all \( x \in \Lambda \). Then for all \( z \in \Lambda \) and \( x \in R(z, \delta') \), we have \( x \in B(z, 3\delta') \) and thus
\[
R(z, \delta') \subset B(z, 3\delta') \subset B(x, 6\delta') \subset R(x, \delta).
\]

Using Lemma 8.4 [15, Lemma 6.10] establishes the desired lower bound on \( R(x, \delta) \), which gives full support and the Gibbs property from (8.11).

### 8.3. Ergodicity via a modified Hopf argument

In this section, we prove that if \( \mu \) is an \( f \)-invariant probability measure on \( \Lambda \) with local product structure, then \( \mu \) is ergodic.

**Definition 8.5.** A point \( z \in \Lambda \) is Birkhoff regular if the Birkhoff averages
\[
\psi^-(z) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \psi(f^{-k}z) \text{ and } \psi^+(z) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \psi(f^kz)
\]
are defined and equal to each other for every continuous function \( \psi \) on \( \Lambda \). In this case we write \( \langle \psi \rangle = \psi^-(z) = \psi^+(z) \) for their common value. The set of Birkhoff regular points is denoted \( \mathcal{B} \).

**Lemma 8.6.** Let \( \psi : \Lambda \to \mathbb{R} \) be continuous. Then
\begin{enumerate}
  \item for every \( x, y \in \mathcal{B} \) with \( y \in V^u_\Lambda(x) \), we have \( \overline{\psi}(x) = \overline{\psi}(y) \); and
\end{enumerate}
(2) for every $\zeta > 0$, there is $\epsilon > 0$ such that for every $x, y \in B$ with $y \in B^{cs}(x, \epsilon)$, we have $|\overline{\psi}(x) - \overline{\psi}(y)| < \zeta$.

Proof. Both statements rely on the following consequence of uniform continuity: for every continuous $\psi: \Lambda \to \mathbb{R}$ and every $\zeta > 0$, there is $\epsilon > 0$ such that if $x_k, y_k \in \Lambda$ are sequences with $\limsup_{k \to \infty} d(x_k, y_k) \leq \epsilon$, then $\limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(x_k) - \frac{1}{n} \sum_{k=1}^{n} f(y_k) \leq \zeta$. For the first claim in the lemma, put $x_k = f^{-k}(x)$ and $y_k = f^{-k}(y)$ so that $d(x_k, y_k) \to 0$ and $\epsilon > 0$ can be taken arbitrarily small. For the second claim in the lemma, use forward iterates together with the fact from (A2) that $f$ does not expand distances along $V^{cs}_{loc}(x)$, so $d(f^k(x), f^k(y)) < \epsilon$ for all $k \geq 0$. □

Now let $\mu$ have local product structure, and consider the set
\[(8.12) \quad A_\mu := \{x \in B : \mu^n_x(V^{cu}_{loc}(x) \setminus B) = 0\}\]
of all points $x$ for which $\mu^n_x$-a.e. point in $V^{cu}_x(x)$ is Birkhoff regular for $\mu$. By the Birkhoff ergodic theorem, we have $\mu(B) = 1$, so $\mu(M \setminus B) = 0$, and thus the disintegration into conditional measures in [24] gives $\mu^n_x(M \setminus B) = 0$ for $\mu$-a.e. $x$; in other words, $\mu(A_\mu) = 1$. Thus to prove that $\mu$ is ergodic, it suffices to prove that $\overline{\psi}$ is constant on $A_\mu$, which we do in the next lemma.

Figure 8.1. Birkhoff averages are essentially constant.

**Lemma 8.7.** Given any $x, \omega \in A_\mu$, we have $\overline{\psi}(\omega) = \overline{\psi}(x)$.

Proof. Fix $\epsilon > 0$ small enough that $B(x, \epsilon) \cap \Lambda$ is contained in a rectangle $R$. By Lemma 6.2, there exists $k \in \mathbb{N}$ such that
\[\tau \lambda^k < \epsilon/2 \text{ and } f^k(B^u(\omega, \epsilon/2)) \cap B^{cs}(x, \epsilon) \neq \emptyset.\]
Let $p$ denote a point in the intersection; then $p \in B^{cs}(x, \epsilon)$ and $f^{-k}(p) \in B^u(\omega, \epsilon)$, so by our choice of $k$ we have $f^{-k}(V^u_R(p)) \subset B^u(f^{-k}p, \epsilon/2) \subset B^u(\omega, \epsilon)$; see Figure 8.1. Since $\mu^u_x(B^u_R(\omega, \epsilon) \setminus B) = 0$, we conclude that $\mu^u_{p}(V^u_R(p) \setminus B) = 0$ by Lemma 2.8. Similarly, $\mu^u_x(V^u_R(x) \setminus B) = 0$; since $\mu$ has local product structure, this implies that $\mu^u_x(\pi_x(p) \setminus \pi_x(B)) = 0$. Therefore, $\overline{\psi}(x) = \overline{\psi}(p) = \overline{\psi}(\omega)$. □
0, and thus $\mu_p(V^u_R(p) \setminus (\mathcal{B} \cap \pi_{xp}\mathcal{B})) = 0$. In particular, there exists $z \in V^u_R(p) \cap \mathcal{B} \cap \pi_{xp}\mathcal{B}$, so that $y = \pi_{px}(z) \in V^u_R(x) \cap \mathcal{B}$. Then we have
\[ \overline{\psi}(x) = \overline{\psi}(y) \] and \( \underline{\psi}(\omega) = \overline{\psi}(f^{-k}z) = \overline{\psi}(z) \)
where the first two equalities use the first part of Lemma 8.6. This gives \( |\overline{\psi}(\omega) - \overline{\psi}(y)| = |\overline{\psi}(z) - \overline{\psi}(y)| \). Moreover, given any $\zeta > 0$, we can use the second part of Lemma 8.6 to choose $\epsilon > 0$ so small that $|\overline{\psi}(z) - \overline{\psi}(y)| < \zeta$. Letting $\zeta \to 0$ we conclude that $\overline{\psi}(\omega) = \overline{\psi}(x)$. \( \square \)

8.4. Uniqueness via Bowen’s argument. In this section we prove the following result.

Proposition 8.8. Let $\Lambda, f, \varphi$ be as in 8.1, and let $\mu$ be an ergodic $f$-invariant probability measure on $\Lambda$ such that the conditional measures $\mu^u_x$ are equivalent to the Carathéodory measures $m^u_x$ for $\mu$-a.e. $x$. Then $\mu$ is the unique equilibrium measure for $\varphi$.

We start by recalling some definitions and facts from 23 regarding entropy along the unstable foliation.

For a partition $\alpha$ of $\Lambda$, let $\alpha(x)$ denote the element of $\alpha$ containing $x$. If $\alpha$ and $\beta$ are two partitions such that $\alpha(x) \subseteq \beta(x)$ for all $x \in \Lambda$, we then write $\alpha \leq \beta$. For a measurable partition $\beta$, we denote $\beta^n_m = \bigvee_{i=m}^n f^{-i}\beta$. Take $\epsilon_0 > 0$ small. Let $\mathcal{Q} = \mathcal{Q}_{\epsilon_0}$ denote the set of finite measurable partitions of $\Lambda$ whose elements have diameters not exceeding $\epsilon_0$. For each $\beta \in \mathcal{Q}$ we define a finer partition $\eta = \mathcal{Q}^{\mu}(\beta)$ such that $\eta(x) = \beta(x) \cap V^u_{loc}(x)$ for each $x \in \Lambda$. Let $\mathcal{Q}^u = \mathcal{Q}^u_{\epsilon_0}$ denote the set of all partitions obtained this way.

Given a measure $\nu$ and measurable partitions $\alpha$ and $\eta$, let
\[ H_\nu(\alpha|\eta) := -\int_{\Lambda} \log \nu_{\eta(x)}(\alpha(x)) \, d\nu(x) \]
(8.13)

denote the conditional entropy of $\alpha$ given $\eta$ with respect to $\nu$, where $\{\nu_{\eta(x)}\}$ is a family of (normalized) conditional measures of $\nu$ relative to $\eta$.

The conditional entropy of $f$ with respect to a measurable partition $\alpha$ given $\eta \in \mathcal{Q}^u$ is defined as
\[ h_\nu(f, \alpha|\eta) = \limsup_{n \to \infty} \frac{1}{n} H_\nu(\alpha^{n-1}|\eta). \]

(8.14)
The following is a direct consequence of 23 Theorem A and Corollary A.1] stated in our setting.

Proposition 8.9. Suppose $\nu$ is an ergodic measure. Then for any $\alpha \in \mathcal{Q}$ and $\gamma \in \mathcal{Q}^u$ one has that
\[ h_\nu(f) = h_\nu(f, \alpha|\gamma). \]

(8.15)

We recall the following technical result.

Lemma 8.10. Let $\alpha$, $\beta$, and $\gamma$ be measurable partitions with $H_\nu(\alpha|\gamma), H_\nu(\beta|\gamma) < \infty$.

1. If $\gamma \geq \beta$, then $H_\nu(\alpha|\gamma) \leq H_\nu(\alpha|\beta)$.

2. $H_\nu(\alpha^{n-1}|\gamma) = H_\nu(\alpha|\gamma) + \sum_{i=1}^{n-1} H_\nu(\alpha|f^i(\alpha^{i-1} \vee \gamma))$.

Statement (1) of Lemma 8.10 is well known and can be found for example in 35. Statement (2) is proved in 23 as Lemma 2.6(i). We use these to prove the following lemma.
Lemma 8.11. For any $\alpha \in \mathcal{Q}$ and $\gamma \in \mathcal{Q}^u$ one has that
\[ h_\mu(f, \alpha | \gamma) \leq H_\mu(\alpha | \mathcal{Q}^u(f\alpha)). \]

Proof. We start with an observation regarding the partition $f^i(\alpha_0^{i-1} \vee \gamma)$ from Lemma 8.10(2): because $\alpha \in \mathcal{Q}$, every element of $\alpha_0^{i-1}$ has diameter $< \epsilon_0$ in the dynamical metric $d_i$, and is thus contained in $B_i(x, \epsilon_0)$ for some $x$. Since $\gamma \in \mathcal{Q}^u$, we have $\gamma(x) \subset V_{\text{loc}}^u(x)$, and thus every element of $\alpha_0^{i-1} \vee \gamma$ is contained in $B_i(x, \epsilon_0)$ for some $x$. It follows that every element of $f^i(\alpha_0^{i-1} \vee \gamma)$ is contained in $f^i(B_i^u(x, \epsilon_0))$ for some $x$. This gives
\[ (f^i(\alpha_0^{i-1} \vee \gamma))(y) \subset (f\alpha)(y) \cap V_{\text{loc}}^u(y) = (\mathcal{Q}^u(f\alpha))(y), \]
and thus $f^i(\alpha_0^{i-1} \vee \gamma) \supset \mathcal{Q}^u(f\alpha)$. We deduce that
\[ H_\mu(\alpha_0^{n-1} | \gamma) = H_\mu(\alpha | \gamma) + \sum_{i=1}^{n-1} H_\mu(\alpha | f^i(\alpha_0^{i-1} \vee \gamma)) \]
\[ \leq H_\mu(\alpha | \gamma) + (n - 1)H_\mu(\alpha | \mathcal{Q}^u(f\alpha)), \]
where the first line uses Statement (2) of Lemma 8.10 and the second line uses Statement (1) of that lemma. Dividing both sides by $n$ and sending $n \to \infty$ concludes the proof of Lemma 8.11. \qed

Now we prove Proposition 8.8. Since every ergodic component of an equilibrium measure is itself an equilibrium measure, it suffices to prove that $\mu$ is the only ergodic equilibrium measure for $\varphi$. Moreover, if $\nu \ll \mu$ is ergodic, then ergodicity of $\mu$ implies that $\nu = \mu$; thus it suffices to consider an ergodic measure $\nu$ such that $\nu \perp \mu$, and prove that $P_\nu := h_\nu(\varphi) + \int_M \varphi \, d\nu < P(\varphi)$.

Figure 8.2. The partitions $\mathcal{R}$ and $\mathcal{R}_n$, and the choice of $\epsilon_R$.

With $\epsilon_0$ as above, let $\mathcal{R} \in \mathcal{Q}_{\epsilon_0}$ be a finite partition by rectangles as in Lemma 2.5 so that each rectangle has diameter $< \epsilon$, is the closure of its interior, and moreover has $(\mu + \nu)$-null boundary, where we use the relative topology on $\Lambda$. Choose one point from the interior of each rectangle and enumerate them as $x_1, \ldots, x_m \in \Lambda$. Fix $\epsilon_R > 0$ such that
\[ B^u_\Lambda(y, \epsilon_R) \subset \mathcal{R}(x_i) \text{ for all } i = 1, \ldots, m \text{ and } y \in V_{\mathcal{R}(x_i)}^{cs}(x_i), \]
as in Figure 8.2. As shown in that figure, for $n \in \mathbb{N}$ let $\mathcal{R}_n$ be a partition by rectangles obtained by dividing each rectangle in $\mathcal{R}$ further into rectangles in such a way that
that guarantees that the partition by rectangles
first of these items guarantees that $Q^u(R) = Q^u(R_n)$ for every $n$, and the second
guarantees that the partition by rectangles $\alpha_n := f^{-n}R_n$ is also contained in $Q$. Observe
that
\begin{equation}
\begin{aligned}
nh_\nu(f) = h_\nu(f^n) &= h_\nu(f^n, \alpha_n|Q^u(\alpha_n)) \\
&\leq H_\nu(\alpha_n|Q^u(\mathcal{R}_n)) = H_\nu(\alpha_n|Q^u(\mathcal{R}))
\end{aligned}
\end{equation}
where the first equality is a standard fact about entropy, the second equality uses Proposition \[8.9\]
and the inequality uses Lemma \[8.11\].

For every element $\eta \in \mathcal{Q}^u(\mathcal{R})$, let $\nu_\eta$ denote the conditional measure of $\nu$ on $\eta$
(relative to the measurable partition $\mathcal{Q}^u(\mathcal{R})$ of $\Lambda$); we also write this as $\nu_\eta$ when $\eta$
is the partition element containing $x$. Let also $m_\eta^C$ denote the Carathéodory measure on $\eta$.

As in Definition \[8.5\] let $B$ denote the set of Birkhoff regular points, and consider as in
\[8.12\] the sets
\[A_\nu = \{x \in B : \nu_\nu^u(V_{\text{loc}}^u(x) \setminus B) = 0\}, \quad A_\mu = \{x \in B : \nu_\nu^u(V_{\text{loc}}^u(x) \setminus B) = 0\}.
\]
The same argument as given there shows that $\nu(A_\nu) = 1$ and $\mu(A_\mu) = 1$. Let $\psi : \Lambda \to \mathbb{R}$
be a continuous function such that $\int \psi \, d\nu \neq \int \psi \, d\mu$, and consider the following sets of generic points:
\[G_\nu^\psi = \left\{ x \in B : \bar{\psi}(x) = \int \psi \, d\nu \right\}, \quad G_\mu^\psi = \left\{ x \in B : \bar{\psi}(x) = \int \psi \, d\mu \right\}.
\]
By the Birkhoff ergodic theorem and the fact that $\nu, \mu$ are ergodic, we have $\nu(G_\nu^\psi) = \mu(G_\mu^\psi) = 1$.

**Lemma 8.12.** For every $x \in A_\nu \cap G_\nu^\psi$, writing $\eta = \eta(x)$, the measures $\nu_\eta = \nu_\nu^u$
and $m_\eta^C$ are mutually singular, and in particular, satisfy $\nu_\eta(\eta \setminus B) = 0$ and $m_\eta^C(\eta \cap B) = 0$.

**Proof.** Suppose to the contrary that $\nu_\nu^u$ and $m_\eta^C$ are not mutually singular; then since
$\nu_\nu^u(\eta(x) \setminus B) = 0$, we must have $m_\eta^C(\eta(x) \cap B) > 0$. Fix $0 < \zeta < |\int \psi \, d\mu - \int \psi \, d\nu|$ and let $\epsilon > 0$ be given by Lemma \[8.6\]. Since $\mu$ gives positive measure to every open set in $\Lambda$, there exists $p \in B(x, \epsilon) \cap A_\mu \cap G_\mu^\psi$. In particular, $\eta(p) \cap B$ has full $\mu_\nu^u$-measure, and thus also has full $m_p^C$-measure by \[4.3\]. As in the proof of Lemma \[8.7\] (see also Figure \[8.1\]),
we conclude that $m_\eta^C(\pi_{\eta}(\eta(x) \cap B) \cap B > 0$, and thus there exists $y \in B \cap \eta$ such that
$z := V_{\text{loc}}^u(y) \setminus \eta(p) \in B$, and Lemma \[8.6\] gives
\[\bar{\psi}(x) = \bar{\psi}(y) = \bar{\psi}(z) \pm \zeta = \bar{\psi}(p) \pm \zeta.\]
Since $x \in G_\nu^\psi$ and $p \in G_\mu^\psi$, we conclude that $\int \psi \, d\nu = \int \psi \, d\mu \pm \zeta$, contradicting our choice of $\zeta$, and we conclude that $\nu_\nu^u$ and $m_\eta^C$ are mutually singular, as claimed. \qed

We continue with the proof of uniqueness. By Lemma \[8.12\] for every $x \in A_\nu \cap G_\nu^\psi$, we
have $\nu_{\eta(x)}(\eta(x) \cap B) = 1$ and $m_{\eta(x)}^C(\eta(x) \cap B) = 0$. Thus for every $k \in \mathbb{N}$ there are sets
$G_k \subset U_k \subset \eta(x)$ such that $G_k$ is compact, $U_k$ is (relatively) open, $\nu_{\eta(x)}(G_k) \to 1$, and $m_{\eta(x)}^C(U_k) \to 0$. Since the diameter of every element in $\alpha_n \cap \eta(x)$ goes to 0 uniformly as $n \to \infty$, we can choose for each $n$ a set $C_{\eta(n)}^n \subset \eta(x)$ and a value of $k = k(n)$ such that

1. $C_{\eta(n)}^n$ is a union of elements of $\alpha_n \cap \eta(x)$,
2. $G_k \subset C_{\eta(n)}^n \subset U_k$, and
3. $k \to \infty$ as $n \to \infty$.

In particular, the sequence of sets $C_{\eta(n)}^n$ satisfies

$$(8.19) \quad \lim_{n \to \infty} \nu_{\eta(x)}(C_{\eta(n)}^n) = 1 \text{ and } \lim_{n \to \infty} m_{\eta(x)}^C(C_{\eta(n)}^n) = 0.$$ 

By our construction of $\alpha_n$ and the definition of $\epsilon_R$ in (8.17), for every $A \in \alpha_n$ there is a point $x_A \in A$ with the property that

$$(8.20) \quad B_n^u(y, \epsilon_R) \cap A \subset A \text{ for all } y \in V^{cs}_{s}(x_A).$$

Given $x \in A$ and $A \in \alpha_n$ such that the intersection $V^{cs}_{loc}(x_A) \cap V^{u}_{loc}(x)$ is nonempty, denote the unique point in this intersection by $x^n_A$. Note that in particular, $x^n_A$ is defined whenever $A \cap \eta \neq \emptyset$. With the convention that $0 \log 0 = 0$, (8.18) gives

$$nP_v = n \left( h_v(f) + \int \varphi \, d\nu \right)$$

$$\leq \int_A \left( - \log \nu_u^x(\alpha_n(x) \cap \eta(x)) + S_n\varphi(x) \right) \, d\nu(x)$$

$$= \int_{A/Q^n(\mathbb{R})} \sum_{A \in \alpha_n} \left( - \nu_{\eta}(A \cap \eta) \log \nu_{\eta}(A \cap \eta) + \int_{A \cap \eta} S_n \varphi \, d\nu_{\eta} \right) \, d\tilde{\nu}(\eta).$$

Whenever $A \cap \eta \neq \emptyset$, the point $x^n_A$ exists, lies on $V^{cs}_{s}(x_A)$, and the $u$-Bowen property gives $S_n \varphi(y) \leq S_n \varphi(x^n_A) + Q_u$ for all $y \in A \cap \eta$. Thus we obtain

$$nP_v \leq \int_{A/Q^n(\mathbb{R})} \sum_{A \in \alpha_n} \nu_{\eta}(A \cap \eta) \left( S_n \varphi(x^n_A) - \log \nu_{\eta}(A \cap \eta) \right) \, d\tilde{\nu}(\eta) + Q_u.$$ 

The next step is to separate the sum into two pieces, one corresponding to $A \subset C^n_u$ and the other to $A \not\subset C^n_u$, and then bound each piece from above by the following general inequality from [24] (20.3.5), which holds for all $x_1, \ldots, x_m \geq 0$ and $b_1, \ldots, b_m \in \mathbb{R}$:

$$(8.22) \quad \sum_{i=1}^m x_i(b_i - \log x_i) \leq \left( \sum_{i=1}^m x_i \right) \log \left( \sum_{j=1}^m e^{b_j} \right) + \frac{1}{e}.$$ 

Fix a choice of $\eta$ and $\alpha^n_R = \{ A \in \alpha_n : A \cap \eta \subset C^n_u \}$. Applying (8.22) to the sum over $\alpha^n_R$, with $\nu_{\eta}(A \cap \eta)$ in place of the $x_i$ terms and $S_n \varphi(x^n_A)$ in place of the $b_i$ terms, we obtain

$$(8.23) \quad \sum_{A \in \alpha^n_R} \nu_{\eta}(A \cap \eta) \left( S_n \varphi(x^n_A) - \log \nu_{\eta}(A \cap \eta) \right) \leq \nu_{\eta}(C^n_u) \log \sum_{A \in \alpha^n_R} e^{S_n \varphi(x^n_A)} + \frac{1}{e}.$$
Since the measures \( m_\nu^C \) satisfy the \( u \)-Gibbs property, there is a constant \( Q_0 \) such that \( e^{S_n \varphi(x)} \leq Q_0 e^{n P(\varphi)} m_x^C(B_n^u(x, \epsilon R)) \) for all \( x \in \Lambda \) and \( n \in \mathbb{N} \), and thus (8.20) gives
\[
\sum_{A \in \alpha_n^\eta} e^{S_n \varphi(x_A^\eta)} \leq \sum_{A \in \alpha_n^\eta} Q_0 e^{n P(\varphi)} m_\eta^C(B_n^u(x_A^\eta, \epsilon R)) \leq Q_0 e^{n P(\varphi)} m_\eta^C(C_n^\eta).
\]
Together with (8.23), we obtain
\[
\int_{\Lambda/Q^u(\mathcal{R})} \sum_{A \in \alpha_n^\eta} \nu_\eta(A \cap \eta) \left( S_n \varphi(x_A^\eta) - \log \nu_\eta(A \cap \eta) \right) d\nu(\eta)
\leq \int_{\Lambda/Q^u(\mathcal{R})} \left( \nu_\eta(C_n^\eta) \left( \log m_\eta^C(C_n^\eta) + n P(\varphi) + \log Q_0 + \frac{1}{\epsilon} \right) \nu_\eta(C_n^\eta) \right) d\nu(\eta).
\]
A similar estimate holds if we replace \( \alpha_n^\eta \) by \( \alpha_n \setminus \alpha_n^\eta \), and thus (8.21) gives
\[
n P_\nu \leq \int_{\Lambda/Q^u(\mathcal{R})} \left( \nu_\eta(C_n^\eta) \log m_\eta^C(C_n^\eta) + \nu_\eta(\eta \setminus C_n^\eta) \log m_\eta^C(\eta \setminus C_n^\eta) \right) d\nu(\eta)
+ n P(\varphi) + \log Q_0 + \frac{2}{\epsilon}.
\]
Recalling (8.19), we see that the second term inside the integral is uniformly bounded independent of \( n \), while the first term goes to \( -\infty \) as \( n \to \infty \) for every \( \eta \); we conclude that the integral goes to \( -\infty \) as \( n \to \infty \), and thus
\[
\lim_{n \to \infty} \left( n(P_\nu - P(\varphi)) - \log Q_0 - \frac{2}{\epsilon} \right) = -\infty.
\]
This implies that \( P_\nu < P(\varphi) \), and thus \( \nu \) is not an equilibrium measure for \( \varphi \). We conclude that \( \mu \) is the unique equilibrium measure for \( \varphi \), as claimed.

**APPENDIX A. LEMMAS ON RECTANGLES AND CONDITIONAL MEASURES**

**Proof of Lemma 2.5.** Let \( \epsilon > 0 \) be small enough that \( R(x, \delta) \) as in (2.2) is defined for all \( x \in \Lambda \) and \( \delta \in (0, \epsilon) \). Then given \( x \in \Lambda \), the function \( \delta \mapsto \mu(R(x, \delta)) \) is monotonic on \((0, \epsilon)\), and hence continuous at all but countably many values of \( \delta \). Let \( \delta(x) \) be a point of continuity; then \( \mu(\partial R(x, \delta(x))) = 0 \). Note that \( \{ \text{int } R(x, \delta(x)) : x \in \Lambda \} \) is an open cover for the compact set \( \Lambda \), so there are \( x_1, \ldots, x_n \in \Lambda \) such that writing \( \delta_i = \delta(x_i) \), the rectangles \( \{ R(x_i, \delta_i) \}_{i=1}^n \) cover \( \Lambda \). Given \( I \subset \{ 1, \ldots, n \} \), the set \( R_I = \bigcap_{i \in I} R(x_i, \delta_i) \) is either empty or is itself a rectangle that is the closure of its interior and has \( \mu \)-null (relative) boundary. Taking \( \mathcal{I} \) to be the collection of all subsets \( I \) of \( \{ 1, \ldots, n \} \) for which \( R_I \) is nonempty, the set of rectangles \( \{ R_I : I \in \mathcal{I} \} \) satisfies the properties required by the lemma. \( \square \)

**Proof of Lemma 2.7.** First we prove the lemma when \( R_1 \subset R_2 \). In this case every \( \psi \in L^1(R_1, \mu) \) extends to a function in \( L^1(R_2, \mu) \) by setting it to 0 on \( R_2 \setminus R_1 \); then fixing \( x \in R_2 \) and defining measures \( \tilde{\mu}^i \) on \( V_{R_i}^\eta(x) \) by
\[
\tilde{\mu}^i(A) = \mu \left( \bigcup_{y \in A} V_{R_i}^\eta(y) \right),
\]
we see from (2.6) that
\[
\int_{R_1} \psi \, d\mu = \int_{R_2} \psi \, d\mu = \int_{V^c_{R_2}} \psi \, d\mu_y \, d\hat{\mu}^2(y) \\
= \int_{V^c_{R_1}} \psi \, d\mu_y \, d\hat{\mu}^2(y),
\]
(A.1)
where the last equality uses the fact that \( \psi \) vanishes on \( R_2 \setminus R_1 \). Let \( N = \{ y \in R_1 : \mu_y^2(R_1) = 0 \} \); then \( N \) is a union of unstable sets in \( R_1 \) and thus
\[
\hat{\mu}^1(N) = \mu(N) = \int_{N \cap V^c_{R_1}} \mu_y^2(R_1) \, d\hat{\mu}^2(y) = 0,
\]
so the function \( c(y) = \mu_y^2(R_1) \) is positive \( \mu \)-a.e. on \( R_1 \), and \( \hat{\mu}^1 \)-a.e. on \( V^c_{R_1} \). Given \( A \subset V^c_{R_1} \), we have
\[
\hat{\mu}^1(A) = \mu \left( \bigcup_{y \in A} V^u_{R_1}(y) \right) = \int_A \mu_y^2(V^u_{R_1}(y)) \, d\hat{\mu}^2(y) = \int_A c(y) \, d\hat{\mu}^2(y);
\]
we conclude that \( \hat{\mu}^1 \ll \hat{\mu}^2 \) with Radon–Nikodym derivative given by \( c \). Since \( c \) is positive \( \hat{\mu}^1 \)-a.e., we conclude that \( \hat{\mu}^2 \ll \hat{\mu}^1 \) with derivative \( 1/c \), so (A.1) gives
\[
\int_{R_1} \psi \, d\mu = \int_{V^c_{R_1}} \int_{V^u_{R_1}} \psi \, \frac{\psi}{c(y)} \, d\mu_y \, d\hat{\mu}^1(y),
\]
By a.e.-uniqueness of the system of conditional measures, this shows that \( \mu_y^1 = \frac{1}{c(y)} \mu_y^2 \big|_{V^c_{R_1}} \) for \( \mu \)-a.e. \( y \in R_1 \).

Now consider two arbitrary rectangles \( R_1, R_2 \). The set \( R_3 = R_1 \cap R_2 \) is also a rectangle, and by the argument given above, given \( x \in R_3 \), the functions \( c_i : V^c_{R_3}(x) \to [0,1] \) defined by \( c_i(y) = \mu_y^2(R_3) \) are positive \( \hat{\mu}^1 \)-a.e., and \( \mu_y^1 = \frac{1}{c(y)} \mu_y^2 \big|_{V^c_{R_3}} \) for \( \mu \)-a.e. \( y \in R_3 \), from which we conclude that
\[
\mu_y^1 \big|_{V^c_{R_3}} = \frac{c_1(y)}{c_2(y)} \mu_y^2 \big|_{V^c_{R_3}},
\]
completing the proof of Lemma 2.7.

**Proof of Lemma 2.8.** From Lemma 2.7 it suffices to prove the result for a single choice of rectangles. Thus we let \( R \) be any rectangle containing \( x \), small enough that \( f(R) \) is also a rectangle. Then since \( \mu \) is \( f \)-invariant, for every \( \psi \in L^1(f(R), \mu) \), we have \( \psi \circ f \in L^1(R, \mu) \), and (2.4) gives
\[
\int_{f(R)} \psi \, d\mu = \int_R \psi \circ f \, d\mu = \int_R \int_{V^u_R(x)} \psi \circ f \, d\mu^u_x \, d\mu(x) \\
= \int_R \int_{f(V^u_R(x))} \psi \, d(f_* \mu^u_x) \, d\mu(x) = \int_{f(R)} \int_{V^u_R(y)} \psi \, d(f_* \mu^u_x) \, d\mu(y).
\]
By uniqueness of the system of conditional measures, this shows that for the rectangles \( R \) and \( f(R) \), we have \( f_* \mu^u_x = \mu^u_x \) for \( \mu \)-a.e. \( x \in R \), and by Lemma 2.7, this completes the proof of Lemma 2.8.
Proof of Lemma 2.9. Since $y,z$ play symmetric roles in (1), we conclude that (1) and (2) are equivalent, and similarly for (3) and (4). We prove that (2) is equivalent to (5) and (6); the proof for (4) is similar. Clearly (6) implies (5). We prove that (2) implies (5), and then that (5) implies both (2) and (6).

First suppose that (2) holds, fix $p \in \mathbb{R}$ such that $(\pi_{cs}^{y} \ast \mu_{p}^{u}) (z) \sim \mu_{u}^{p}$ for $\mu$-a.e. $z \in \mathbb{R}$, and define $h: \mathbb{R} \to (0, \infty)$ by

$$h(z) = \frac{d\mu_{u}^{y}}{d(\pi_{cs}^{y} \ast \mu_{p}^{u})}(z).$$

Define $\tilde{\mu}_{p}$ on $V_{cs}^{u}(p)$ by $\tilde{\mu}_{p}(E) = \mu(\bigcup_{y \in E} V_{cs}^{u}(y))$, so that (2.6) and (A.2) give

$$\int_{V_{cs}^{u}(p)} \int_{V_{cs}^{u}(y)} \psi(z) h(z) d(\pi_{cs}^{y} \ast \mu_{p}^{u}) (z) d\tilde{\mu}_{p}(y) = \int_{\mathbb{R}} \psi(z) h(z) d(\mu_{u}^{p} \otimes \tilde{\mu}_{p})(z).$$

This proves (5). Now we suppose that (5) holds, so that writing

$$h(z) = \frac{d\mu_{u}^{y}}{d(\tilde{\mu}_{p}^{u} \otimes \tilde{\mu}_{cs}^{y})}(z),$$

we have

$$\int_{V_{cs}^{u}(p)} \int_{V_{cs}^{u}(y)} \psi(z) h(z) d(\pi_{cs}^{y} \tilde{\mu}_{p}^{u}) (z) d\tilde{\mu}_{cs}^{y}(y).$$

Recall that $\mu_{u}^{y}$ is uniquely determined (up to a scalar) for $\mu$-a.e. $y$ by the condition that there is a measure $\nu$ on $V_{cs}^{u}(p)$ with

$$\int_{V_{cs}^{u}(p)} \int_{V_{cs}^{u}(y)} \psi(z) d\mu_{u}^{y}(z) d\nu(y)$$

for every integrable $\psi$. Comparing this to (A.5) we conclude that $\mu_{u}^{y} \sim \pi_{cs}^{y} \tilde{\mu}_{p}^{y}$ for $\mu$-a.e. $y \in \mathbb{R}$, which establishes both (2) and (6).

\[\Box\]

References


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