SRB measures for non-uniformly hyperbolic systems

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Joint work with Dmitry Dolgopyat and Yakov Pesin
1. Introduction and classical results
   - Definition of SRB measure
   - Some known results

2. General method to build an SRB measure
   - Decomposing the space of invariant measures
   - Recurrence to compact sets

3. A non-uniform Hadamard–Perron theorem
   - Sequences of local diffeomorphisms
   - Frequency of large admissible manifolds

4. Sufficient conditions for existence of an SRB measure
   - Usable hyperbolicity
   - Existence of an SRB measure
Physically meaningful invariant measures

- $M$ a compact Riemannian manifold
- $f : M \to M$ a $C^{1+\varepsilon}$ local diffeomorphism
- $\mathcal{M}$ the space of Borel measures on $M$
- $\mathcal{M}(f) = \{ \mu \in \mathcal{M} \mid \mu \text{ is } f\text{-invariant} \}$
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**Birkhoff ergodic theorem.** If $\mu \in \mathcal{M}(f)$ is ergodic then it describes the statistics of $\mu$-a.e. trajectory of $f$: for every integrable $\varphi$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)) = \int \varphi \, d\mu$$
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To be “physically meaningful”, a measure should describe the statistics of *Lebesgue*-a.e. trajectory.
SRB measures

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"absolutely continuous" $\implies$ “a.c. on unstable manifolds”

$\mu \in \mathcal{M}(f)$ is an SRB measure if
- all Lyapunov exponents non-zero;
- $\mu$ has a.c. conditional measures on unstable manifolds.

SRB measures are physically meaningful. **Goal:** Prove existence of an SRB measure.
SRB measures are known to exist in the following settings.

- Uniformly hyperbolic $f$ (Sinai, Ruelle, Bowen)
- Partially hyperbolic $f$ with positive/negative central exponents (Alves–Bonatti–Viana, Burns–Dolgopyat–Pesin–Pollicott)

Key tool is a dominated splitting $T_x M = E^s(x) \oplus E^u(x)$.

1. $E^s, E^u$ depend continuously on $x$.
2. $\angle(E^s, E^u)$ is bounded away from 0.

Both conditions fail for non-uniformly hyperbolic $f$. 
Non-uniformly hyperbolic maps

The Hénon maps $f_{a,b}(x, y) = (y + 1 - ax^2, bx)$ are a perturbation of the family of logistic maps $g_a(x) = 1 - ax^2$.

1. $g_a$ has an absolutely continuous invariant measure for “many” values of $a$. (Jakobson)

2. For $b$ small, $f_{a,b}$ has an SRB measure for “many” values of $a$. (Benedicks–Carleson, Benedicks–Young)

3. Similar results for “rank one attractors” – small perturbations of one-dimensional maps with non-recurrent critical points. (Wang–Young)

Genuine non-uniform hyperbolicity, but only one unstable direction, and stable direction must be strongly contracting.
Constructing invariant measures

- $f$ acts on $\mathcal{M}$ by $f_* : m \mapsto m \circ f^{-1}$.
- Fixed points of $f_*$ are invariant measures.
- Césaro averages + weak* compactness $\Rightarrow$ invariant measures:
  $$\mu_n = \frac{1}{n} \sum_{k=0}^{n-1} f_*^k m$$  
  $$\mu_n \to \mu \in \mathcal{M}(f)$$

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  Idea: $m = \text{ volume } \Rightarrow \mu$ is an SRB measure.

  \[ H = \{ x \in M \mid \text{ all Lyapunov exponents non-zero at } x \} \]
  \[ S = \{ \nu \in \mathcal{M} \mid \nu(H) = 1, \nu \text{ a.c. on unstable manifolds} \} \]

- $S \cap \mathcal{M}(f) = \{ \text{SRB measures} \}$
- $S$ is $f_*$-invariant, so $\mu_n \in S$ for all $n$.
- $S$ is not compact. So why should $\mu$ be in $S$?
Non-uniform hyperbolicity in $\mathcal{M}$

Theme in NUH: choose between invariance and compactness. Replace unstable manifolds with $n$-admissible manifolds $V$:

$$d(f^{-k}(x), f^{-k}(y)) \leq C e^{-\lambda k} d(x, y)$$

for all $0 \leq k \leq n$ and $x, y \in V$. 
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\( S_n = \{ \nu \text{ supp. on and a.c. on } n \text{-admissible manifolds, } \nu(H) = 1 \} \).

This set of measures has various non-uniformities.

1. Value of \( C, \lambda \) in definition of \( n \)-admissibility.
2. Size and curvature of admissible manifolds.
3. \( \| \rho \| \) and \( \| 1/\rho \| \), where \( \rho \) is density wrt. leaf volume.
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3. $\|\rho\|$ and $\|1/\rho\|$, where $\rho$ is density wrt. leaf volume.

Given $K > 0$, let $S_n(K)$ be the set of measures for which these non-uniformities are all controlled by $K$.

large $K \Rightarrow$ worse non-uniformity

$S_n(K)$ is compact, but not $f_\ast$-invariant.
Conditions for existence of an SRB measure

- $M$ be a compact Riemannian manifold, $U \subset M$ open, $f : U \to M$ a local diffeomorphism with $f(U) \subset U$.
- Let $\mu_n$ be a sequence of measures whose limit measures are all invariant.
- Fix $K > 0$, write $\mu_n = \nu_n + \zeta_n$, where $\nu_n \in S_n(K)$.

Theorem (C.–Dolgopyat–Pesin, 2010)

If $\lim_{n \to \infty} \|\nu_n\| > 0$, then some limit measure of $\{\mu_n\}$ has an ergodic component that is an SRB measure for $f$. 
Conditions for existence of an SRB measure

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The question now becomes: How do we obtain recurrence to the set $S_n(K)$?
Coordinates in $TM$

We use local coordinates to write the map $f$ along a trajectory as a sequence of local diffeomorphisms.

- $\{f^n(x) \mid n \geq 0\}$ is a trajectory of $f$
- $U_n \subset T_{f^n(x)}M$ is a neighbourhood of 0 small enough so that the exponential map $\exp_{f^n(x)} : U_n \rightarrow M$ is injective
- $f_n : U_n \rightarrow \mathbb{R}^d = T_{f^{n+1}(x)}M$ is the map $f$ in local coordinates
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Suppose $\mathbb{R}^d = T_{f^n(x)}M$ has an invariant decomposition $E^u_n \oplus E^s_n$ with asymptotic expansion (contraction) along $E^u_n$ ($E^s_n$).

\[
Df_n(0) = A_n \oplus B_n
\]

\[
f_n = Df_n(0) + s_n
\]

\[
f_n(v, w) = (A_n v + g_n(v, w), B_n w + h_n(v, w))
\]
Controlling hyperbolicity and regularity

\[ \mathbb{R}^d = E_n^u \oplus E_n^s \quad f_n = (A_n \oplus B_n) + s_n \]

Start with an admissible manifold \( V_0 \) tangent to \( E_0^u \) at 0, push it forward and define an invariant sequence of admissible manifolds by \( V_{n+1} = f_n(V_n) \).

\[ V_n = \text{graph } \psi_n = \{ v + \psi_n(v) \} \quad \psi_n : B(E_n^u, r_n) \to E_n^s \]

Need to control the size \( r_n \) and the regularity \( ||D\psi_n||, |\psi_n|_\epsilon \).
Controlling hyperbolicity and regularity

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Consider the following quantities:

\[ \lambda^u_n = \log(\| A_n^{-1} \|^{-1}) \quad \lambda^s_n = \log \| B_n \| \]
\[ \alpha_n = \angle(E^u_n, E^s_n) \quad C_n = \| s_n \|_{C^{1+\varepsilon}} \]
Classical Hadamard–Perron results

Uniform case: Constants such that

- \( \lambda^s_n \leq \bar{\lambda}^s < 0 < \bar{\lambda}^u < \lambda^u_n \)
- \( \alpha_n \geq \bar{\alpha} > 0 \)
- \( C_n \leq \bar{C} < \infty \)

Then \( V_n \) has uniformly large size: \( r_n \geq \bar{r} > 0 \).
**Classical Hadamard–Perron results**

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**Non-uniform case:** $\lambda_n^s, \lambda_n^u, \alpha_n$ still uniform, but $C_n$ not.

$$C_n \text{ grows slowly } \Rightarrow r_n \text{ decays slowly}$$
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We want to consider the case where
- $\lambda_n^s < 0 < \lambda_n^u$ may fail (may even have $\lambda_n^u < \lambda_n^s$)
- $\alpha_n$ may become arbitrarily small
- $C_n$ may become arbitrarily large (no control on speed)
Usable hyperbolicity

In order to define $\psi_{n+1}$ implicitly, we need control of the regularity of $\psi_n$. Control $\|D\psi_n\|$ and $|D\psi_n|_\varepsilon$ by decreasing $r_n$ if necessary. So how do we guarantee that $r_n$ becomes “large” again?
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$$\beta_n = C_n (\sin \alpha_{n+1})^{-1}$$

Fix a threshold value $\bar{\beta}$ and define the usable hyperbolicity:

$$\lambda_n = \begin{cases} 
\min \left( \lambda_n^u, \lambda_n^u + \frac{1}{\varepsilon} (\lambda_n^u - \lambda_n^s) \right) & \text{if } \beta_n \leq \bar{\beta}, \\
\min \left( \lambda_n^u, \lambda_n^u + \frac{1}{\varepsilon} (\lambda_n^u - \lambda_n^s), \frac{1}{\varepsilon} \log \frac{\beta_n}{\beta_{n+1}} \right) & \text{if } \beta_n > \bar{\beta}.
\end{cases}$$
A Hadamard–Perron theorem

Write $F_n = f_{n-1} \circ \cdots \circ f_1 \circ f_0 : U_0 \to \mathbb{R}^d = T_{f^n(x)} M$. Let $V_0 \subset \mathbb{R}^d$ be a $C^{1+\varepsilon}$ manifold tangent to $E^u_0$ at 0, and let $V_n(r)$ be the connected component of $F_n(V_0) \cap (B(E^u_n, r) \times E^s_n)$ containing 0.

Theorem (C.–Dolgopyat–Pesin, 2010)

Suppose $\bar{\beta}$ and $\bar{\chi} > 0$ are such that $\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \lambda_k > \bar{\chi} > 0$. Then there exist constants $\tilde{\gamma}, \tilde{\kappa}, \bar{r} > 0$ and a set $\Gamma \subset \mathbb{N}$ with positive lower asymptotic frequency such that for every $n \in \Gamma$,

1. $V_n(\bar{r})$ is the graph of a $C^{1+\varepsilon}$ function $\psi_n : B_{E^u_n}(\bar{r}) \to E^s_n$ satisfying $\|D\psi_n\| \leq \tilde{\gamma}$ and $|D\psi_n|_{\varepsilon} \leq \tilde{\kappa}$;

2. if $F_n(x), F_n(y) \in V_n(\bar{r})$, then for every $0 \leq k \leq n$, $\|F_n(x) - F_n(y)\| \geq e^{k\bar{\chi}} \|F_{n-k}(x) - F_{n-k}(y)\|$. 
Cone families

Return to a local diffeomorphism $f: U \to M$. Given $x \in M$, a subspace $E \subset T_x M$, and an angle $\theta$, we have a cone

$$K(x, E, \theta) = \{ v \in T_x M \mid \angle(v, E) < \theta \}.$$  

If $E, \theta$ depend measurably on $x$, this defines a measurable cone family. Suppose $A \subset U$ has positive Lebesgue measure, is forward invariant, and has two measurable cone families $K^s(x), K^u(x)$ s.t.

1. $Df(K^u(x)) \subset K^u(f(x))$ for all $x \in A$
2. $Df^{-1}(K^s(f(x))) \subset K^s(x)$ for all $x \in f(A)$
3. $T_x M = E^s(x) \oplus E^u(x)$

SRB measures for non-uniformly hyperbolic systems
Usable hyperbolicity (again)

Define \( \lambda^u, \lambda^s : A \to \mathbb{R} \) by

\[
\lambda^u(x) = \inf \{ \log \| Df(v) \| \mid v \in K^u(x), \| v \| = 1 \}, \\
\lambda^s(x) = \sup \{ \log \| Df(v) \| \mid v \in K^s(x), \| v \| = 1 \}.
\]

Let \( \alpha(x) \) be the angle between the boundaries of \( K^s(x) \) and \( K^u(x) \). Fix \( \bar{\alpha} > 0 \) and consider the quantities

\[
\zeta(x) = \begin{cases} 
\frac{1}{\varepsilon} \log \frac{\alpha(f(x))}{\alpha(x)} & \text{if } \alpha(x) < \bar{\alpha}, \\
+\infty & \text{if } \alpha(x) \geq \bar{\alpha}.
\end{cases}
\]

\[
\lambda(x) = \min \left\{ \lambda^u(x), \lambda^u(x) + \frac{1}{\varepsilon}(\lambda^u(x) - \lambda^s(x)), \zeta(x) \right\}
\]
An existence result

Consider points with positive asymptotic usable hyperbolicity:

\[ S = \left\{ x \in A \mid \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \lambda(f^k(x)) > 0 \text{ and } \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \lambda^s(f^k(x)) < 0 \right\}. \]
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**Theorem (C.–Dolgopyat–Pesin, 2010)**

*If there exists \( \bar{\alpha} > 0 \) such that \( \text{Leb} \ S > 0 \), then \( f \) has a hyperbolic SRB measure supported on \( \Lambda \).*
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**Theorem (C.–Dolgopyat–Pesin, 2010)**

*Fix \( x \in U \). Let \( V(x) \) be an embedded submanifold such that \( T_x V(x) \subset K^u(x) \), and let \( m_V \) be leaf volume on \( V(x) \). Suppose that there exists \( \bar{\alpha} > 0 \) such that \( \lim_{r \to 0} m_V(S \cap B(x, r)) > 0 \). Then \( f \) has a hyperbolic SRB measure supported on \( \Lambda \).*