Effective hyperbolicity and applications of new Hadamard–Perron theorems

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The talk in one slide

Hadamard–Perron theorem: linear data governs non-linear behaviour on small scales

Consequences: SRB measures, closing lemmas, etc.

Uniform hyperbolicity: well-understood, rare

Non-uniform hyp.: understood **if asymptotic behaviour known**

- Depends on ergodic theory/infinite information
- SRB measure: need measure-independent approach
- Closing lemma: want finite-information

Get these with effective hyperbolicity

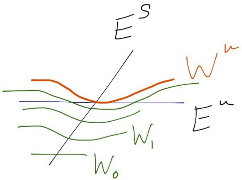
The simplest case

Assumption: $f : \mathbb{R}^d \to \mathbb{R}^d$ with $f(0) = 0$ and $Df(0)$ hyperbolic

- E^s stable subspace, E^u unstable subspace
- $|Df(0)(v^s)| \leq e^{\lambda^s}|v^s|$ and $|Df(0)(v^u)| \geq e^{\lambda^u}|v^u|$
- $\max(\lambda^s, 0) < \lambda^u$

Conclusion: There exists $W^u = \operatorname{graph} \psi$ tangent to E^u such that

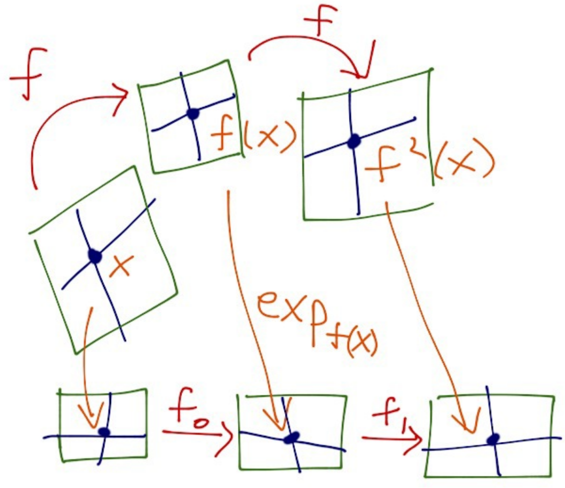
- $\|D\psi(x^u)\| \approx 0$ for $x^u \approx 0$
- $x, y \in W^u \Rightarrow$
  $d(f^{-n}x, f^{-n}y) \leq e^{-n\chi} d(x, y)$
- $\chi < \lambda^u$ is arbitrary



Proof uses graph transform $W_0 \mapsto W_1 \mapsto W_2 \cdots$

Sequences of germs

Away from fixed points, use local coordinates to get sequence f_n



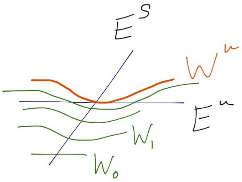
Uniform hyperbolicity

Assumption: $f_n\colon \mathbb{R}^d \to \mathbb{R}^d$ with $f_n(0) = 0$

- $E_n^{s,u}$ invariant under $Df_n(0)$, uniformly transverse
- $|Df_n(0)(v^s)| \leq e^{\lambda^s}|v^s|$ and $|Df_n(0)(v^u)| \geq e^{\lambda^u}|v^u|$
- $\max(\lambda^s, 0) < \lambda^u$

Conclusion: There exists $W_n^u = \operatorname{graph} \psi_n$ tangent to E_n^u such that

- $\|D\psi_n(x^u)\| \leq \gamma$ for $|x^u| \leq r$
- $x, y \in W^u \Rightarrow$
  $d(f^{-n}x, f^{-n}y) \leq e^{-n\chi} d(x, y)$
- $\chi < \lambda^u$ is arbitrary



SRB measures

$f : M \circlearrowleft$ a diffeo, U a trapping region: $\overline{f(U)} \subset U$

- Describe asymptotics of Lebesgue-typical trajectories
- Absolutely continuous invariant measure? May not exist
- Look for SRB measure: non-zero Lyapunov exponents and absolutely continuous on unstable manifolds

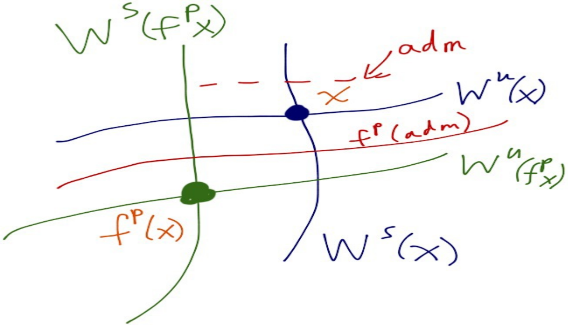
Need Hadamard–Perron theorem to define. How to find?

- $m =$ Lebesgue measure (volume) on some admissible manifold
- Cesàro averages $\mu_n = \frac{1}{n} \sum_{k=0}^{n-1} f_*^k m$, then $\mu_{n_j} \to \mu$ invariant

Is μ SRB? Yes if f is uniformly hyperbolic – continuous splitting $T_x M = E^u(x) \oplus E^s(x)$, uniform expansion/contraction

Closing lemma

Orbit segment $x, f(x), \ldots, f^p(x) \approx x$. Periodic point nearby?



f^p induces graph transform on space of u-admissible manifolds

- Contraction $\Rightarrow$ fixed point, similarly for s-admissibles
- Intersection is periodic point

Non-uniform hyperbolicity

Assumption: $f_n\colon \mathbb{R}^d \to \mathbb{R}^d$ is $C^{1+\alpha}$ with $f_n(0) = 0$

- $E_n^{s,u}$ invariant under $Df_n(0)$, **not** uniformly transverse
- $|Df_n(0)(v^s)| \leq e^{\lambda_n^s}|v^s|$ and $|Df_n(0)(v^u)| \geq e^{\lambda_n^u}|v^u|$
- $\overline{\lim}\, \frac{1}{n}\sum \lambda_k^s < 0 < \chi < \underline{\lim}\, \frac{1}{n}\sum \lambda_k^u$

Conclusion: There exists $W_n^u = \operatorname{graph}\psi_n$ tangent to E_n^u such that

- $\|D\psi_n(x^u)\| \leq \gamma$ for $|x^u| \leq r/C$
- $x, y \in W^u \Rightarrow d(f^{-n}x, f^{-n}y) \leq Ce^{-n\chi}d(x,y)$
- C depends on asymptotic behaviour of $\lambda_n^{s,u}$ and θ_n

Non-uniform set $\Lambda = \bigcup_C \Lambda_C$ is union of regular sets (Pesin sets)

- μ hyperbolic invariant $\Rightarrow \mu(\Lambda) = 1$
- Λ invariant, non-compact, Λ_C compact, non-invariant

SRB measures and closing lemmas

NUH lets us define SRB measures, but not find them

- Recall Cesàro averages μ_n – need to know how big the images of admissible manifolds are at $f^n(x)$, so need good recurrence properties to Λ_C
- Recurrence properties come from ergodic theory

Closing lemma for NUH as long as both $x, f^p(x) \in \Lambda_C$ and $d(f^p(x), x) < \varepsilon(C)$.

- Determining C requires an infinite amount of information – knowledge of entire trajectory

Effective hyperbolicity

$f_n \colon E^u_n \oplus E^s_n \to E^u_{n+1} \oplus E^s_{n+1}$ a $C^{1+\alpha}$ germ with $f_n(0) = 0$

- $|Df_n(0)(v^s)| \leq e^{\lambda^s_n}|v^s|$ and $|Df_n(0)(v^u)| \geq e^{\lambda^u_n}|v^u|$
- $\theta_n = \measuredangle(E^u_n, E^s_n)$, write $B(\theta) = \{n \mid \theta_n < \theta\}$

Splitting is dominated if $\lambda^s_n < \lambda^u_n$.

Defect from domination: $\Delta_n = \max(0, \frac{1}{\alpha}(\lambda^s_n - \lambda^u_n))$

Definition

$\{f_n \mid n \geq 0\}$ is effectively hyperbolic if

1. $\lim_{\theta \to 0} \overline{\delta}(B(\theta)) = 0$, and
2. $\underline{\lim}_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}(\lambda^u_k - \Delta_k) > 0$.

Return to large scale

Sequence of admissible manifolds $W_n = \operatorname{graph} \psi_n$, through 0 and tangent to E^u_n, with $f_n(W_n) \supset W_{n+1}$

- $\psi_n \colon B^u_n(r_n) \to E^s_n$, think of r_n as 'size' of admissible manifold
- $|D\psi_n|_\alpha \leq \kappa_n$, think of κ_n as 'curvature'
- $\kappa_n r_n^\alpha \leq \gamma \Rightarrow \|D\psi_n\| \leq \gamma$

Theorem (C.–Pesin)

If $\{f_n \mid n \geq 0\}$ is effectively hyperbolic, then there exists $r > 0, \kappa > 0$, and $\Gamma \subset \mathbb{N}$ such that

1. $\underline{\delta}(\Gamma) > 0$, *and*
2. *for every $n \in \Gamma$ we have $r_n \geq r$ and $\kappa_n \leq \kappa$.*

Γ is the set of effective hyperbolic times:

$$\sum_{k=m}^{n-1} \lambda^e_k \geq (n-m)\chi \text{ for all } 0 \leq m < n$$

Construction of SRB measures

f a $C^{1+\alpha}$ diffeo, U a trapping region, $X \subset U$ an invariant set with invariant cone families $K^{u,s}(x)$

- $S = \{x \in X \mid \text{forward trajectory of } x \text{ is effectively hyperbolic}\}$
  $\cap\{x \in X \mid K^s(x) \text{ has negative Lyapunov exponent}\}$

Theorem (C.–Dolgopyat–Pesin)

If $\text{Leb}(S) > 0$ *then* f *has an SRB measure.*

Explicit computation of constants

Consider finite orbit segment $\{f_n \mid 0 \leq n < p\}$

- $L = \max(|Df_n|_\alpha, \ |\log(\frac{\theta_{n+1}}{\theta_n})|, \ |\log(\frac{\|Df_n(0)(v)\|}{\|v\|})|)$
- $\lambda_n^e = \lambda_n^u - \Delta_n - L\mathbf{1}_{\{\theta_n < \theta\}}$
- $M_n^u = \max_{0 \leq m < n} \left((n-m)\chi^u - \sum_{k=m}^{n-1} \lambda_k^e \right)$, similarly M_n^s

Definition

Orbit segment is completely effectively hyperbolic with parameters $M, \theta > 0$ and rates $\chi^s < 0 < \chi^u$ if $\theta_0, \theta_p > \theta$ and

$$M \geq \max(M_p^u, M_p^s, M_0^u, M_0^s),$$

$$M \geq M_n^u + \sum_{k=0}^{n-1} (\lambda_k^s - \chi^s) \text{ for all } 0 \leq n \leq p,$$

and similarly for M_n^s.

Finite-information closing lemma

Theorem (C.–Pesin)

Fix parameters M, θ and rates $\chi^{s,u}$. Given $\delta > 0$ there is $\varepsilon > 0$ and $p_0 \in \mathbb{N}$ such that if

1. *$p \geq p_0$ and $\{x, \ldots, f^p(x)\}$ is completely effectively hyperbolic with these parameters and rates;*
2. *$d(x, f^p x) < \varepsilon$, and $E^\sigma \subset K^\sigma(x)$ have $d(Df^p(E^\sigma), E^\sigma) < \varepsilon$,*

then there exists a hyperbolic periodic point $z = f^p z$ such that $d(x, z) < \delta$.

Example – sheared Katok map

f an Axiom A surface diffeo, p a hyperbolic fixed point

- Near p the map f is time-1 for linear vector field $\dot{x} = Ax$
- Slow-down: $\dot{x} = Axr^\varepsilon$ where $r = d(x, p)$ (Katok example)
- Add shear term: if $A = \begin{pmatrix} \gamma & 0 \\ 0 & -\beta \end{pmatrix}$ then get ODEs

$$\dot{x} = \gamma r^\varepsilon x + y$$
$$\dot{y} = -\beta r^\varepsilon y$$

Parameters M for effective hyperbolicity can be computed directly from how much time orbit segment spends near p.

- SRB measure exists (takes some argument to show $\mathrm{Leb}(S) > 0$)
- Closing lemma applies based on time spent near shear

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