Usable hyperbolicity and SRB measures

Vaughn Climenhaga

University of Toronto

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Joint work with Dmitry Dolgopyat and Yakov Pesin

The talk in one slide

- Behaviour of dissipative maps described by a physical measure.
- SRB measures are physical, have non-zero exponents.
- Want a condition to check at Lebesgue-typical points: non-zero Lyapunov exponent does not seem to be enough.
- Need to control geometric structures (local manifolds).

New result: non-zero usable hyperbolicity $\Rightarrow$ SRB measure

Key tool: new version of Hadamard–Perron theorem on existence of local stable and unstable manifolds.

Applications: use this to study maps on boundary of Axiom A – dissipative Katok example; neutral fixed point with shear (stable and unstable directions become degenerate at the fixed point).

Describing statistics of orbits

- M a compact Riemannian manifold
- $f\colon M \to M$ a $C^{1+\varepsilon}$ diffeomorphism

Goal: Study case when f is chaotic – trajectories of f appear "random" over long time scales.

Describe statistical properties of orbits using appropriate measures.

- $\mathcal{M}$ = the space of finite Borel measures on M
- $\mathcal{M}(f) = \{\mu \in \mathcal{M} \mid \mu \text{ is } f\text{-invariant}\}$

Definition

x is generic for $\mu \in \mathcal{M}(f)$ if $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)) = \int \varphi\, d\mu$ for every continuous $\varphi\colon M \to \mathbb{R}$. The set of generic points for μ is its basin of attraction, denoted G_μ.

Physically meaningful measures

- An ergodic measure $\mu \in \mathcal{M}(f)$ describes the statistics of μ-a.e. trajectory: $\mu(G_\mu) = 1$.
- To be "physically meaningful", a measure should describe the statistics of *Lebesgue*-a.e. trajectory.

Definition

$\mu \in \mathcal{M}(f)$ is a physical measure if $\mathrm{Leb}(G_\mu) > 0$.

- Smooth or absolutely continuous invariant measures $\left(\mu(E) = \int_E \rho(x) \cdot d\,\mathrm{Leb}(x)\right)$ are physical

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- Smooth or absolutely continuous invariant measures ($\mu(E) = \int_E \rho(x) \cdot d\,\mathrm{Leb}(x)$) are physical, but may not exist. . .
- . . . interesting dynamics often happen on a set of Lebesgue measure zero. (Solenoid, etc.)

Lyapunov exponents

"chaotic" $\rightsquigarrow$ hyperbolic $\approx$ "has stable and unstable directions"

- Don't need absolute continuity in stable directions – a δ-measure on a stable fixed point is a physical measure.
- Only need absolute continuity in unstable directions.

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Theorem (Oseledec)

If $\mu \in \mathcal{M}(f)$ is ergodic, then there exist

- *Lyapunov exponents* $\lambda_1 < \lambda_2 < \cdots < \lambda_p$,
- *an f-invariant decomposition* $TM = E_1 \oplus E_2 \oplus \cdots \oplus E_p$, *and*
- *a set* Z_μ *with* $\mu(Z_\mu) = 1$

s.t. $\lim_{n\to\infty} \frac{1}{n} \log \|Df^n(x)(v)\| = \lambda_i$ *for all* $x \in Z_\mu$, $v \in E_i(x)$.

Definition

μ is hyperbolic if all its Lyapunov exponents are non-zero.

SRB measures as physical measures

If μ is hyperbolic and $x \in Z_\mu$, then $T_xM = E^s(x) \oplus E^u(x)$, where

$$v \in E^s(x) \Rightarrow \lambda(v) < 0 \qquad \text{and} \qquad v \in E^u(x) \Rightarrow \lambda(v) > 0.$$

Pesin theory: For hyperbolic μ, then f has stable and unstable manifolds at μ-a.e. point, tangent to $E^s(x)$ and $E^u(x)$.

Definition

$\mu \in \mathcal{M}(f)$ is an SRB measure if it is hyperbolic and has absolutely continuous conditional measures on unstable manifolds.

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Ergodic SRB measures are physical measures.

- If μ is an SRB measure, then $\mathrm{Leb}\{x \mid \lambda \neq 0\} > 0$.
- Is the converse true? Does $\mathrm{Leb}\{x \mid \lambda \neq 0\} > 0$ imply existence of an SRB measure?

Uniform geometric structure

SRB measures are known to exist in the following settings.

- Uniformly hyperbolic f (Sinai, Ruelle, Bowen)
- Partially hyperbolic f with positive/negative central exponents (Alves–Bonatti–Viana, Burns–Dolgopyat–Pesin–Pollicott)

Key tool is a dominated splitting $T_xM = E^s(x) \oplus E^u(x)$.

1. E^s, E^u depend continuously on x.
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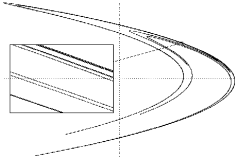
$$\Big(\text{Dominated splitting}\Big) + \Big(\mathrm{Leb}\{x \mid \lambda \neq 0\} > 0\Big) \Rightarrow \text{SRB measure}$$

For non-uniformly hyperbolic f, the splitting is not dominated.

Non-uniformly hyperbolic maps

The Hénon maps $f_{a,b}(x,y) = (a - x^2 - by, x)$ are a perturbation of the family of logistic maps $g_a(x) = a - x^2$.



1. g_a has an absolutely continuous invariant measure for "many" values of a. (Jakobson)
2. For b small, $f_{a,b}$ has an SRB measure for "many" values of a. (Benedicks–Carleson, Benedicks–Young)
3. Similar results for "rank one attractors" – small perturbations of one-dimensional maps with non-recurrent critical points. (Wang–Young)

Genuine non-uniform hyperbolicity, but only one unstable direction, and stable direction must be strongly contracting.

Other non-uniformly hyperbolic maps

Other examples:

1. Hénon $f_{a,b}(x, y) = (a - x^2 - by, x)$ for $b \gg 0$.
2. Generalised Hénon $f_{a,b}(x, y, z) = (a - y^2 - bz, x, y)$: expect to have two unstable directions, so not rank one.
3. Large perturbations of Axiom A maps: Katok construction (slowdown near hyperbolic fixed point), no dominated splitting; slowdown + shear, no continuous splitting.
4. Small perturbations of maps with SRB measures: either local or global.

Goal: Develop a method for establishing the existence of an SRB measure that can be applied to these and other examples.

Cone families

SRB measures for diffeomorphisms with usable hyperbolicity

$$\Big(x \in M\Big) + \Big(\text{subspace } E \subset T_x M\Big) + \Big(\text{angle } \theta\Big) \rightsquigarrow \text{cone}$$

$$K(x, E, \theta) = \{v \in T_x M \mid \measuredangle(v, E) < \theta\}.$$

E, θ depend measurably on x $\rightsquigarrow$ measurable cone family.

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E, θ depend measurably on x $\rightsquigarrow$ measurable cone family.

Cone families $K^s(x), K^u(x)$ are invariant and transverse if

1. $\overline{Df(K^u(x))} \subset K^u(f(x))$
2. $\overline{Df^{-1}(K^s(f(x)))} \subset K^s(x)$
3. $T_x M = E^s(x) \oplus E^u(x)$

Usable hyperbolicity

Consider invariant transverse cone families $K^s(x), K^u(x) \subset T_xM$.

$$\lambda^u(x) = \inf\{\log \|Df(v)\| \mid v \in K^u(x), \|v\| = 1\},$$
$$\lambda^s(x) = \sup\{\log \|Df(v)\| \mid v \in K^s(x), \|v\| = 1\}.$$

Defect: $d(x) = \max\left(0, \frac{1}{\varepsilon}(\lambda^s(x) - \lambda^u(x)\right)$

Usable hyperbolicity = expansion − defect

Definition

The usable hyperbolicity at x is $\lambda(x) = \lambda^u(x) - d(x)$.

Let $\alpha(x) = \measuredangle(K^s(x), K^u(x))$. Fix $\bar{\alpha} > 0$ and consider

$$\rho_{\bar{\alpha}}(x) = \lim_{n\to\infty} \frac{1}{n} \#\{0 \leq k < n \mid \alpha(f^k(x)) < \bar{\alpha}\}.$$

An existence result

"non-zero exponents" $\rightsquigarrow$ "non-zero usable hyperbolicity"

$$S = \left\{ x \,\middle|\, \underline{\lim}_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \lambda(f^k(x)) > 0 \text{ and } \lim_{\bar{\alpha}\to 0} \rho_{\bar{\alpha}}(x) = 0 \right.$$
$$\left. \text{and } \overline{\lim}_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \lambda^s(f^k(x)) < 0 \right\}$$

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Theorem (C.–Dolgopyat–Pesin 2011)

If $\operatorname{Leb} S > 0$, *then* f *has an SRB measure.*

Remark: Same result holds if S has positive Lebesgue measure along some manifold tangent to the unstable cones $K^u(x)$.

Large perturbations: an indifferent fixed point

f an Axiom A diffeomorphism.

- f has an SRB measure.
- Small perturbations of f are Axiom A.
- Consider perturbation on boundary of "small".

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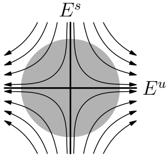
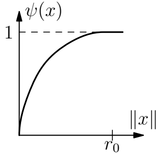


Near a fixed point p, f is the time-1 map of $\dot{x} = Ax$.



Slow down dynamics of f near p: let $g =$ time-1 map for $\dot{x} = \psi(x)Ax$, with $g = f$ outside of $V = B(p, r_0)$.

Theorem (C.–Dolgopyat–Pesin 2011)

g has an SRB measure.

Usable hyperbolicity for g

- If f has a smooth invariant measure μ, then $\psi(x)^{-1}d\mu$ defines a smooth invariant measure for g.
- If the SRB measure for f is not smooth, then the attractor for f is not g-invariant.

f is Axiom A $\Rightarrow f$ has invariant cone families $K^u(x)$ and $K^s(x)$

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f is Axiom A $\Rightarrow$ f has invariant cone families $K^u(x)$ and $K^s(x)$

- $K^u(x)$ and $K^s(x)$ are g-invariant.
- $\lambda^u(x) \geq 0 \geq \lambda^s(x)$ and $\alpha(x) \gg 0$ for every x.
- $\lambda(x) = \lambda^u(x) \geq \chi > 0$ for every $x \notin V$.

$$\frac{1}{n}\sum_{k=0}^{n-1} \lambda(g^k(x)) \geq \chi \cdot \frac{1}{n}\#\{0 \leq k < n \mid g^k(x) \notin V\}$$

$$\Big(\text{usable hyperbolicity}\Big) \geq \chi \cdot \Big(\text{time spent outside } V\Big)$$

Average sojourn times

- $\tau(x) = \min\{t \mid g^t(x) \notin V\}$ sojourn time spent in V
- $G(x) = g^{\tau(x)}(x)$ first return map to outside of V
- $\tau_n(x) = \tau(G^{n-1}(x))$ sojourn time after n returns

Claim: $\exists\ R > 0$ such that $\overline{\lim}\ \frac{1}{n}\sum_{k=1}^{n} \tau_k(x) \leq R$ for Leb-a.e. x.

Average sojourn time is bounded for Lebesgue typical trajectories.

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Average sojourn time is bounded for Lebesgue typical trajectories.

- $\Omega(t_1,\ldots,t_n) = \{x \mid \tau_k(x) = t_k \text{ for } 1 \leq k \leq n\}$
- $\mathrm{Leb}\,\Omega(\vec{t}) \leq C^n \prod_{k=1}^{n} t_k^{-\gamma}$ with $\gamma > 2$
- Model (τ_k) with i.i.d. (T_k) such that $P(T_k = t) = Ct^{-\gamma}$
- Claim holds using fact that $E(T_k) < \infty$

An indifferent fixed point with a shear

Let f be Axiom A with $\dim E^u = 1$. Slow down f near $p = f(p)$ as before, then add shear.

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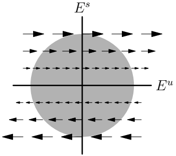
Let $N\colon \mathbb{R}^d \to \mathbb{R}^d$ be linear such that

- $N(\mathbb{R}^d) \subset E^u \subset \ker N$,

and ξ a bump function near p.



Near p, let g = time-1 map for $\dot{x} = (\psi(x)A + \xi(x)N)x$, with $g = f$ outside of $V = B(p, r_0)$.

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Stable cones for g

Shear $\Rightarrow$ stable cone for f is no longer g-invariant. Need to

1. establish existence of stable invariant cones $K^s(x)$ for g;
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Claim: This boils down to estimating average sojourn times.

- $A = V \setminus g(V)$ (just entered neighbourhood of p)
- $B = g(V) \setminus V$ (just left the neighbourhood of p)
- Let $G\colon A \to B$ and $F\colon B \to A$ be the induced maps

Need to understand action of DG and DF on the space of s-dimensional subspaces of $\mathbb{R}^d$ transverse to E^u.

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Need to understand action of DG and DF on the space of s-dimensional subspaces of $\mathbb{R}^d$ transverse to E^u.

- Identify with $\mathbb{R}^s$ (intersections with translations of E^u)
- DG acts as a translation (parabolically)
- DF acts as multiplication (hyperbolically)

Stable cones for g (ctd.)

$$\{E \subset \mathbb{R}^d \mid E \text{ transverse to } E^u\} \quad \leftrightarrow \quad \mathbb{R}^s$$
$$E \to E^u \quad \leftrightarrow \quad \vec{v} \to \infty$$

Goal: $\vec{v}$ such that

$$\vec{v},\ DG(\vec{v}),\ DF \circ DG(\vec{v}),\ DG \circ DF \circ DG(\vec{v}), \dots$$

does not go to ∞. This corresponds to $E \subset \mathbb{R}^d$ such that

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does not go to ∞. This corresponds to $E \subset \mathbb{R}^d$ such that

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does not go to E^u. Given $\vec{v} \in \mathbb{R}^s$, we have

- $|DG_x(\vec{v})_j| \geq |v_j| - C\tau(x)$, (translation)
- $|DF_x(\vec{v})_j| \geq \lambda |v_j|$, where $\lambda > 1$. (multiplication)

Usable hyperbolicity

Let $R_n(x) := \sum_{k=0}^{\infty} C\lambda^{-k}\tau_{n+k+1}(x)$, so $R_n = \lambda(R_{n-1} - C\tau_n)$.

- Then $(DF \circ DG)B(R_{n-1}(x)) \supset B(R_n(F \circ G(x)))$.

If $\tau_n(x)$ is bounded in average, then $\underline{\lim}\, R_n(x) < \infty$, so:

- $B(R_n(x))$ contains a $\vec{v}$ whose iterates do not go to ∞
- This shows the existence of $E^s(x)$
- $\alpha_n = \measuredangle(E^s(g^n(x)), E^u(g^n(x)))$ is controlled by R_n
- $\{n \mid \alpha_n \leq \bar{\alpha}\}$ arbitrarily sparse when $\bar{\alpha}$ arbitrarily small

Together with the fact that $\lambda(x) = \lambda^u(x)$, we get

bounded average sojourn time

$\Rightarrow$ positive asymptotic rate of usable hyperbolicity

$\Rightarrow$ g has an SRB measure

Constructing invariant measures

Build invariant measures using action on $\mathcal{M}$

- f acts on $\mathcal{M}$ by $f_*\colon m \mapsto m \circ f^{-1}$.
- Fixed points of f_* are invariant measures.
- Césaro averages + weak* compactness $\Rightarrow$ invariant measures:

$$\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} f_*^k m \qquad\qquad \mu_{n_j} \to \mu \in \mathcal{M}(f)$$

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Idea: m = volume $\Rightarrow \mu$ is an SRB measure.

$$H = \{x \in M \mid \text{all Lyapunov exponents non-zero at } x\}$$

$$\mathcal{S} = \{\nu \in \mathcal{M} \mid \nu(H) = 1, \nu \text{ a.c. on unstable manifolds}\}$$

- $\mathcal{S} \cap \mathcal{M}(f) = \{\text{SRB measures}\}$
- $\mathcal{S}$ is f_*-invariant, so $m \in \mathcal{S} \Rightarrow \mu_n \in \mathcal{S}$ for all n.
- $\mathcal{S}$ is *not* compact. So why should μ be in $\mathcal{S}$?

Non-uniform hyperbolicity in $\mathcal{M}$

Theme in NUH: choose between invariance and compactness.

Replace unstable manifolds with n-admissible manifolds V.

$$d(f^{-k}(x), f^{-k}(y)) \leq Ce^{-\lambda k} d(x,y) \text{ for all } 0 \leq k \leq n \text{ and } x, y \in V$$

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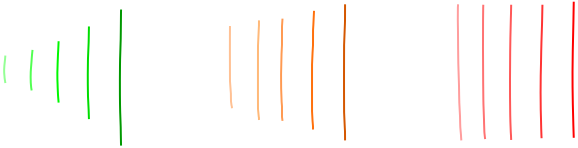
1. Value of C, λ in definition of n-admissibility.



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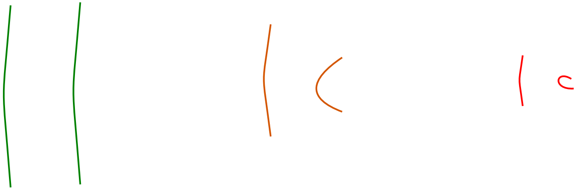

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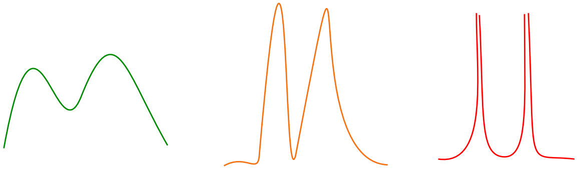

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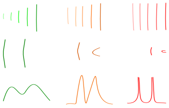

Given $K > 0$, let $\mathcal{S}_n(K)$ be the set of measures for which these non-uniformities are all controlled by K.

$$\text{large } K \Rightarrow \text{worse non-uniformity}$$

$\mathcal{S}_n(K)$ is compact, but not f_*-invariant.

Non-uniformities controlled by K

Admissible manifold V near x defined by

- decomposition $T_x M = G \oplus F$ with $\alpha = \measuredangle(G, F)$,
- $C^{1+\varepsilon}$ function $\psi \colon G \cap B(0, r) \to F$ with $\|D\psi\| \leq \gamma$ and $|D\psi|_\varepsilon \leq \kappa$ such that $V = \exp_x(\operatorname{graph} \psi)$.

Density $\rho \in C^\varepsilon(V)$ and backwards dynamics satisfy

- $L^{-1} \leq \rho(x) \leq L$ and $\|\rho\|_{C^\varepsilon} \leq L$,
- $d(f^{-k}(x), f^{-k}(y)) \leq C e^{-\lambda k} d(x, y)$.

K controls all the quantities $\alpha, r, \gamma, \kappa$ (geometry of the admissible manifold), L (density function), and C, λ (dynamics).

Conditions for existence of an SRB measure

- M be a compact Riemannian manifold, $U \subset M$ open, $f : U \to M$ a local diffeomorphism with $\overline{f(U)} \subset U$.
- Let μ_n be a sequence of measures whose limit measures are all invariant. (In applications, $\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} f_*^k \text{Leb.}$)
- Fix $K > 0$, write $\mu_n = \nu_n + \zeta_n$, where $\nu_n \in \mathcal{S}_n(K)$.

Theorem (C.–Dolgopyat–Pesin 2011)

If $\mu_{n_k} \to \mu$ and $\overline{\lim}_{n_k \to \infty} \|\nu_{n_k}\| > 0$, then some ergodic component of μ is an SRB measure for f.

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The question now becomes: How do we obtain recurrence to the set $\mathcal{S}_n(K)$?

Coordinates in TM

We use local coordinates to write the map f along a trajectory as a sequence of local diffeomorphisms.

- $\{f^n(x) \mid n \geq 0\}$ is a trajectory of f
- $U_n \subset T_{f^n(x)}M$ is a small neighbourhood of 0
- $f_n \colon U_n \to \mathbb{R}^d = T_{f^{n+1}(x)}M$ is the map f in local coordinates

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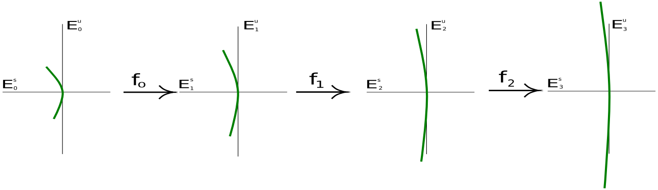
Decompose $\mathbb{R}^d = T_xM = E_0^u \oplus E_0^s$, let $E_{n+1}^{u,s} = Df_n(E_n^{u,s})$.

- Want E_n^u and E_n^s asymptotically expanding and contracting.
- Want $\overline{\lim}_n \measuredangle(E_n^u, E_n^s) > 0$.
- ($\underline{\lim}_n \measuredangle(E_n^u, E_n^s) > 0$ is probably unavoidable.)

Controlling hyperbolicity and regularity

$$\mathbb{R}^d = T_{f^n(x)}M = E^u_n \oplus E^s_n \qquad f_n = (A_n \oplus B_n) + s_n$$

Start with an admissible manifold V_0 tangent to E^u_0 at 0 and push it forward: $V_{n+1} = f_n(V_n)$.



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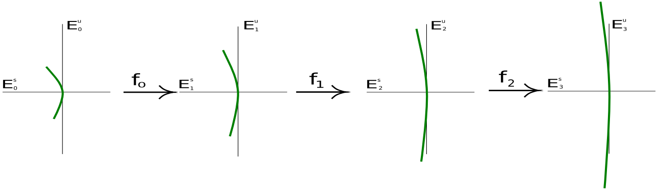
$$V_n = \operatorname{graph} \psi_n = \{v + \psi_n(v)\} \qquad \psi_n \colon B(E_n^u, r_n) \to E_n^s$$

Need to control the size r_n and the regularity $\|D\psi_n\|$, $|D\psi_n|_\varepsilon$.



Controlling hyperbolicity and regularity

Consider the following quantities:

$$\lambda_n^u = \log(\|A_n^{-1}\|^{-1}) \qquad \lambda_n^s = \log\|B_n\|$$

$$\alpha_n = \measuredangle(E_n^u, E_n^s) \qquad C_n = |Ds_n|_\varepsilon$$

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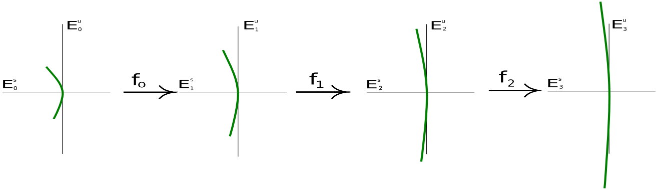
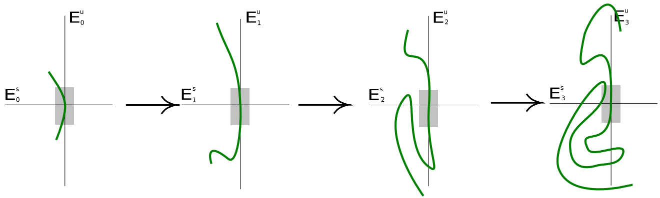


Classical Hadamard–Perron results

Uniform case: Constants such that

- $\lambda_n^s \leq \bar{\lambda}^s < 0 < \bar{\lambda}^u < \lambda_n^u$
- $\alpha_n \geq \bar{\alpha} > 0$
- $C_n \leq \bar{C} < \infty$

Then V_n has uniformly large size: $r_n \geq \bar{r} > 0$.



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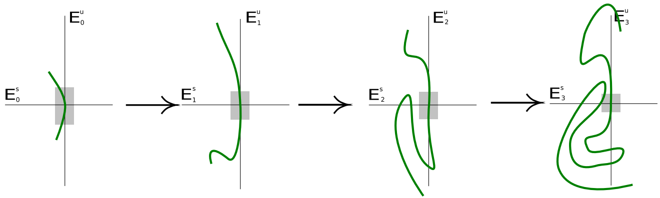
C_n grows slowly $\Rightarrow$ r_n decays slowly



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Non-uniform case: $\lambda_n^s, \lambda_n^u, \alpha_n$ still uniform, but C_n not.

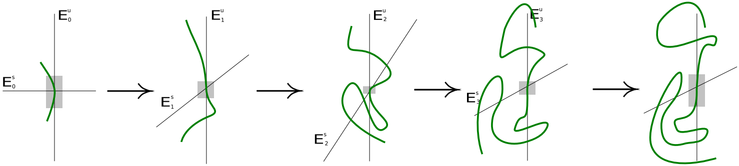
$$C_n \text{ grows slowly} \Rightarrow r_n \text{ decays slowly}$$

We want to consider the case where

- $\lambda_n^s < 0 < \lambda_n^u$ may fail (may even have $\lambda_n^u < \lambda_n^s$)
- α_n may become arbitrarily small
- C_n may become arbitrarily large (no control on speed)

Usable hyperbolicity

In order to define ψ_{n+1} implicitly, we need control of the regularity of ψ_n. Control $\|D\psi_n\|$ and $|D\psi_n|_\varepsilon$ by decreasing r_n if necessary. So how do we guarantee that r_n becomes "large" again?



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Defect – splitting not dominated: $d_n = \max\left(0, \frac{1}{\varepsilon}(\lambda_n^s - \lambda_n^u)\right)$

Distortion – large nonlinearity, small angle: $\beta_n = C_n(\sin \alpha_{n+1})^{-1}$

Fix a threshold value $\bar{\beta}$ and define the usable hyperbolicity:

$$\lambda_n = \begin{cases} \lambda_n^u - d_n & \text{if } \beta_n \leq \bar{\beta}, \\ \min\left(\lambda_n^u - d_n, \frac{1}{\varepsilon}\log\frac{\beta_{n-1}}{\beta_n}\right) & \text{if } \beta_n > \bar{\beta}. \end{cases}$$

Continuous dominated splitting $\Rightarrow$ $\lambda_n = \lambda_n^u$

Positive usable hyperbolicity

Key criterion is positive usable hyperbolicity:

$$\varliminf_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \lambda_k > 0 \text{ for some } \bar{\beta}$$

One way to establish this is to have both of the following:

1. Expansion beats defect:

$$\varliminf_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \lambda_k^u - d_k > 0$$

2. Distortion is almost bounded: Let $\Gamma^{\bar{\beta}} = \{n \mid \beta_n > \bar{\beta}\}$. Then $\Gamma^{\bar{\beta}}$ has arbitrarily small upper asymptotic density.

A Hadamard–Perron theorem

- $F_n = f_{n-1} \circ \cdots \circ f_1 \circ f_0 \colon U_0 \to \mathbb{R}^d = T_{f^n(x)} M$
- $V_0 \subset \mathbb{R}^d$ a $C^{1+\varepsilon}$ manifold tangent to E_0^u at 0
- $V_n(r) =$ connected component of $F_n(V_0) \cap B(r)$ containing 0

Theorem (C.–Dolgopyat–Pesin 2011)

Suppose $\underline{\lim}\, \frac{1}{n} \sum_{k=0}^{n-1} \lambda_k > \bar{\chi} > 0$ for some $\bar{\beta}$. Then there exist constants $\bar{\alpha}, \bar{\gamma}, \bar{\kappa}, \bar{r} > 0$ and a set $\Gamma \subset \mathbb{N}$ with positive lower asymptotic frequency such that for every $n \in \Gamma$,

1. $\measuredangle(E_n^u, E_n^s) \geq \bar{\alpha}$;
2. $V_n(\bar{r}) = \operatorname{graph} \psi_n$ *and* $\|D\psi_n\| \leq \bar{\gamma}$, $|D\psi_n|_\varepsilon \leq \bar{\kappa}$;
3. *if* $F_n(x), F_n(y) \in V_n(\bar{r})$, *then for every* $0 \leq k \leq n$,

$$\|F_n(x) - F_n(y)\| \geq e^{(n-k)\bar{\chi}} \|F_k(x) - F_k(y)\|.$$

Idea of proof

Start with V_0, study $V_n = F_n(V_0)$. Choose r_n, γ_n, κ_n such that

- $V_n(r_n) = \operatorname{graph} \psi_n$
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Can improve γ_n, κ_n at the cost of reducing r_n, or vice versa. Give conditions on "goodness parameters" r_n, γ_n, κ_n; inequalities in terms of λ_n^u, λ_n^s, and β_n.

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Truncate parameters at threshold values $\bar{r}, \bar{\gamma}, \bar{\kappa}$:

- define goodness g_n by $g_0 = 1$ and $g_{n+1} = \min(1, e^{\lambda_n} g_n)$;
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positive asymptotic rate of usable hyperbolicity

$\Rightarrow$ positive frequency of usable hyperbolic times (Pliss' lemma)

$\Rightarrow$ thresholded parameters spend enough time at threshold

Completion of proof

- $\mu_0 = \mathrm{Leb}\,|_{V_0}$
- $\mu_n = (f_*^n \mu_0)|_{V_n(r_n)}$ (normalised)
- $\mu_n \in \mathcal{S}_n(K)$ for $n \in \Gamma$
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