Non-uniform specification, thermodynamic formalism, and towers

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The talk in one slide

Goal: Hyperbolicity $\rightsquigarrow$ equilibrium states / SRB measures

- Existence, uniqueness
- Statistical properties $\begin{cases} \text{decay of correlations, CLT} \\ \text{large deviations, multifractal} \end{cases}$

Known:

- Markov partition $\Rightarrow$ all of these
- Non-uniform version of Markov partition $\rightsquigarrow$ towers
- Specification $\Rightarrow$ some of these

Questions:

- Specification $\Rightarrow$ EDC, CLT? Non-uniform specification?

Answers:

- Non-uniform specification $\Rightarrow$ uniqueness, large deviations
- (NU) specification $\Rightarrow$ tower

General setting

X a compact metric space, $f : X \to X$ continuous

$\mathcal{M} = \{\text{Borel probability measures on } X\}$

- $\mathcal{M}_f = \{f\text{-invariant}\}$, $\mathcal{M}_f^e = \{\text{ergodic}\}$

$\varphi \in C(X) \quad \rightsquigarrow \quad P(\varphi) = \sup\{h(\mu) + \int \varphi \, d\mu \mid \mu \in \mathcal{M}_f\}$

- Topological pressure, also admits definition as dimension
- Supremum achieved by equilibrium state
- SRB (physical) measures are equilibrium states for $-\log J^u$

Expansive $\Rightarrow$ equilibrium states exist

- For now, assume expansive (weaken this assumption later)

Shift spaces

Shift space: closed, shift-invariant set $X \subset A^{\mathbb{N}}$

- $A = \{1, \dots, p\}$ a finite alphabet

Every finite word $w \in A^* = \bigcup_{n\geq 0} A^n$ determines a cylinder

$$[w] = \{x \in X \mid x_1 \cdots x_n = w\} \qquad (n = |w|)$$

The language of X is $\mathcal{L} = \{w \in A^* \mid [w] \neq \emptyset\}$.

Transitive $\Leftrightarrow$ for all $u, v \in \mathcal{L}$ there exists $w \in \mathcal{L}$ s.t. $uwv \in \mathcal{L}$

- X has specification if there exists $\tau \in \mathbb{N}$ such that w can be chosen with $|w| \leq \tau$, **independently of the length of** u, v

Pressure as growth rate

Given $\mathcal{D} \subset \mathcal{L}$ and $\varphi \in C(X)$, partition sums for $\mathcal{D}$, φ are

$$\Lambda_n(\mathcal{D}, \varphi) = \sum_{w \in \mathcal{D}_n} e^{\varphi_n(w)},$$

where $\mathcal{D}_n = \{w \in \mathcal{D} \mid |w| = n\}$ and $\varphi_n(w) = \sup_{x \in [w]} S_n\varphi(x)$.

$S_n\varphi(x) = \varphi(x) + \varphi(\sigma x) + \cdots + \varphi(\sigma^{n-1}x)$

Variational principle: $P(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \Lambda_n(\mathcal{L}, \varphi)$

- For $\mathcal{D} \subset \mathcal{L}$, also consider $P(\mathcal{D}, \varphi) = \overline{\lim}_{n\to\infty} \frac{1}{n} \log \Lambda_n(\mathcal{D}, \varphi)$.

Unique equilibrium states

φ has bounded distortions if there exists $V \in \mathbb{R}$ such that

$$|S_n\varphi(x) - S_n\varphi(y)| \leq V \text{ for all } w \in \mathcal{L},\ x, y \in [w] \qquad (n = |w|)$$

$\mu \in \mathcal{M}_\sigma(X)$ is Gibbs if there are $K, K' > 0$ such that

$$K \leq \frac{\mu[w]}{e^{-nP(\varphi)+S_n\varphi(x)}} \leq K'$$

for all $w \in \mathcal{L},\ n = |w|,\ x \in [w]$.

Theorem (Bowen, 1974)

If X has specification and φ has bounded distortions, then φ has a unique equilibrium state μ, and μ has the Gibbs property.

Large deviations

- $(x, n) \in X \times \mathbb{N} \rightsquigarrow$ empirical measure $\mathcal{E}_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} \delta_{\sigma^k x}$

$m \in \mathcal{M}$ has large deviations principle with rate $q\colon \mathcal{M} \to [-\infty, 0]$ if

$$U \subset \mathcal{M} \text{ open } \Rightarrow \liminf_{n\to\infty} \frac{1}{n} \log m\{x \mid \mathcal{E}_n(x) \in U\} \geq \sup_{\mu \in U} q(\mu)$$

and similar upper bound on lim sup when U closed.

Theorem (Young, 1990)

If X has specification and m is Gibbs for φ, then X satisfies a large deviations principle with reference measure m and rate function

$$q(\mu) = \begin{cases} h(\mu) + \int \varphi\, d\mu - P(\varphi) & \mu \in \mathcal{M}_\sigma(X) \\ -\infty & \mu \notin \mathcal{M}_\sigma(X) \end{cases}$$

Other statistical properties

(X, σ, μ) has exponential decay of correlations on a class of functions $\mathcal{F}$ if there is $\gamma < 1$ s.t. $\forall \varphi, \psi \in \mathcal{F}\ \exists C = C(\varphi, \psi)$ s.t.

$$\left| \int (\varphi \circ \sigma^n)\psi \, d\mu - \int \varphi \, d\mu \int \psi \, d\mu \right| \leq C\gamma^n$$

Question: Specification $\Rightarrow$ μ_φ has EDC? What about CLT?

Known: Both follow if (X, σ, μ) has a tower with exponential tails

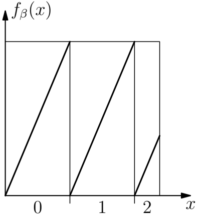
Revised question: Specification $\Rightarrow$ a tower with exponential tails?

- Return to this after discussing non-uniform specification

β-shifts

For $\beta > 1$, Σ_β is the coding space for the map

$$f_\beta\colon [0,1] \to [0,1], \qquad x \mapsto \beta x \pmod 1$$

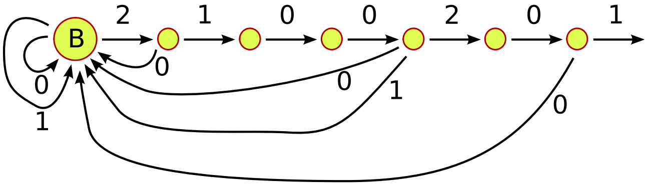


$1_\beta = a_1 a_2 \cdots$, where $1 = \sum_{n=1}^{\infty} a_n \beta^{-n}$

Fact: $x \in \Sigma_\beta \Leftrightarrow \sigma^n x \preceq 1_\beta$ for all n

$\Leftrightarrow x$ labels a walk starting at **B** on this graph:

(Here $1_\beta = 2100201\ldots$)



Properties of β-shifts

Σ_β has specification iff $1_\beta \not\supset$ arbitrarily long sequences of 0s

Schmeling (1997): For Leb-a.e. β, Σ_β does not have specification

Hofbauer (1979): Σ_β has a unique measure of maximal entropy

Walters (1978): Every Lipschitz potential has a unique eq. state

Equilibrium state is not Gibbs – so what about large deviations?
And what about more general bounded distortion potentials?

Collections of words

$\mathcal{D} \subset \mathcal{L}$ has specification if there exists $\tau \in \mathbb{N}$ such that for all $u, v \in \mathcal{D}$, there exists $w \in \mathcal{L}$ with $|w| \leq \tau$ such that $uwv \in \mathcal{L}$.

- Σ_β: $\mathcal{G} = \{$words starting and ending at B$\}$ has specification

φ has bounded distortion on $\mathcal{D}$ if there exists $V \in \mathbb{R}$ such that for all $w \in \mathcal{D}$, $n = |w|$, $x, y \in [w]$, we have $|S_n\varphi(x) - S_n\varphi(y)| \leq V$.

μ has the Gibbs property on $\mathcal{D}$ if there are $K, K' > 0$ such that for all $w \in \mathcal{D}$, $n = |w|$, $x \in [w]$, we have $K \leq \frac{\mu[w]}{e^{-nP(\varphi)+S_n\varphi(x)}} \leq K'$.

Decompositions

Idea: Unique ES if spec and bdd dist on "large enough" $\mathcal{G} \subset \mathcal{L}$

What does "large enough" mean?

Decomposition of $\mathcal{L}$: sets $\mathcal{C}^p, \mathcal{G}, \mathcal{C}^s \subset \mathcal{L}$ such that $\mathcal{L} = \mathcal{C}^p \mathcal{G} \mathcal{C}^s$.

$$\mathcal{G}^M = \{uvw \in \mathcal{L} \mid u \in \mathcal{C}^p, v \in \mathcal{G}, w \in \mathcal{C}^s, |u|, |w| \leq M\}$$

Theorem (C.–Thompson, 2012)

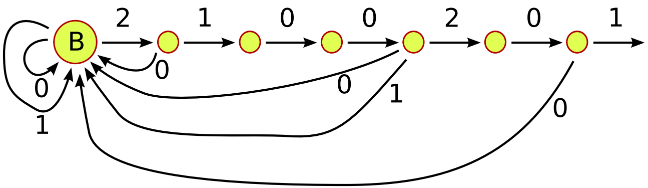
Suppose $\mathcal{L}$ has a decomposition such that

1. φ *has bounded distortion on* $\mathcal{G}$
2. $\mathcal{G}^M$ *has specification for every* M
3. $P(\mathcal{C}^p \cup \mathcal{C}^s, \varphi) < P(\varphi)$

Then φ has a unique equilibrium state μ. It is Gibbs on each $\mathcal{G}^M$.

Example: β-shift

$$\mathcal{C}^p = \emptyset$$
$$\mathcal{G} = \{\text{words (paths) starting and ending at } B\}$$
$$\mathcal{C}^s = \{\text{words (paths) starting at } B \text{ and never returning}\}$$



- $\mathcal{L} = \mathcal{C}^p \mathcal{G} \mathcal{C}^s$
- $\mathcal{G}^M$ corresponds to paths ending in first M vertices, so $\mathcal{G}^M$ has specification for each M
- $h(\mathcal{C}) = 0$, where $\mathcal{C} = \mathcal{C}^p \cup \mathcal{C}^s$

Hölder potentials

To get unique equilibrium state for φ, need $P(\mathcal{C}, \varphi) < P(\varphi)$.

Equivalent conditions: (hyperbolic potential)

- $\sup_x \overline{\lim} \frac{1}{n} S_n\varphi(x) < P(\varphi)$
- $\exists n$ such that $\sup_x \frac{1}{n} S_n\varphi(x) < P(\varphi)$
- Every equilibrium state for φ has $h(\mu) > 0$

Theorem (C.–Thompson, 2012)

When X is a β-shift, every Hölder continuous potential is hyperbolic. In particular, it has a unique equilibrium state μ, and μ is Gibbs on each $\mathcal{G}^M$.

Interval maps

Let f be a piecewise expanding interval map, X the coding space

- Graph presentation gives decomposition: F a finite subset
$$\begin{cases} \mathcal{C}^p = \text{paths entering } F \text{ only on last step, or never} \\ \mathcal{G} = \text{paths starting and ending in } F \\ \mathcal{C}^s = \text{paths starting in } F \text{ and never returning} \end{cases}$$

$h(\mathcal{C}) > 0$, but can be made arbitrarily small by taking F large

Unique equilibrium state for φ, Gibbs on each $\mathcal{G}^M$, if

- $\sup_x \overline{\lim} \frac{1}{n} S_n\varphi(x) < P(\varphi)$ (or other equiv. condition)

Question: Hölder $\Rightarrow$ unique ES for all such interval maps?

- $\exists$ shift space with $h(\mathcal{C}) = 0$ but above properties fail (Conrad)

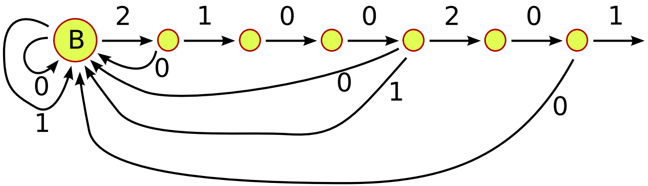
Statistical specification properties

Large deviations results have been obtained for β-shift and other systems by using statistical specification properties.

- Pfister, Sullivan (2005)
- Yamamoto (2009)
- Varandas (2012)

All reflect idea that the gluing procedure can be weakened in a way that does not interfere too much with Birkhoff averages.

β-shifts



Given any $v \in \mathcal{L}$, can transform v into a word $u \in \mathcal{G}$ by making a single change. (Change last non-zero symbol to 0).

Thus given any $v, w \in \mathcal{L}$, the word vw may not be in $\mathcal{L}$, but can be transformed into a word in $\mathcal{L}$ by making a single change.

General method for getting a word that concatenates statistical properties of v and w, as long as $\frac{\text{number of changes}}{\text{length of word}} \to 0$.

Edit metric

Goal: Define a metric on A^* (set of all finite words) that controls how much Birkhoff sums can vary.

An edit of a word w is any of the following:

- Substitution: $w = uav \mapsto w' = ubv$ $\quad u, v \in A^*, \ a, b \in A$
- Insertion: $w = uv \mapsto w' = ubv$ $\quad u, v \in A^*, \ b \in A$
- Deletion: $w = uav \mapsto w' = uv$ $\quad u, v \in A^*, \ a \in A$

$\hat{d}(v, w) =$ minimum number of edits required to go from v to w.

Key property: Let D be a metric inducing the weak* topology on $\mathcal{M}(X)$. Then for every $\eta > 0$ there is $\delta > 0$ such that if $\frac{\hat{d}(v,w)}{|v|} < \delta$, then $D(\mathcal{E}_{|v|}(x), \mathcal{E}_{|w|}(y)) < \eta$ for all $x \in [v]$ and $y \in [w]$.

Edit approachability

mistake function: a non-increasing sub-linear function $g: \mathbb{N} \to \mathbb{N}$.
$(\frac{g(n)}{n} \to 0)$

$\mathcal{L}$ is edit approachable by $\mathcal{G} \subset \mathcal{L}$ if there exists a mistake function g such that for every $v \in \mathcal{L}$, there is $w \in \mathcal{G}$ with $\hat{d}(v, w) < g(|v|)$.

Equivalently, $\mathcal{L} = \bigcup_{w \in \mathcal{G}} B_{\hat{d}}(w, g(|w|))$.

Example: For β-shifts, $\mathcal{L}$ is edit approachable by $\mathcal{G}$.

Theorem (C.–Thompson–Yamamoto, 2013)

X a shift space on a finite alphabet, $\mathcal{L}$ its language. Suppose

1. *$\mathcal{L}$ is edit approachable by $\mathcal{G}$,*
2. *$\mathcal{G}$ has specification (with good concatenations),*
3. *$m \in \mathcal{M}(X)$ is Gibbs for φ on $\mathcal{G}$.*

Then X satisfies a LDP with reference measure m and rate f'n

$$q(\mu) = \begin{cases} h(\mu) + \int \varphi \, d\mu - P(\varphi) & \mu \in \mathcal{M}_\sigma(X) \\ -\infty & \mu \notin \mathcal{M}_\sigma(X) \end{cases}$$

In particular, every Hölder continuous φ on a β-shift.

Key tool in proof

The bulk of the proof is in the following "horseshoe" proposition.

X a shift space, $\mathcal{L}$ edit approachable by $\mathcal{G}$ with specification

Then $\exists$ an increasing sequence $X_n \subset X$ of subshifts s.t.

1. Each X_n has specification
2. If m is Gibbs on $\mathcal{G}$, then it is Gibbs on every $\mathcal{L}(X_n)$
3. For every $\mu \in \mathcal{M}_\sigma(X)$ there are subshifts $Y_n \subset X_n$ s.t. $\mathcal{M}_\sigma(Y_n) \to \{\mu\}$ and $\underline{\lim}\, h(Y_n) \geq h(\mu)$

In particular, ergodic measures are entropy-dense in $\mathcal{M}_\sigma(X)$

Towers

Tower: enough of system coded by full shift on **countable** alphabet

For our purposes, (X, σ, μ) has tower if $\exists G \subset \mathcal{L}$ such that

- $\mu(G^{\mathbb{N}}) = 1$ (or $\mu(G^{\mathbb{Z}}) = 1$ for two-sided shifts)
- $v, w \in G \Rightarrow w \neq v\square$

Tower is $\Omega = \{(\underline{w}, n) \in G^{\mathbb{N}} \times \mathbb{N} \mid n \leq |w_0|\}$

$$F \colon \Omega \to \Omega \text{ given by } F(\underline{w}, n) = \begin{cases} (\underline{w}, n+1) & n < |w_0| \\ (\sigma(\underline{w}), 0) & n = |w_0| \end{cases}$$

Return time: $R(w_0 w_1 w_2 \cdots) = |w_0|$

- Exponential tails: $\mu\{R \geq n\} \leq C\gamma^n$ $\qquad \gamma < 1$

Guarantees exponential decay of correlations, CLT

Synchronised and coded shifts

Well-known: specification $\Rightarrow$ synchronised $\Rightarrow$ coded

Synchronised: $\exists\ v \in \mathcal{L}$ such that $uv \in \mathcal{L}, vw \in \mathcal{L} \Rightarrow uvw \in \mathcal{L}$

Coded: there exists $G \subset \mathcal{L}$ such that $\mathcal{L} = (G^*)^{\leq}$

- Equivalent: strongly connected countable graph presentation

Proof that synchronised $\Rightarrow$ coded: $G = \{vu \mid vuv \in \mathcal{L}\}$

- Next slides: spec $\Rightarrow$ sync ($\Rightarrow$ coded) $\Rightarrow$ tower

Dynamical interpretation: $x.v\Box \leftrightarrow W^u, \quad \Box.vy \leftrightarrow W^s$

- Synchronised: local product structure on $[v]$ for some v
- Markov: local product structure on $[v]$ for all (suff. long) v

A synchronising word

Specification $\Rightarrow$ synchronised (Bertrand 1988). Given $u, w \in \mathcal{L}$, let

$$C(u, w) = \{y \in \mathcal{L} \mid uyw \in \mathcal{L}, |y| \leq \tau\}.$$

Specification implies non-empty.

- Start with any u, w. Note that $C(\square u, w\square) \subset C(u, w)$.
- Extend to $\square u$ and $w\square$ such that $C(\square u, w\square) \neq C(u, w)$.
- Iterate. $C(u, w)$ finite $\Rightarrow$ process terminates.
- Let $v = uyw$ for some $y \in C(u, w) = C(\square u, w\square)$

Claim: v is a synchronising word

- $av \in \mathcal{L}$, $vb \in \mathcal{L} \Rightarrow auyw \in \mathcal{L}, uywb \in \mathcal{L}$
- By choice of u, w, get $y \in C(au, wb)$, so $avb = auywb \in \mathcal{L}$

Towers from specification

Specification $\Rightarrow$ unique equilibrium state μ_φ for Hölder φ

Also implies synchronised, hence coded with $G = \{vu \mid vuv \in \mathcal{L}\}$

- μ-a.e. x has v occur infinitely often, hence in Z
- $\{R \geq n\} \subset \{x \mid x_k \cdots x_{k+n} \not\supset v\}$
- Partition sum over this set grows like $e^{nP'}$ for $P' < P(\varphi)$
- Gibbs property for μ_φ gives exponential tail

Theorem (C. 2013)

If X is a shift with specification on a finite alphabet and μ is the unique equilibrium state for a Hölder potential, then μ has EDC and CLT.

Non-uniform specification

Theorem (C. 2013)

Let X be a shift with a decomposition $\mathcal{L} = \mathcal{C}^p \mathcal{G} \mathcal{C}^s$ s.t.

1. *every $\mathcal{G}^M$ has specification;*
2. $v \in \mathcal{G} \Rightarrow vw \in \mathcal{G}\mathcal{C}^s$,

and let $\varphi \in C(X)$ be a potential such that

3. *φ has bounded distortions on $\mathcal{G}$;*
4. $P(\mathcal{C}^p \cup \mathcal{C}^s, \varphi) < P(\varphi)$.

Let μ be the unique equilibrium state for φ. Then (X, σ, μ) has a tower with exponential tails, so that μ has EDC and CLT.

Proof follows similar idea, but X need not be synchronised.

- Get a word y that synchronises $\mathcal{G}$, not $\mathcal{L}$, then build tower around 'good' returns to $[y]$, instead of all returns.

Weakened expansivity condition

(X, f) expansive $\Leftrightarrow$ $\Gamma_\epsilon(x) := \bigcap_n B_n(x, \epsilon) = \{x\}$ for all $x \in X$.

- Let $N_f^\epsilon = \{x \mid \Gamma_\epsilon(x) \neq \{x\}\}$ be the non-expansive set.

Pressure of obstructions to expansivity is

$$P_{\exp}^\perp(\varphi) = \lim_{\epsilon\to 0} \sup\left\{h(\mu) + \int \varphi\, d\mu \mid \mu(N_f^\epsilon) > 0, \mu \in \mathcal{M}_f\right\}.$$

Replace language $\mathcal{L}$ with space of orbit segments $X \times \mathbb{N}$, consider pressure of obstructions to φ-specification

$$P_{\mathrm{spec},\varphi}^\perp(\varphi) = \lim_{\epsilon\to 0} \inf\{P(\mathcal{C}^p \cup \mathcal{C}^s, \varphi, \epsilon) \mid X \times \mathbb{N} = \mathcal{C}^p\mathcal{G}\mathcal{C}^s,$$
$$\text{every } \mathcal{G}^M \text{ has } \epsilon\text{-specification with bounded } \varphi\text{-distortion}\}$$

A uniqueness result

Theorem (C.–Thompson, 2013)

Let X be a compact metric space, $f: X \to X$ a continuous map, and $\varphi \in C(X)$. Suppose that $P^{\perp}_{\mathrm{exp}}(\varphi) < P(\varphi)$ and $P^{\perp}_{\mathrm{spec},\varphi}(\varphi) < P(\varphi)$. Then φ has a unique equilibrium state μ.

Question: Does μ have EDC and CLT? That is, can the tower construction from the symbolic setting be abstracted to this setting?

Towers from specification

Axiom A systems have towers (Young 1998): key ingredients are

1. bounded distortion;
2. local product structure;
3. uniform transitivity.

Axiom A $\Rightarrow$ local product structure everywhere

Expansive $+$ specification $\Rightarrow$ local product structure somewhere

1. Expected theorem: Young's construction goes through ok
2. Question: What about non-uniform spec / expansivity?