Hyperbolic behaviour in smooth ergodic theory

Vaughn Climenhaga

University of Houston

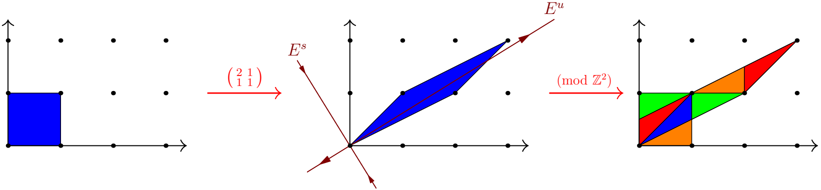
June 22, 2017

Invariant measures for Anosov diffeomorphisms

Let M be a compact manifold, $f\colon M \to M$ a $C^{1+\alpha}$ Anosov diffeo

$$T_xM = E^u_x \oplus E^s_x \text{ with } \|Df|_{E^s_x}\| \le \lambda < 1 \text{ and } \|Df|^{-1}_{E^u_x}\| \ge \lambda^{-1}$$

Example: $M = \mathbb{T}^d$ and f a hyperbolic automorphism from $SL(d,\mathbb{Z})$



$\mathcal{M}(f) = \{f\text{-inv. Borel prob. measures on } M\}$ is a (huge) simplex

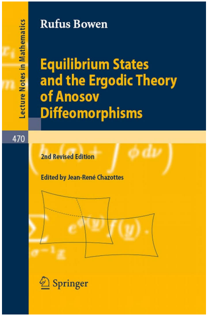
- Extreme points (ergodic meas.) are dense – *Poulsen simplex*
- Contains copies of every m.p.t. with $h_\mu(T) < h_{\rm top}(f)$

Thermodynamic formalism for Anosov diffeomorphisms

$\mathcal{M}(f)$ may have ‘distinguished’ measures

- Measure of maximal entropy (MME):
  $h_\mu(f) = h_{\text{top}}(f) = \sup_{\nu \in \mathcal{M}(f)} h_\nu(f)$
- Sinai–Ruelle–Bowen (SRB) measure:
  abs. cts. conditionals on unstable leaves
- Equilibrium state (ES) for $\varphi\colon M \to \mathbb{R}$:
  achieves $\sup_\nu \left(h_\nu(f) + \int \varphi \, d\mu\right) =: P(\varphi)$

Existence? Uniqueness? Properties?



Theorem (Sinai, Ruelle, Bowen; see Bowen’s 1975 monograph)

If $f\colon M \to M$ Anosov, then there is a unique ES μ_φ for every Hölder $\varphi\colon M \to \mathbb{R}$. Moreover, μ_φ is Bernoulli, has EDC, and CLT.

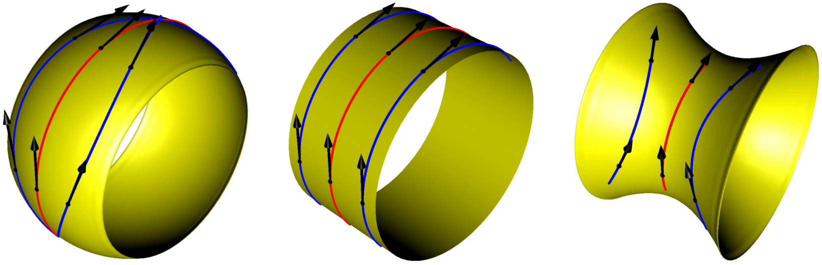
When $\varphi(x) = -\log|\det Df|_{E^u_x}|$, the ES μ_φ is the SRB measure

Geodesic flow, curvature, and convexity

Let M be a smooth Riemannian manifold

- Every $v \in T^1M$ has a unique geodesic $\gamma_v(t)$ with $\dot\gamma_v(0) = v$
- Geodesic flow $f_t \colon T^1M \to T^1M$ takes $v \mapsto \gamma_v(t)$

Consider distance between two nearby geodesics... (Jacobi fields)



Positive curvature	Zero curvature	Negative curvature
concave	**linear**	**convex**

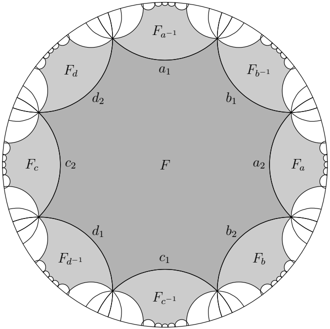
Negative curvature and hyperbolicity

For Anosov systems, distance between nearby trajectories is convex

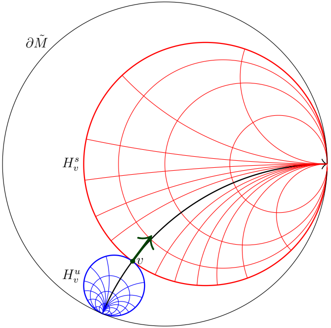
Theorem (Anosov 1967/69, *Proc. Steklov Inst. Math.*)

If M is compact and has negative curvature everywhere, then the geodesic flow $f_t \colon T^1M \to T^1M$ is Anosov.

1. Go to universal cover $\tilde{M}$



2. Get $E^{s,u}$ from horospheres



Decay of correlations

Each μ_φ for an Anosov diffeo has exponential decay of correlations:

$$\int \psi_1(x)\psi_2(f^n x)\, d\mu_\varphi(x) - \int \psi_1\, d\mu_\varphi \int \psi_2\, d\mu_\varphi \leq C(\psi_1, \psi_2)\theta^n$$

for some $\theta < 1$ as long as ψ_1, ψ_2 are both Hölder. Mechanism for this is exponential divergence of nearby trajectories.

Anosov flows have unique equilibrium states for Hölder potentials

- EDC is harder: no expansion or contraction in flow direction

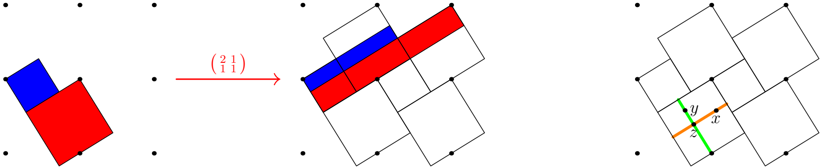
Theorem (Dolgopyat 1998 and Liverani 2004, *Annals of Math.*)

For geodesic flow in negative curvature, every μ_φ has EDC.

Uses the **contact structure** of geodesic flows.

Approach I: Markov partitions

First proofs (cf. Bowen's monograph) came via **Markov partitions**



f an Anosov diffeo $\Rightarrow \exists$ topological Markov chain Σ on a finite alphabet and a semi-conjugacy from (Σ, σ) to (M, f)

1. Problems for f can be translated into problems about Σ
2. Go from Σ to one-sided shift Σ^+ (quotient along stables)
3. **Transfer operator** $\mathcal{L}\colon C^\alpha(\Sigma^+) \to C^\alpha(\Sigma^+)$ has a spectral gap

Approach II: Anisotropic Banach spaces

An alternate approach is to replace $C^\alpha(\Sigma^+)$ with a Banach space $\mathcal{B}$ whose definition does not require a Markov partition. (Liverani used this to extend Dolgopyat's result to higher dimensions)

Simpler case: expanding dynamics (no contracting directions)

- Direct functional analytic approach goes back at least to Lasota and Yorke, 1973, *TAMS*
- "Expansion makes the functions smoother"

The contracting direction complicates matters

- Elements of $\mathcal{B}$ should behave like smooth/Hölder functions along unstable leaves, and like measures along stable leaves
- The appropriate anisotropic Banach spaces were introduced by Blank, Keller, and Liverani, 2002, *Nonlinearity*
- Further results by Baladi, Gouëzel, Tsujii

Approach III: Specification

Some Systems with Unique Equilibrium States

by

RUFUS BOWEN*

Department of Mathematics
University of California
Berkeley, California 94720

We shall be dealing with a homeomorphism $f: X \to X$ of a compact metric space and a continuous $\varphi: X \to R$. Let $M_f(X)$ denote the set of all f-invariant Borel probability measures on X. $\mu \in M_f(X)$ is called an *equilibrium state* (for f and φ) if

$$h_\mu(f)+\mu(\varphi) = \sup_{\nu \in M_f(X)} (h_\nu(f)+\nu(\varphi)),$$

where $h_\mu(f)$ is the entropy of μ. We want conditions on f and φ which guarantee a unique equilibrium state.

f is called *expansive* if there is an $\epsilon > 0$ such that for any two points $x \neq y$ in X there is an $n \in Z$ so that $d(f^n(x), f^n(y)) > \epsilon$. f satisfies *specification* if for each $\delta > 0$ there is an integer $p(\delta)$ for which the following is true: if $I_1, \cdots, I_n$ are intervals of integers contained in $[a, b]$ with $d(I_i, I_j) \geq p(\delta)$ for $i \neq j$ and $x_1, \cdots, x_n \in X$, then there is a point $x \in X$ with $f^{b-a+p(\delta)}(x) = x$ and $d(f^k(x), f^k(x_i)) < \delta$ for $k \in I_i$. This condition allows us to construct a lot of periodic points.

For $\varphi \in C(X)$ and $n \geq 1$ let

$$(S_n\varphi)(x) = \varphi(x)+\varphi(f(x))+\cdots+\varphi(f^{n-1}(x)).$$

Let $V(f)$ be the set of $\varphi \in C(X)$ for which an $\epsilon > 0$ and a K exist for which the following is true: $d(f^k(x), f^k(y)) \leq \epsilon$ for all $0 \leq k < n \Rightarrow |S_n\varphi(x) - S_n\varphi(y)| \leq K$.

THEOREM. *Let f: $X \to X$ be an expansive homeomorphism of a compact metric space satisfying specification. Then each $\varphi \in V(f)$ has a unique equilibrium state μ_φ.*

Remark. Let δ be any expansive constant for f. Then, if $\varphi \in V(f)$, $|\varphi|_f = \sup \{|S_n\varphi(x) - S_n\varphi(y)|: n \geq 1 \text{ and } d(f^k(x), f^k(y)) \leq \delta \forall k \in [0, n)\}$ is finite (if ϵ, K are as in the definition of $\varphi \in V(f)$ and $d(x, y) \leq \epsilon$ follows from $d(f^j(x), f^j(y)) \leq \delta$ for all $|j| \leq N$ (there is such an N by expansiveness), then $|\varphi|_f \leq K+2N\|\varphi\|$).

* Partially supported by NSF grant GP-14519.

MATHEMATICAL SYSTEMS THEORY, Vol. 8, No. 3.

If f is transitive, then any sequence of finite-length orbit segments can be shadowed by a single orbit.

f has **specification** if the transition time is bounded, independently of the lengths of the orbits to be shadowed.

Anosov $\Rightarrow$ expansive + specification

Theorem (Bowen 1974/75)

Exp. + spec. $\Rightarrow$ unique ES for every φ with bounded distortion

- Nothing about correlation decay -

Thermodynamics in nonuniform hyperbolicity

Anosov condition is very restrictive. **Nonuniform hyperbolicity** is more widespread: exponential contraction and expansion happen for some/most orbits (nonzero Lyapunov exponents), but we may have to wait arbitrarily long to see them.

For non-Anosov systems, even with some hyperbolicity, it is possible for existence, uniqueness, and/or EDC to fail.

- Can have phase transitions, multiple equilibrium states
- Rates of correlation decay can be subexponential

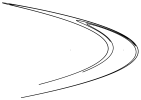
Instead of a single theorem that is universally valid, focus on **techniques** and on the **examples** that they handle

Four examples (all images from Wikipedia)

Hénon map

$a = 1.4,\ b = .3$

(Delayed logistic)



$f(x, y) = (1 - ax^2 + y, bx)$

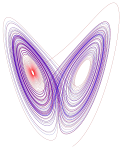
Lorenz flow

$\dot{x} = \sigma(y - x)$

$\dot{y} = x(\rho - z) - y$

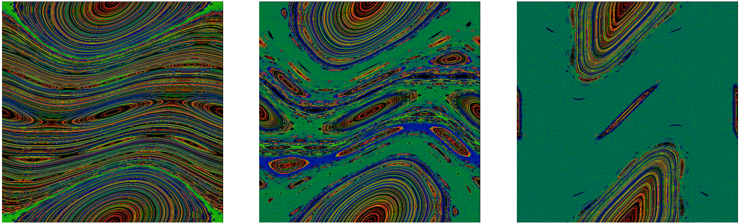
$\dot{z} = xy - \beta z$

$\sigma = 10,\ \beta = 8/3,\ \rho = 28$



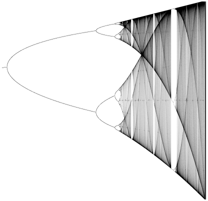
Standard map (Chirikov–Taylor) $K = .6,\ .97,\ 2$

$(p, x) \mapsto (\bar{p}, \bar{x})$ where $\bar{p} = p + K \sin x$ and $\bar{x} = x + \bar{p}$



Geodesic flow over a manifold with nonpositive curvature

Questions: Hénon map



Start with logistic map $f(x) = 1 - ax^2$.
Does it have an SRB measure?

- YES for $a \in A$ with $\mathrm{Leb}(A) > 0$:
  Jakobson, 1981, *Comm. Math. Phys.*

For Hénon: for sufficiently small $b > 0$ $\exists A$ s.t. $\mathrm{Leb}(A) > 0$ and for every $a \in A$, there is an SRB measure with EDC.

- Benedicks & Young (1993 *Invent.*, 2000 *Astérisque*)
- Young (1998 *Annals*): develops **towers** (ctbl state symbolics)

Equilibrium states? Some results by Berger, Senti, Takahasi, but for the most part still an open problem

Remark: no rigorous results for classical parameter values

- Galias and Tucker (2015, *Chaos*) found sinks for nearby a, b

Questions: Lorenz flow

Theorem (Tucker, 1999, *C. R. Acad. Sci. Paris Sér. I Math.*)

The Lorenz flow has a unique SRB measure.

Tucker's proof involves rigorous numerics and normal form theory

Theorem (Araújo, Melbourne, 2016, *Ann. Henri Poincaré*)

The Lorenz SRB measure has exponential decay of correlations.

EDC uses Young towers, careful estimates on foliation regularity

Question: what about other equilibrium states for Lorenz?

Questions: Standard map

Lebesgue measure is preserved for the standard map, so existence of an SRB is no problem... but is it hyperbolic? ergodic?

Numerical evidence suggests that for large enough K, the set X of points with nonzero Lyapunov exponents has positive Lebesgue measure. No rigorous proof available. Closest so far:

- Gorodetski 2012, *Comm. Math. Phys.*: $\dim_H(X) = 2$
- Blumenthal, Xue, Young, 2017, *Annals*: nonzero Lyapunov exponents for random perturbations of the standard map

Questions: Geodesic flows in nonpositive curvature

Let M be a smooth Riemannian manifold with nonpositive curvature, then $T^1M = \text{Reg} \cup \text{Sing}$, where (roughly speaking)

- all Lyapunov exponents are nonzero on Reg;
- on Sing there is always a zero exponent.

M has rank 1 if $\text{Reg} \neq \emptyset$. (In dim 2, this means genus ≥ 2.)
Geodesic flow preserves Liouville measure m, so existence of SRB is no problem. **Question:** Is it ergodic?

- Ergodicity was claimed several times in the 1980s, but these were based on an incorrect proof that $m(\text{Sing}) = 0$.

Theorem (Knieper, 1998, *Annals*)

A geodesic flow in rank 1 has a unique MME.

Burns, C., Fisher, Thompson: unique ES if $P(\text{Sing}, \varphi) < P(\varphi)$

- **Question:** Do these unique eq. states have EDC?

The big picture: techniques

Techniques from unif. hyp. must be replaced by more general ones

1. Finite Markov partitions for Anosov $\rightsquigarrow$ Young towers or countable Markov partitions for NUH
   - Sarig, 2013, *JAMS*: nonzero Lyap. exp. $\Rightarrow$ ctbl Markov part.
2. Anisotropic Banach spaces for Anosov $\rightsquigarrow$??????
   (May need this to deduce EDC for rank 1 eq. states)
3. Specification for Anosov $\rightsquigarrow$ non-uniform specification (C., Thompson, 2016, *Adv. Math.*), main tool for rank 1 result

General (vague) question: to what extent are these approaches interchangeable? Must the equilibrium states from non-uniform specification admit towers and have EDC? Can Sarig's Markov partitions be used for results on uniqueness and correlation decay?