Non-uniform specification properties and large deviations

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The talk in one slide

Setting: X a shift space on a finite alphabet (generalises naturally)

Theorem (Known results)

Suppose X has specification. Then

1. *bounded distortion $\Rightarrow$ unique equilibrium state.*
2. *A large deviations principle holds for every Gibbs measure.*

Goal: Same results with non-uniform versions of above properties

Key idea:

- $\mathcal{L}$ the language of X (space of finite orbit segments)
- Only require properties for $\mathcal{G} \subset \mathcal{L}$
- Get results if $\mathcal{G}$ is "big enough"

Specification for shift spaces

Shift space: closed, shift-invariant set $X \subset \mathcal{A}^{\mathbb{N}}$

- $A = \{1, \dots, p\}$ a finite alphabet

Every finite word $w \in A^* = \bigcup_{n \geq 0} A^n$ determines a cylinder

$$[w] = \{x \in X \mid x_1 \dots x_n = w\} \qquad (n = |w|)$$

The language of X is $\mathcal{L} = \{w \in A^* \mid [w] \neq \emptyset\}$.

X is topologically transitive iff

- for all $u, v \in \mathcal{L}$ there exists $w \in \mathcal{L}$ such that $uwv \in \mathcal{L}$

X has specification if

- there exists $\tau \in \mathbb{N}$ such that w can be chosen with $|w| \leq \tau$, **independently of the length of** u, v

Pressure and equilibrium states

Topological pressure of $\varphi \colon X \to \mathbb{R}$ is

$$P(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{w\in\mathcal{L}_n} e^{\varphi_n(w)} \right),$$

where $\mathcal{L}_n = \{w \in \mathcal{L} \mid |w| = n\}$ and $\varphi_n(w) = \sup_{x\in[w]} S_n\varphi(x)$.

$S_n\varphi(x) = \varphi(x) + \varphi(\sigma x) + \cdots + \varphi(\sigma^{n-1}x)$

Variational principle: $P(\varphi) = \sup\{h(\mu) + \int \varphi \, d\mu \mid \mu \in \mathcal{M}_\sigma(X)\}$

- $\mathcal{M}_\sigma(X) = \{\sigma\text{-invariant probability measures on } X\}$

A measure achieving the supremum is an equilibrium state.

Unique equilibrium states

φ has bounded distortions if there exists $V \in \mathbb{R}$ such that

$$|S_n\varphi(x) - S_n\varphi(y)| \leq V \text{ for all } w \in \mathcal{L},\ x, y \in [w] \qquad (n = |w|)$$

$\mu \in \mathcal{M}_\sigma(X)$ is Gibbs if there are $K, K' > 0$ such that

$$K \leq \frac{\mu[w]}{e^{-nP(\varphi)+S_n\varphi(x)}} \leq K'$$

for all $w \in \mathcal{L},\ n = |w|,\ x \in [w]$.

Theorem (Bowen, 1974)

If X has specification and φ has bounded distortions, then φ has a unique equilibrium state μ, and μ has the Gibbs property.

Empirical measures

$\mathcal{M}(X) = \{\text{Borel probability measures on } X\}$

Given $x \in X$ and $n \in \mathbb{N}$, get empirical measure

$$\mathcal{E}_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} \delta_{\sigma^k x} \qquad \mathcal{E}_n(x)(\varphi) = S_n\varphi(x)$$

Recall $\mathcal{E}_n(x) \to m$ for m-a.e. x if m ergodic

Large deviations studies rate of decay of $m\{x \mid \mathcal{E}_n(x) \in U\}$ for sets $U \subset \mathcal{M}(X)$ not containing m.

Large deviations

X satisfies a large deviations principle with reference measure m and rate function $q\colon \mathcal{M}(X) \to [-\infty, 0]$ if

$$U \subset \mathcal{M}_\sigma(X) \text{ open } \Rightarrow \liminf_{n\to\infty} \frac{1}{n} \log m\{x \mid \mathcal{E}_n(x) \in U\} \geq \sup_{\mu \in U} q(\mu)$$

$$F \subset \mathcal{M}_\sigma(X) \text{ closed } \Rightarrow \limsup_{n\to\infty} \frac{1}{n} \log m\{x \mid \mathcal{E}_n(x) \in F\} \leq \sup_{\mu \in F} q(\mu)$$

Theorem

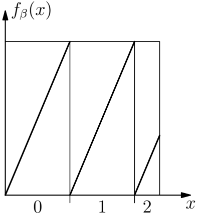
If X has specification and m is Gibbs for φ, then X satisfies a large deviations principle with reference measure m and rate function

$$q(\mu) = \begin{cases} h(\mu) + \int \varphi \, d\mu - P(\varphi) & \mu \in \mathcal{M}_\sigma(X) \\ -\infty & \mu \notin \mathcal{M}_\sigma(X) \end{cases}$$

β-shifts

For $\beta > 1$, Σ_β is the coding space for the map

$$f_\beta \colon [0,1] \to [0,1], \qquad x \mapsto \beta x \pmod 1$$

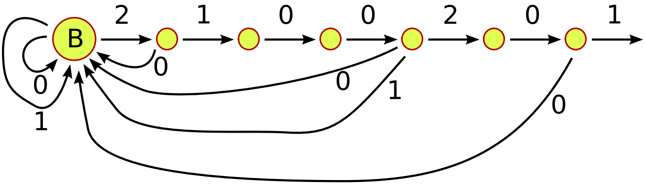


$1_\beta = a_1 a_2 \cdots$, where $1 = \sum_{n=1}^\infty a_n \beta^{-n}$

Fact: $x \in \Sigma_\beta \Leftrightarrow \sigma^n x \preceq 1_\beta$ for all n

$\Leftrightarrow x$ labels a walk starting at **B** on this graph:

(Here $1_\beta = 2100201\ldots$)



Properties of β-shifts

Σ_β has specification iff 1_β does not contain arbitrarily long sequences of 0s.

Schmeling (1997): For Leb-a.e. β, Σ_β does not have specification

Hofbauer (1979): Σ_β has a unique measure of maximal entropy

Walters (1978): Every Lipschitz potential has a unique eq. state

Equilibrium state is not Gibbs – so what about large deviations?
And what about more general bounded distortion potentials?

Coded systems

Given $\beta > 1$, $\alpha \in (0,1)$, consider coding space for

$$f_{\alpha,\beta}\colon x \mapsto \alpha + \beta x \pmod 1.$$

Can be presented on a countable graph, but more complicated.
Similarly with any piecewise expanding interval map.

General class of coded systems: two equivalent characterisations

- Can be presented on a countable graph (finitely many labels)
- Countable set G of generating words w^j that can be freely concatenated: $\mathcal{L} = \overline{G^*} = \{\text{subwords of } w^{j_1} \cdots w^{j_n}\}$

Given $S \subset \mathbb{N}$, consider the words $w^n = 0^n 1$ for each $n \in S$.

- Σ_S is the coded system with generators $\{w^n \mid n \in S\}$.

Σ_S has specification iff S is syndetic (bounded gaps)

Potentials with unbounded distortion

Manneville–Pomeau / Hofbauer-type potentials on Σ_β:

- $x \in \Sigma_\beta$: $k(x)$ = number of initial 0s in x
- $\varphi(x) = a_{k(x)}$ where $a_n \to 0$ as $n \to \infty$
- If $|\sum a_n| = \infty$, then φ has unbounded distortion and $P(t\varphi)$ can exhibit phase transitions
- Arises from $f(x) = x + \gamma x^{1+\varepsilon} \pmod 1$ for $\gamma > 0$, $\varphi = -\log f'$

Grid potentials:

- $\varphi(x) = \psi(x) + a_{k(x)}$, where ψ has bounded distortion and $2^{-k(x)}$ is distance from x to some subshift $Y \subset X$

Collections of words

X a shift space, $\mathcal{L}$ its language, $\mathcal{D} \subset \mathcal{L}$

Pressure of φ on $\mathcal{D}$. Let $\mathcal{D}_n = \{w \in \mathcal{D} \mid |w| = n\}$, then

$$P(\mathcal{D}, \varphi) = \overline{\lim} \frac{1}{n} \log \sum_{w \in \mathcal{D}_n} e^{\varphi_n(w)} \qquad h(\mathcal{D}) = P(\mathcal{D}, 0)$$

$\mathcal{D}$ has specification if there exists $\tau \in \mathbb{N}$ such that for all $w^1, \dots, w^k \in \mathcal{D}$, there exist $v^1, \dots, v^{k-1} \in \mathcal{L}$ with $|v^j| \leq \tau$ such that $w^1 v^1 w^2 \cdots v^{k-1} w^k \in \mathcal{L}$.

φ has bounded distortion on $\mathcal{D}$ if there exists $V \in \mathbb{R}$ such that for all $w \in \mathcal{D}$, $n = |w|$, $x, y \in [w]$, we have $|S_n\varphi(x) - S_n\varphi(y)| \leq V$.

μ has the Gibbs property on $\mathcal{D}$ if there are $K, K' > 0$ such that for all $w \in \mathcal{D}$, $n = |w|$, $x \in [w]$, we have $K \leq \frac{\mu[w]}{e^{-nP(\varphi) + S_n\varphi(x)}} \leq K'$.

Decompositions

Idea: Unique equilibrium state for φ if there is a "large enough" $\mathcal{G} \subset \mathcal{L}$ with specification such that φ has bounded distortion on $\mathcal{D}$.

What does "large enough" mean?

Decomposition of $\mathcal{L}$: sets $\mathcal{C}^p, \mathcal{G}, \mathcal{C}^s \subset \mathcal{L}$ such that $\mathcal{L} = \mathcal{C}^p \mathcal{G} \mathcal{C}^s$.

$$\mathcal{G}^M = \{uvw \in \mathcal{L} \mid u \in \mathcal{C}^p, v \in \mathcal{G}, w \in \mathcal{C}^s, |u|, |w| \leq M\}$$

Theorem (C.–Thompson, 2012)

Suppose $\mathcal{L}$ has a decomposition such that

1. *φ has bounded distortion on $\mathcal{G}$*
2. *$\mathcal{G}^M$ has specification for every M*
3. $P(\mathcal{C}^p \cup \mathcal{C}^s, \varphi) < P(\varphi)$

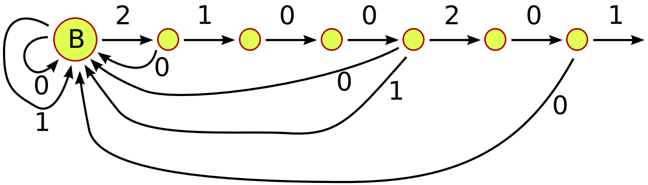
Then φ has a unique equilibrium state μ. It is Gibbs on each $\mathcal{G}^M$.

Example: β-shift

$\mathcal{C}^p = \emptyset$

$\mathcal{G} = \{$words (paths) starting and ending at $B\}$

$\mathcal{C}^s = \{$words (paths) starting at B and never returning$\}$



- $\mathcal{L} = \mathcal{C}^p \mathcal{G} \mathcal{C}^s$
- $\mathcal{G}^M$ corresponds to paths ending in first M vertices, so $\mathcal{G}^M$ has specification for each M
- $h(\mathcal{C}) = 0$, where $\mathcal{C} = \mathcal{C}^p \cup \mathcal{C}^s$

Hölder potentials

To get unique equilibrium state for φ, need $P(\mathcal{C},\varphi) < P(\varphi)$.

Suppose we know that $h(\mathcal{C}) + \sup_{x\in X} \left(\overline{\lim}\, \frac{1}{n} S_n\varphi(x)\right) < P(\varphi)$.
Then get $P(\mathcal{C},\varphi) < P(\varphi)$.

Equivalent conditions:

- $\sup_x \overline{\lim}\, \frac{1}{n} S_n\varphi(x) < P(\varphi) - h(\mathcal{C})$
- $\exists n$ such that $\sup_x \frac{1}{n} S_n\varphi(x) < P(\varphi) - h(\mathcal{C})$
- Every equilibrium state for φ has $h(\mu) > h(\mathcal{C})$

Theorem (C.–Thompson, 2012)

When X is a β-shift, every Hölder continuous potential satisfies the above condition. In particular, it has a unique equilibrium state μ, and μ is Gibbs on each $\mathcal{G}^M$.

Decompositions for coded systems

Let X be a coded shift: two natural decompositions of $\mathcal{L}$.

In terms of countable graph presentation

- Fix a finite subset F of the graph
- $\mathcal{C}^p =$ paths starting outside F and entering it only on the last step, or never
- $\mathcal{G} =$ paths starting and ending in F
- $\mathcal{C}^s =$ paths starting in F and never returning

In terms of generators

- $G \subset A^*$ a set of generators
- $\mathcal{G} = G^* = \{w^1 \cdots w^n \mid w^j \in \mathcal{G}\}$
- $\mathcal{C}^p =$ suffixes of generators
- $\mathcal{C}^s =$ prefixes of generators

Interval maps

Let f be a piecewise expanding interval map, X the coding space

- Graph presentation gives decomposition of $\mathcal{L}$
- $h(\mathcal{C}) > 0$, but can be made arbitrarily small by taking F large

Definition

The entropy of obstructions to specification is

$$h_{\mathrm{spec}}^{\perp}(X) = \inf\{h(\mathcal{C}^p \cup \mathcal{C}^s) \mid \text{there exists } \mathcal{G} \text{ such that } \mathcal{L} = \mathcal{C}^p \mathcal{G} \mathcal{C}^s \text{ and every } \mathcal{G}^M \text{ has specification}\}$$

Unique equilibrium state for φ, Gibbs on each $\mathcal{G}^M$, if any (all) of

- $\sup_x \overline{\lim} \frac{1}{n} S_n\varphi(x) < P(\varphi) - h_{\mathrm{spec}}^{\perp}$
- $\exists n$ such that $\sup_x \frac{1}{n} S_n\varphi(x) < P(\varphi) - h_{\mathrm{spec}}^{\perp}$
- Every equilibrium state for φ has $h(\mu) > h_{\mathrm{spec}}^{\perp}$

Positive entropy equilibrium states

For S-gap shifts the natural decomposition from generators gives

- $\mathcal{C}^p = \{0^n 1 \mid n \in \mathbb{N}\}$
- $\mathcal{G} = \{0^{n_1} 1 \cdots 0^{n_k} 1 \mid n_j \in S\}$
- $\mathcal{C}^s = \{0^n \mid n \in \mathbb{N}\}$

So $h^{\perp}_{\text{spec}} = 0$, as with other examples

For S-gap shifts, every Hölder potential has $P(\varphi) > \sup \overline{\lim} \frac{1}{n} S_n \varphi$.

This gives same set of results as for β-shifts: unique equilibrium state, Gibbs on $\mathcal{G}^M$

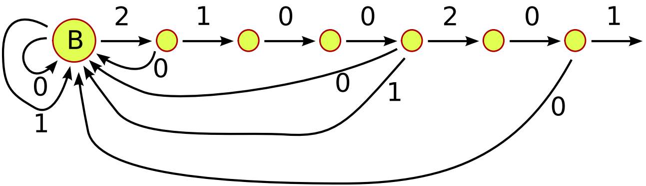
Open questions: What about piecewise expanding interval maps? Are there coded systems with $h^{\perp}_{\text{spec}} = 0$ for which some Hölder potentials have zero entropy equilibrium states?

Unbounded distortion

X a β-shift, $\varphi(x) = \psi(x) + a_{k(x)}$

$\mathcal{G}$ = paths starting at B and ending at the next vertex from B

$\mathcal{C}^s$ = paths starting at the second vertex and never returning to B, and paths starting at B and never getting to the next vertex



- $\mathcal{G}^M$ has specification for each M
- φ has bounded distortion on $\mathcal{G}$
- $P(\mathcal{C}^s, \varphi) < P(\varphi)$ whenever $\varphi(0) < P(\varphi)$

Manneville–Pomeau for β-transformations

Conclusion: If $P(\varphi) > \varphi(0)$ then φ has a unique equilibrium state.

Corollary: Let $f(x) = x + \gamma x^{1+\varepsilon} \pmod 1$ and $\varphi(x) = -\log f'(x)$, where $\gamma > 0$ and $0 < \varepsilon < 1$. Then $t\varphi$ has a unique equilibrium state for every $t < 1$.

Can get similar results with grid potentials if X has specification.

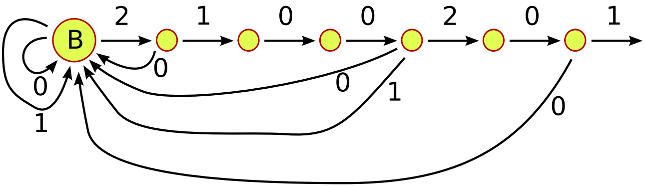
Statistical specification properties

Large deviations results have been obtained for β-shift and other systems by using statistical specification properties.

- Pfister, Sullivan (2005)
- Yamamoto (2009)
- Varandas (2012)

All reflect idea that the gluing procedure can be weakened in a way that does not interfere too much with Birkhoff averages.

β-shifts



Given any $v \in \mathcal{L}$, can transform v into a word $u \in \mathcal{G}$ by making a single change. (Change last non-zero symbol to 0).

Thus given any $v, w \in \mathcal{L}$, the word vw may not be in $\mathcal{L}$, but can be transformed into a word in $\mathcal{L}$ by making a single change.

General method for getting a word that concatenates statistical properties of v and w, as long as $\frac{\text{number of changes}}{\text{length of word}} \to 0$.

Edit metric

Goal: Define a metric on A^* (set of all finite words) that controls how much Birkhoff sums can vary.

An edit of a word w is any of the following:

- Substition: $w = uav \mapsto w' = ubv$ $\quad u, v \in A^*, \ a, b \in A$
- Insertion: $w = uv \mapsto w' = ubv$ $\quad u, v \in A^*, \ b \in A$
- Deletion: $w = uav \mapsto w' = uv$ $\quad u, v \in A^*, \ a \in A$

$\hat{d}(v, w) =$ minimum number of edits required to go from v to w.

Key property: Let D be a metric inducing the weak* topology on $\mathcal{M}(X)$. Then for every $\eta > 0$ there is $\delta > 0$ such that if $\frac{\hat{d}(v,w)}{|v|} < \delta$, then $D(\mathcal{E}_{|v|}(x), \mathcal{E}_{|w|}(y)) < \eta$ for all $x \in [v]$ and $y \in [w]$.

Edit approachability

mistake function: a non-increasing sub-linear function $g\colon \mathbb{N} \to \mathbb{N}$.
$(\frac{g(n)}{n} \to 0)$

$\mathcal{L}$ is edit approachable by $\mathcal{G} \subset \mathcal{L}$ if there exists a mistake function g such that for every $v \in \mathcal{L}$, there is $w \in \mathcal{G}$ with $\hat{d}(v, w) < g(|v|)$.

Equivalently, $\mathcal{L} = \bigcup_{w \in \mathcal{G}} B_{\hat{d}}(w, g(|w|))$.

Examples: For both the β-shifts and the S-gap shifts, $\mathcal{L}$ is edit approachable by the natural choice of $\mathcal{G}$.

Large deviations

Theorem (C.–Thompson–Yamamoto, 2013)

X a shift space on a finite alphabet, $\mathcal{L}$ its language. Suppose

1. *$\mathcal{L}$ is edit approachable by $\mathcal{G}$,*
2. *$\mathcal{G}$ has specification (with good concatenations),*
3. *$m \in \mathcal{M}(X)$ is Gibbs for φ on $\mathcal{G}$.*

Then X satisfies a LDP with reference measure m and rate f'n

$$q(\mu) = \begin{cases} h(\mu) + \int \varphi \, d\mu - P(\varphi) & \mu \in \mathcal{M}_\sigma(X) \\ -\infty & \mu \notin \mathcal{M}_\sigma(X) \end{cases}$$

In particular, every Hölder continuous φ on a β-shift or S-gap shift.

Key tool in proof

The bulk of the proof is in the following proposition.

X a shift space, $\mathcal{L}$ edit approachable by $\mathcal{G}$ with specification

Then $\exists$ an increasing sequence $X_n \subset X$ of subshifts s.t.

1. Each X_n has specification
2. If m is Gibbs on $\mathcal{G}$, then it is Gibbs on every $\mathcal{L}(X_n)$
3. For every $\mu \in \mathcal{M}_\sigma(X)$ there are subshifts $Y_n \subset X_n$ s.t. $\mathcal{M}_\sigma(Y_n) \to \{\mu\}$ and $\underline{\lim}\, h(Y_n) \geq h(\mu)$

In particular, ergodic measures are entropy-dense in $\mathcal{M}_\sigma(X)$

Moral

One moral of the story:

Many good consequences of specification (and other properties) can still be obtained as long as properties hold on a "large enough" set of words (orbit segments)

"Large enough" means the ability to get from $\mathcal{L}$ to $\mathcal{G}$ with some "small" tinkering, where meaning of "small" depends on context

- Unique equilibrium state: only need to remove a prefix and a suffix from the word in $\mathcal{L}$, and these come from "small" lists
- Large deviations: only need to make a small number of edits
- Hölder $\Rightarrow$ only positive entropy equilibrium states: **????**