Unique equilibrium states for some robustly transitive systems

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January 11, 2015

Joint work with Todd Fisher (BYU) and Daniel J. Thompson (Ohio State)
Pressure and equilibrium states

\( \mathcal{X} \) a compact metric space, \( f : \mathcal{X} \to \mathcal{X} \) continuous

- \( \mathcal{M}_f(\mathcal{X}) = \{ f\text{-inv. Borel prob. measures} \} \) Often very large…
- Fix a potential function \( \varphi : \mathcal{X} \to \mathbb{R} \)
- Equilibrium state maximises \( h_\mu(f) + \int \varphi \, d\mu \) over \( \mathcal{M}_f(\mathcal{X}) \)
- Existence? Uniqueness? Statistical properties?
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Variational principle: Supremum is topological pressure

\[
P(\varphi) = \lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n} \log \Lambda_n(X, \varphi, \delta)
\]

- \( X \times \mathbb{N} = \) ‘space of orbit segments’ \( (x, n) \leftrightarrow (x, fx, \ldots, f^{n-1}x) \)
- weight function \( \Phi : X \times \mathbb{N} \to \mathbb{R} \) given by \( \Phi(x, n) = S_n \varphi(x) \)
- Bowen ball: \( B_n(x, \delta) = \{ y \in X \mid \max_{0 \leq k < n} d(f^kx, f^ky) < \delta \} \)
- \( \Lambda_n(X, \varphi, \delta) = \sup \{ \sum_{x \in E} e^{\Phi(x, n)} \mid x, y \in E \Rightarrow y \notin B_n(x, \delta) \} \)
Uniqueness for uniform hyperbolicity

Most complete results for Anosov systems

- $M$ compact Riemannian manifold, $f: M \to M$ diffeomorphism
- $TM = E^u \oplus E^s$, $\|Df|_{E^s}\| \leq \lambda < 1$, $\|Df^{-1}|_{E^u}\| \leq \lambda < 1$

Theorem (Bowen, Ruelle, Sinai, 1970s)

$(M, f)$ mixing Anosov $+ \varphi$ Hölder $\Rightarrow \exists$ unique eq. state $\mu$

- Strong statistical properties for $(M, f, \mu)$: exponential decay of correlations, central limit theorem, etc.
- $\varphi = -\log |\det Df|_{E^u}| \Rightarrow$ eq. state is ‘physical’ measure (SRB)

Two techniques: Markov partitions and specification
Mañé’s example

Want to understand ‘large’ classes of systems; in particular, get behaviour stable under $C^1$ perturbation of $f$.

Anosov maps are $C^1$ stable; gives $C^1$-open set of transitive diffeos. Non-Anosov examples of robust transitivity given by Mañé.
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- $A \in SL(3, \mathbb{Z}) \mapsto f_A : \mathbb{T}^3 \to \mathbb{T}^3$
- Eigvals $0 < \lambda_1 < \lambda_2 < 1 < \lambda_3 \Rightarrow f_A$ is Anosov
- $TM = E^s \oplus E^c \oplus E^u$, with $E^s \oplus E^c$ uniformly contracting
- Make $C^0$ perturbation in $E^s \oplus E^c$ in $\eta$-nbhd of fixed point

$f_0$ is partially hyperbolic, has both expansion and contraction in $E^c$. 
A uniqueness result

$g$ $C^1$-close to $f_0$: still partially hyperbolic, $E^{s,c,u}$ integrate to foliations, local product structure, dense leaves

- $\lambda_c(g) := \sup \{ \| Dg \|_{E^c(x)} : x \in \text{nbhd } B \text{ of perturbation} \}$
- $\lambda_s(g) := \sup \{ \| Dg \|_{E^c(x)} : x \notin \text{nbhd } B \text{ of perturbation} \}$
- $\lambda_s(g) < 1 < \lambda_c(g) \Rightarrow \lambda_c(g)^{1-\gamma} \lambda_s(g)^\gamma = 1$ for some $\gamma > 0$
- For every $r > \gamma$ we have $\lambda_c^{1-r} \lambda_s^r < 1$: uniform contraction in $E^c$ along $(x,n)$ spending at least $rn$ iterates outside nbhd $B$
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Theorem (C.–Fisher–Thompson)

$g$ has a unique equilibrium state for a Hölder potential $\varphi$ if

$$(1 - \gamma)\sup_B \varphi + \gamma(\sup_M \varphi + C(f_A) - \log \gamma) < P(g, \varphi). \quad (\star)$$
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($\star$)

$\varphi(q) + \gamma(\sup_M \varphi - \varphi(q) + C - \log \gamma) + 2|\varphi|_{\alpha} \eta^\alpha < P(f_A, \varphi) \Rightarrow (\star)$

Each $\varphi$ has $C^1$-open set of $g$ with uniqueness (and vice versa)
Bonatti–Viana example

Bonatti–Viana: robust transitivity without partial hyperbolicity

- $A \in \text{SL}(4, \mathbb{Z}) \leadsto f_A : \mathbb{T}^4 \to \mathbb{T}^4$
- Eigvals $0 < \lambda_1 < \lambda_2 < 1 < \lambda_3 < \lambda_4 \Rightarrow f_A$ is Anosov
- $C^0$ perturbation in $E^s = E^1 \oplus E^2$ around fixed pt, get $E^{cs}$

Similar perturbation in $E^u$ around another fixed point, get $f_0$ with a dominated splitting $TM = E^{cs} \oplus E^{cu}$, not partially hyperbolic
Another uniqueness result

$g$ $C^1$-close to $f_0$: still dominated splitting, $E^{cs}, E^{cu}$ integrate to foliations, local product structure, dense leaves

- Similar: $\lambda_s(g) < 1 < \lambda_{cs}(g)$ and $\lambda_{cu}(g) < 1 < \lambda_u(g)$.
- $\gamma = \gamma(g)$ such that $r > \gamma$ gives $\lambda_{cs}^{1-r} \lambda_s^r < 1$ and $\lambda_{cu}^{1-r} \lambda_u^r > 1$
- Put $\lambda_c = \max(\lambda_{cs}, \lambda_{cu}^{-1}) > 1$, controls tail entropy

Theorem (C.–Fisher–Thompson)

$g$ has a unique equilibrium state for a Hölder potential $\varphi$ if

$$(1 - \gamma) \sup_B \varphi + 2 \log \lambda_c + \gamma (\sup_M \varphi + C - \log \gamma) + |\varphi|_a \eta^a < P(g, \varphi).$$

As before, can get sufficient condition in terms of $P(f_A, \varphi)$, so each $\varphi$ has $C^1$-open set of $g$ with uniqueness (and vice versa).
SRB measures

Uniqueness criterion for Bonatti–Viana:

\[ \sup_B \varphi + 2 \log \lambda_c + \gamma (\sup_M \varphi - \sup_B \varphi + C - \log \gamma) + \eta^\alpha |\varphi|_\alpha < P(g, \varphi) \]

Assume \( g \) is \( C^2 \), put \( \varphi = -\log |\det Dg|_{E^{cu}} \) to get SRB

\( \bullet \) Bifurcation in \( E^{cu} \) at \( q \), put \( \chi = |\det Dg|_{E^{cu}(q)} \), get \( \sup_M \varphi = \sup_B \varphi = -\log \chi \) since \( E^{cu} \) expands outside \( B \).

\( \bullet \) For small perturbations get \( \chi > 1 \).

Theorem (C.–Fisher–Thompson)

If \( g \) is a \( C^2 \) Mañé or Bonatti–Viana example and

\[-\log \chi + 2 \log \lambda_c + \gamma (C - \log \gamma) + \eta |g|_{C^2} < 0,\]

then \( P(g, -\log |\det Dg|_{E^{cu}}) = 0 \), there is a unique eq. state \( \mu \) for \( -\log |\det Dg|_{E^{cu}} \), and \( \mu \) is the unique SRB measure for \( g \).
Uniform specification

Transitivity: $\forall \delta > 0, \{ (x_i, n_i) \}_{i=1}^{k} \subset X \times \mathbb{N}, \exists t_i \in \mathbb{N}$ and $x \in X$ s.t.

$x \in B_{n_1}(x_1, \delta), \quad f^{n_1+t_1}x \in B_{n_2}(x_2, \delta), \quad f^{\sum_{i=1}^{j-1}(n_i+t_i)}x \in B_{n_j}(x_j, \delta), \ldots$

Trans. Anosov $\Rightarrow$ specification: can take $t_i \leq T = T(\delta)$ for each $i$

- Any collection of orbit segments can be ‘$(\delta, T)$-glued’
Uniform specification

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Trans. Anosov \( \Rightarrow \) specification: can take \( t_i \leq T = T(\delta) \) for each \( i \)

- Any collection of orbit segments can be ‘\((\delta, T)\)-glued’

For Anosov \( f \) and Hölder \( \varphi \) we also get

- expansive: \( B_\infty(x, \varepsilon) := \{y \mid d(f^kx, f^ky) < \varepsilon \ \forall k \in \mathbb{Z}\} = \{x\} \)

- Bowen property: \( \sup_{(x,n)} \sup_{y \in B_n(x,\varepsilon)} |S_n\varphi(y) - S_n\varphi(x)| < \infty \)

Theorem (Bowen, 1974)

If \((X, f)\) has specification and expansivity, and \( \varphi \) has the Bowen property, then \((X, f, \varphi)\) has a unique equilibrium state \( \mu \).
Non-uniform properties

Mechanism for specification, expansivity, and Bowen property is uniform contraction/expansion + density of stable/unstable leaves

For non-uniformly hyperbolic systems, restrict attention to some $\mathcal{G} \subset X \times \mathbb{N}$ where hyperbolicity is uniform
Non-uniform properties

Mechanism for specification, expansivity, and Bowen property is uniform contraction/expansion + density of stable/unstable leaves

For non-uniformly hyperbolic systems, restrict attention to some \( \mathcal{G} \subset X \times \mathbb{N} \) where hyperbolicity is uniform

- \( \mathcal{G} \) has \( \delta \)-specification if any \( \{(x_i, n_i)\}_i \subset \mathcal{G} \) can be \( (\delta, T) \)-glued
- \( \varphi \) is \( \varepsilon \)-Bowen on \( \mathcal{G} \) if \( \sup_{(x,n) \in \mathcal{G}} \sup_{y \in B_n(x,\varepsilon)} |S_n \varphi(y) - S_n \varphi(x)| < \infty \)
- Later will require that \( \mathcal{G} \) be ‘large’

Non-expansive set: \( NE(\varepsilon) = \{x \mid B_{\infty}(x, \varepsilon) \neq \{x\} \} \)

- \( \mathcal{M}^{ne}(\varepsilon) = \{ \text{ergodic } \mu \in \mathcal{M}_f(X) \mid \mu(NE(\varepsilon)) > 0 \} \)
- \( P_{\text{exp}}(\varphi, \varepsilon) = \sup\{ h_\mu(f) + \int \varphi \, d\mu \mid \mu \in \mathcal{M}^{ne}(\varepsilon) \} \)
An abstract uniqueness result

Given $C^p, C^s \subset X \times \mathbb{N}$ and $M \in \mathbb{N}$, let

$$G^M = \{(x, n) \mid \exists p + g + s = n \text{ s.t. } p, s \leq M, (x, p) \in C^p, (f^p x, g) \in G, (f^{p+g} x, s) \in C^s\}$$

$$C = C^p \cup C^s \cup (X \times \mathbb{N} \setminus \bigcup_M G^M)$$

Quantify ‘pressure of obstructions to specification’ by

$$\Phi_\varepsilon(x, n) = \sup\{S_n \varphi(y) \mid y \in B_n(x, \varepsilon)\},$$

$$P(C, \varphi, \delta, \varepsilon) = \sup\{\sum_{x \in E} e^{\Phi_\varepsilon(x, n)} \mid E \times \{n\} \subset C, \text{ and } E \text{ is } (n, \delta)-\text{sep}\}$$

Theorem (C.–Thompson)

$(X, f, \varphi)$ has a unique eq. state if $\varepsilon > 20\delta > 0$ and $G, C^p, C^s$ are s.t.

1. $P_{\text{exp}}(\varphi, \varepsilon) < P(\varphi)$
2. every $G^M$ has $\delta$-specification
3. $\varphi$ is $\varepsilon$-Bowen on $G$
4. $P(C, \varphi, \delta, \varepsilon) < P(\varphi)$
Application to examples

Unif. hyperbolic outside $B = \text{nbhd where perturbation is made}$

- Fix $r > 0$, let $\mathcal{D} = \{(x, n) \mid \text{at least } rn \text{ iterates outside } B\}$
- $(x, n) \in \mathcal{D} \Rightarrow Dg^n(x) \text{ contracts } E^{cs} \text{ and expands } E^{cu}$
- $\mathcal{G} = \{(x, n) \mid (x, k), (f^{n-k}x, k) \in \mathcal{D} \text{ for all } 0 \leq k \leq n\}$
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- $\mathcal{G} = \{ (x, n) \mid (x, k), (f^{n-k}x, k) \in \mathcal{D} \text{ for all } 0 \leq k \leq n \}$
- $\mathcal{C}_p = \mathcal{C}_s = (X \times \mathbb{N}) \setminus \mathcal{D}$ (‘almost all time in $B$’)
- Given $(x, n)$, let $p$ be maximal such that $(x, p) \in \mathcal{C}_p$, and $s$ maximal such that $(f^{n-s}x, s) \in \mathcal{C}_s$, then $(f^px, n-s-p) \in \mathcal{G}$.
Application to examples

Unif. hyperbolic outside $B = \text{nbhd}$ where perturbation is made

- Fix $r > 0$, let $\mathcal{D} = \{(x, n) \mid \text{at least } rn \text{ iterates outside } B\}$
- $(x, n) \in \mathcal{D} \Rightarrow D g^n(x)$ contracts $E^{cs}$ and expands $E^{cu}$
- $\mathcal{G} = \{(x, n) \mid (x, k), (f^{n-k}x, k) \in \mathcal{D} \text{ for all } 0 \leq k \leq n\}$
- $C^p = C^s = (X \times \mathbb{N}) \setminus \mathcal{D}$ (‘almost all time in $B$’)
- Given $(x, n)$, let $p$ be maximal such that $(x, p) \in C^p$, and $s$ maximal such that $(f^{n-s}x, s) \in C^s$, then $(f^p x, n - s - p) \in \mathcal{G}$.

Can verify conditions of theorem if perturbation is small enough

- local product structure + Hölder gives Bowen property on $\mathcal{G}$
- product structure + density of leaves gives $\mathcal{G}^M$ specification
- $P_{\exp}^\perp(\varphi, \varepsilon), P(C, \varphi, \delta, \varepsilon) \approx "P(B, \varphi) \text{ up to } \gamma n \text{ escapes}"
  \approx (1 - \gamma) \sup_B \varphi + 2 \log \lambda_c + \gamma (\sup_M \varphi + C - \log \gamma) + \eta^\alpha |\varphi|_\alpha$
Onwards to towers?

Specification approach gives uniqueness of equilibrium state, but not stronger statistical properties like exponential decay of correlations, central limit theorem, ASIP, etc.

- Can get these using Young towers provided (1) tail of tower decays exponentially, (2) equilibrium state lifts to tower.
- Liftability is often difficult to establish

**Theorem (C., 2014)**

If $(X, \sigma)$ is a shift space on a finite alphabet and $\varphi$ a Hölder potential such that obstructions to specification have small pressure, then $(X, \sigma)$ contains a Young tower such that every equilibrium state lifts to the tower (in particular, there is a unique equilibrium state), and the tower has exponential tails.

Question: Can this be generalized to smooth systems?