Unique equilibrium states for some robustly transitive systems

Vaughn Climenhaga
University of Houston

January 11, 2015

Joint work with Todd Fisher (BYU)
and Daniel J. Thompson (Ohio State)

Pressure and equilibrium states

X a compact metric space, $f\colon X \to X$ continuous

- $\mathcal{M}_f(X) = \{f\text{-inv. Borel prob. measures}\}$ Often very large...
- Fix a potential function $\varphi\colon X \to \mathbb{R}$
- Equilibrium state maximises $h_\mu(f) + \int \varphi \, d\mu$ over $\mathcal{M}_f(X)$
- Existence? Uniqueness? Statistical properties?

Pressure and equilibrium states

X a compact metric space, $f\colon X \to X$ continuous

- $\mathcal{M}_f(X) = \{f\text{-inv. Borel prob. measures}\}$ Often very large...
- Fix a potential function $\varphi\colon X \to \mathbb{R}$
- Equilibrium state maximises $h_\mu(f) + \int \varphi \, d\mu$ over $\mathcal{M}_f(X)$
- Existence? Uniqueness? Statistical properties?

Variational principle: Supremum is topological pressure

$$P(\varphi) = \lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n} \log \Lambda_n(X, \varphi, \delta)$$

- $X \times \mathbb{N} =$ 'space of orbit segments' $(x, n) \leftrightarrow (x, fx, \dots, f^{n-1}x)$
- weight function $\Phi\colon X \times \mathbb{N} \to \mathbb{R}$ given by $\Phi(x, n) = S_n\varphi(x)$
- Bowen ball: $B_n(x, \delta) = \{y \in X \mid \max_{0 \leq k < n} d(f^k x, f^k y) < \delta\}$
- $\Lambda_n(X, \varphi, \delta) = \sup\{\sum_{x \in E} e^{\Phi(x,n)} \mid x, y \in E \Rightarrow y \notin B_n(x, \delta)\}$

Uniqueness for uniform hyperbolicity

Most complete results for Anosov systems

- M compact Riemannian manifold, $f : M \to M$ diffeomorphism
- $TM = E^u \oplus E^s$, $\|Df|_{E^s}\| \leq \lambda < 1$, $\|Df^{-1}|_{E^u}\| \leq \lambda < 1$

Theorem (Bowen, Ruelle, Sinai, 1970s)

(M, f) mixing Anosov $+$ φ Hölder $\Rightarrow \exists$ unique eq. state μ

- *Strong statistical properties for (M, f, μ): exponential decay of correlations, central limit theorem, etc.*
- *$\varphi = -\log|\det Df|_{E^u}| \Rightarrow$ eq. state is 'physical' measure (SRB)*

Two techniques: Markov partitions and specification

Mañé's example

Want to understand 'large' classes of systems; in particular, get behaviour stable under C^1 perturbation of f.

Anosov maps are C^1 stable; gives C^1-open set of transitive diffeos.

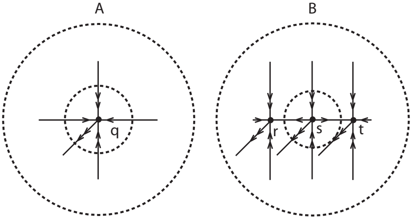
Non-Anosov examples of robust transitivity given by Mañé.

Mañé's example

Want to understand 'large' classes of systems; in particular, get behaviour stable under C^1 perturbation of f.

Anosov maps are C^1 stable; gives C^1-open set of transitive diffeos. Non-Anosov examples of robust transitivity given by Mañé.

- $A \in SL(3, \mathbb{Z}) \rightsquigarrow f_A \colon \mathbb{T}^3 \to \mathbb{T}^3$
- Eigvals $0 < \lambda_1 < \lambda_2 < 1 < \lambda_3 \Rightarrow f_A$ is Anosov
- $TM = E^s \oplus E^c \oplus E^u$, with $E^s \oplus E^c$ uniformly contracting
- Make C^0 perturbation in $E^s \oplus E^c$ in η-nbhd of fixed point



f_0 is partially hyperbolic, has both expansion and contraction in E^c.

A uniqueness result

g C^1-close to f_0: still partially hyperbolic, $E^{s,c,u}$ integrate to foliations, local product structure, dense leaves

- $\lambda_c(g) := \sup\{\|Dg|_{E^c(x)}\| : x \in \text{nbhd } B \text{ of perturbation}\}$
- $\lambda_s(g) := \sup\{\|Dg|_{E^c(x)}\| : x \notin \text{nbhd } B \text{ of perturbation}\}$
- $\lambda_s(g) < 1 < \lambda_c(g) \Rightarrow \lambda_c(g)^{1-\gamma}\lambda_s(g)^\gamma = 1$ for some $\gamma > 0$
- For every $r > \gamma$ we have $\lambda_c^{1-r}\lambda_s^r < 1$: uniform contraction in E^c along (x, n) spending at least rn iterates outside nbhd B

A uniqueness result

g C^1-close to f_0: still partially hyperbolic, $E^{s,c,u}$ integrate to foliations, local product structure, dense leaves

- $\lambda_c(g) := \sup\{\|Dg|_{E^c(x)}\| : x \in \text{nbhd } B \text{ of perturbation}\}$
- $\lambda_s(g) := \sup\{\|Dg|_{E^c(x)}\| : x \notin \text{nbhd } B \text{ of perturbation}\}$
- $\lambda_s(g) < 1 < \lambda_c(g) \Rightarrow \lambda_c(g)^{1-\gamma}\lambda_s(g)^\gamma = 1$ for some $\gamma > 0$
- For every $r > \gamma$ we have $\lambda_c^{1-r}\lambda_s^r < 1$: uniform contraction in E^c along (x, n) spending at least rn iterates outside nbhd B

Theorem (C.–Fisher–Thompson)

g has a unique equilibrium state for a Hölder potential φ if

$$(1-\gamma)\sup_B\varphi + \gamma(\sup_M\varphi + C(f_A) - \log\gamma) < P(g,\varphi). \qquad (\star)$$

A uniqueness result

g C^1-close to f_0: still partially hyperbolic, $E^{s,c,u}$ integrate to foliations, local product structure, dense leaves

- $\lambda_c(g) := \sup\{\|Dg|_{E^c(x)}\| : x \in \text{nbhd } B \text{ of perturbation}\}$
- $\lambda_s(g) := \sup\{\|Dg|_{E^c(x)}\| : x \notin \text{nbhd } B \text{ of perturbation}\}$
- $\lambda_s(g) < 1 < \lambda_c(g) \Rightarrow \lambda_c(g)^{1-\gamma}\lambda_s(g)^{\gamma} = 1$ for some $\gamma > 0$
- For every $r > \gamma$ we have $\lambda_c^{1-r}\lambda_s^r < 1$: uniform contraction in E^c along (x, n) spending at least rn iterates outside nbhd B

Theorem (C.–Fisher–Thompson)

g has a unique equilibrium state for a Hölder potential φ if

$$(1-\gamma)\sup_B\varphi + \gamma(\sup_M\varphi + C(f_A) - \log\gamma) < P(g,\varphi). \qquad (\star)$$

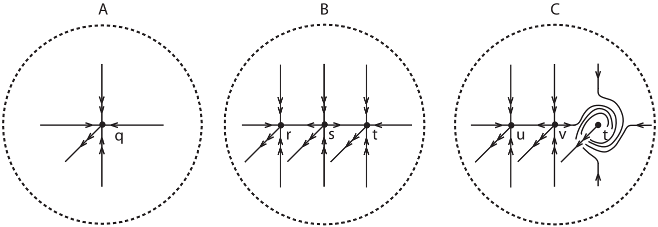
$$\varphi(q) + \gamma(\sup_M\varphi - \varphi(q) + C - \log\gamma) + 2|\varphi|_\alpha\eta^\alpha < P(f_A,\varphi) \Rightarrow (\star)$$

- Each φ has C^1-open set of g with uniqueness (and vice versa)

Bonatti–Viana example

Bonatti–Viana: robust transitivity without partial hyperbolicity

- $A \in SL(4, \mathbb{Z}) \rightsquigarrow f_A \colon \mathbb{T}^4 \to \mathbb{T}^4$
- Eigvals $0 < \lambda_1 < \lambda_2 < 1 < \lambda_3 < \lambda_4 \Rightarrow f_A$ is Anosov
- C^0 perturbation in $E^s = E^1 \oplus E^2$ around fixed pt, get E^{cs}



Similar perturbation in E^u around another fixed point, get f_0 with a dominated splitting $TM = E^{cs} \oplus E^{cu}$, not partially hyperbolic

Another uniqueness result

g C^1-close to f_0: still dominated splitting, E^{cs}, E^{cu} integrate to foliations, local product structure, dense leaves

- Similar: $\lambda_s(g) < 1 < \lambda_{cs}(g)$ and $\lambda_{cu}(g) < 1 < \lambda_u(g)$.
- $\gamma = \gamma(g)$ such that $r > \gamma$ gives $\lambda_{cs}^{1-r}\lambda_s^r < 1$ and $\lambda_{cu}^{1-r}\lambda_u^r > 1$
- Put $\lambda_c = \max(\lambda_{cs}, \lambda_{cu}^{-1}) > 1$, controls tail entropy

Theorem (C.–Fisher–Thompson)

g has a unique equilibrium state for a Hölder potential φ if

$$(1-\gamma)\sup_B \varphi + 2\log\lambda_c + \gamma(\sup_M \varphi + C - \log\gamma) + |\varphi|_\alpha \eta^\alpha < P(g,\varphi).$$

As before, can get sufficient condition in terms of $P(f_A, \varphi)$, so each φ has C^1-open set of g with uniqueness (and vice versa).

SRB measures

Uniqueness criterion for Bonatti–Viana:

$$\sup_B \varphi + 2\log\lambda_c + \gamma(\sup_M \varphi - \sup_B \varphi + C - \log\gamma) + \eta^\alpha |\varphi|_\alpha < P(g,\varphi)$$

Assume g is C^2, put $\varphi = -\log|\det Dg|_{E^{cu}}|$ to get SRB

- Bifurcation in E^{cu} at q, put $\chi = |\det Dg|_{E^{cu}(q)}|$, get $\sup_M \varphi = \sup_B \varphi = -\log\chi$ since E^{cu} expands outside B.
- For small perturbations get $\chi > 1$.

Theorem (C.–Fisher–Thompson)

If g is a C^2 Mañé or Bonatti–Viana example and

$$-\log\chi + 2\log\lambda_c + \gamma(C - \log\gamma) + \eta|g|_{C^2} < 0,$$

then $P(g, -\log|\det Dg|_{E^{cu}}|) = 0$, there is a unique eq. state μ for $-\log|\det Dg|_{E^{cu}}|$, and μ is the unique SRB measure for g.

Uniform specification

Transitivity: $\forall \delta > 0, \{(x_i, n_i)\}_{i=1}^k \subset X \times \mathbb{N}$, $\exists t_i \in \mathbb{N}$ and $x \in X$ s.t.

$$x \in B_{n_1}(x_1, \delta), \quad f^{n_1+t_1}x \in B_{n_2}(x_2, \delta), \quad f^{\sum_{i=1}^{j-1}(n_i+t_i)}x \in B_{n_j}(x_j, \delta), \dots$$

Trans. Anosov $\Rightarrow$ specification: can take $t_i \leq T = T(\delta)$ for each i

- Any collection of orbit segments can be '(δ, T)-glued'

Uniform specification

Transitivity: $\forall \delta > 0, \{(x_i, n_i)\}_{i=1}^k \subset X \times \mathbb{N}$, $\exists t_i \in \mathbb{N}$ and $x \in X$ s.t.

$$x \in B_{n_1}(x_1, \delta), \quad f^{n_1+t_1}x \in B_{n_2}(x_2, \delta), \quad f^{\sum_{i=1}^{j-1}(n_i+t_i)}x \in B_{n_j}(x_j, \delta), \dots$$

Trans. Anosov $\Rightarrow$ specification: can take $t_i \leq T = T(\delta)$ for each i

- Any collection of orbit segments can be '(δ, T)-glued'

For Anosov f and Hölder φ we also get

- expansive: $B_\infty(x, \varepsilon) := \{y \mid d(f^k x, f^k y) < \varepsilon \ \forall k \in \mathbb{Z}\} = \{x\}$
- Bowen property: $\sup_{(x,n)} \sup_{y \in B_n(x,\varepsilon)} |S_n\varphi(y) - S_n\varphi(x)| < \infty$

Theorem (Bowen, 1974)

If (X, f) has specification and expansivity, and φ has the Bowen property, then (X, f, φ) has a unique equilibrium state μ.

Non-uniform properties

Mechanism for specification, expansivity, and Bowen property is uniform contraction/expansion + density of stable/unstable leaves

For non-uniformly hyperbolic systems, restrict attention to some $\mathcal{G} \subset X \times \mathbb{N}$ where hyperbolicity is uniform

Non-uniform properties

Mechanism for specification, expansivity, and Bowen property is uniform contraction/expansion + density of stable/unstable leaves

For non-uniformly hyperbolic systems, restrict attention to some $\mathcal{G} \subset X \times \mathbb{N}$ where hyperbolicity is uniform

- $\mathcal{G}$ has δ-specification if any $\{(x_i, n_i)\}_i \subset \mathcal{G}$ can be (δ, T)-glued
- φ is ε-Bowen on $\mathcal{G}$ if $\sup_{(x,n)\in\mathcal{G}} \sup_{y\in B_n(x,\varepsilon)} |S_n\varphi(y) - S_n\varphi(x)| < \infty$
- Later will require that $\mathcal{G}$ be 'large'

Non-expansive set: $NE(\varepsilon) = \{x \mid B_\infty(x,\varepsilon) \neq \{x\}\}$

- $\mathcal{M}^{ne}(\varepsilon) = \{\text{ergodic } \mu \in \mathcal{M}_f(X) \mid \mu(NE(\varepsilon)) > 0\}$
- $P^\perp_{\exp}(\varphi,\varepsilon) = \sup\{h_\mu(f) + \int \varphi \, d\mu \mid \mu \in \mathcal{M}^{ne}(\varepsilon)\}$

An abstract uniqueness result

Given $\mathcal{C}^p, \mathcal{C}^s \subset X \times \mathbb{N}$ and $M \in \mathbb{N}$, let

$$\mathcal{G}^M = \{(x,n) \mid \exists p + g + s = n \text{ s.t. } p, s \leq M, \\ (x,p) \in \mathcal{C}^p,\ (f^p x, g) \in \mathcal{G},\ (f^{p+g} x, s) \in \mathcal{C}^s\}$$

$$\mathcal{C} = \mathcal{C}^p \cup \mathcal{C}^s \cup \left(X \times \mathbb{N} \setminus \bigcup_M \mathcal{G}^M\right)$$

Quantify 'pressure of obstructions to specification' by

$$\Phi_\varepsilon(x,n) = \sup\{S_n\varphi(y) \mid y \in B_n(x,\varepsilon)\},$$

$$P(\mathcal{C},\varphi,\delta,\varepsilon) = \sup\{\textstyle\sum_{x\in E} e^{\Phi_\varepsilon(x,n)} \mid E \times \{n\} \subset \mathcal{C}, \text{ and } E \text{ is } (n,\delta)\text{-sep}\}$$

Theorem (C.–Thompson)

(X,f,φ) has a unique eq. state if $\varepsilon > 20\delta > 0$ and $\mathcal{G}, \mathcal{C}^{p,s}$ are s.t.

(1) $P^\perp_{\exp}(\varphi,\varepsilon) < P(\varphi)$

(2) *every $\mathcal{G}^M$ has δ-specification*

(3) *φ is ε-Bowen on $\mathcal{G}$*

(4) $P(\mathcal{C},\varphi,\delta,\varepsilon) < P(\varphi)$

Application to examples

Unif. hyperbolic outside $B =$ nbhd where perturbation is made

- Fix $r > 0$, let $\mathcal{D} = \{(x, n) \mid \text{at least } rn \text{ iterates outside } B\}$
- $(x, n) \in \mathcal{D} \Rightarrow Dg^n(x)$ contracts E^{cs} and expands E^{cu}
- $\mathcal{G} = \{(x, n) \mid (x, k), (f^{n-k}x, k) \in \mathcal{D} \text{ for all } 0 \leq k \leq n\}$

Application to examples

Unif. hyperbolic outside $B =$ nbhd where perturbation is made

- Fix $r > 0$, let $\mathcal{D} = \{(x, n) \mid \text{at least } rn \text{ iterates outside } B\}$
- $(x, n) \in \mathcal{D} \Rightarrow Dg^n(x)$ contracts E^{cs} and expands E^{cu}
- $\mathcal{G} = \{(x, n) \mid (x, k), (f^{n-k}x, k) \in \mathcal{D} \text{ for all } 0 \le k \le n\}$
- $\mathcal{C}^p = \mathcal{C}^s = (X \times \mathbb{N}) \setminus \mathcal{D}$ ('almost all time in B')
- Given (x, n), let p be maximal such that $(x, p) \in \mathcal{C}^p$, and s maximal such that $(f^{n-s}x, s) \in \mathcal{C}^s$, then $(f^p x, n - s - p) \in \mathcal{G}$.

Application to examples

Unif. hyperbolic outside B = nbhd where perturbation is made

- Fix $r > 0$, let $\mathcal{D} = \{(x, n) \mid \text{at least } rn \text{ iterates outside } B\}$
- $(x, n) \in \mathcal{D} \Rightarrow Dg^n(x)$ contracts E^{cs} and expands E^{cu}
- $\mathcal{G} = \{(x, n) \mid (x, k), (f^{n-k}x, k) \in \mathcal{D} \text{ for all } 0 \le k \le n\}$
- $\mathcal{C}^p = \mathcal{C}^s = (X \times \mathbb{N}) \setminus \mathcal{D}$ ('almost all time in B')
- Given (x, n), let p be maximal such that $(x, p) \in \mathcal{C}^p$, and s maximal such that $(f^{n-s}x, s) \in \mathcal{C}^s$, then $(f^p x, n - s - p) \in \mathcal{G}$.

Can verify conditions of theorem if perturbation is small enough

- local product structure + Hölder gives Bowen property on $\mathcal{G}$
- product structure + density of leaves gives $\mathcal{G}^M$ specification
- $P^{\perp}_{\exp}(\varphi, \varepsilon), P(\mathcal{C}, \varphi, \delta, \varepsilon) \approx$ "$P(B, \varphi)$ up to γn escapes"

  $\approx (1 - \gamma)\sup_B \varphi + 2 \log \lambda_c + \gamma(\sup_M \varphi + C - \log \gamma) + \eta^\alpha |\varphi|_\alpha$

Onwards to towers?

Specification approach gives uniqueness of equilibrium state, but not stronger statistical properties like exponential decay of correlations, central limit theorem, ASIP, etc.

- Can get these using Young towers provided (1) tail of tower decays exponentially, (2) equilibrium state lifts to tower.
- Liftability is often difficult to establish

Theorem (C., 2014)

If (X, σ) is a shift space on a finite alphabet and φ a Hölder potential such that obstructions to specification have small pressure, then (X, σ) contains a Young tower such that every equilibrium state lifts to the tower (in particular, there is a unique equilibrium state), and the tower has exponential tails.

Question: Can this be generalized to smooth systems?