1 Natural relationship between smooth dynamics & symbolic dynamics

M a cpt Riem mfd, $f : M \circlearrowleft$ diffeo

Given $\mathcal{R} = \{R_a \mid a \in A\}$, $R_a \subset M$, relation between $A^{\mathbb{Z}}$ & M:

$$\underline{u} \in A^{\mathbb{Z}}, \quad \pi(\underline{u}) = \bigcap_{n \in \mathbb{Z}} f^{-n}(R_{u_n}) = \{x \in M \mid f^n(x) \in R_{u_n} \ \forall n \in \mathbb{Z}\}$$

If each R_a is closed then $\Sigma = \{\underline{u} \mid \pi(\underline{u}) \neq \emptyset\}$ is closed, σ-inv.

If $\pi(\underline{u})$ is a singleton $\forall \underline{u} \in \Sigma$ then we abuse notation and write $\pi : \Sigma \to M$. Easy to see that

$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma} & \Sigma \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{f} & M \end{array}$$

$\boxed{\text{NB}}$ π is onto $\Leftrightarrow \bigcup_{a \in A} R_a = M$

π is 1-1 $\Leftrightarrow R_a$ are pairwise disjoint

$\boxed{\text{GOAL}}$ Use results from symbolic dynamics to say useful things about (M, f): equilibrium states, statistical properties, etc.

This is particularly powerful if Σ is Markov. (paths on labelled graph)

Guiding principle:

Markov = nearest neighbour condition on symbolic.

nearest neighbour on smooth = pseudo-orbit

- $\{x_n\} \subset M$ is a ε-p.o. if $d(f(x_n), x_{n+1}) < \varepsilon \ \forall n \in \mathbb{Z}$.

2 Uniform hyperbolicity

$\boxed{\text{I}}$ Anosov / Axiom A $\Rightarrow$ $\boxed{\text{II}}$ local product structure $\Rightarrow$ $\boxed{\text{III}}$ shadowing lemma

$\boxed{\text{I}}$ $TM = E^s \oplus E^u$, $\|Df|_{E^s}\| \le \lambda < 1$, $\|Df^{-1}|_{E^u}\| \le \lambda < 1$

$\boxed{\text{II}}$ $\Delta_\varepsilon = \{(x,y) \mid d(x,y) \le \varepsilon\}$ $\quad [\cdot,\cdot] : \Delta_\varepsilon \to M$

$\{[x,y]\} = W^u(x) \cap W^s(y)$ (Smale bracket)

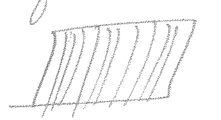
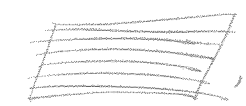
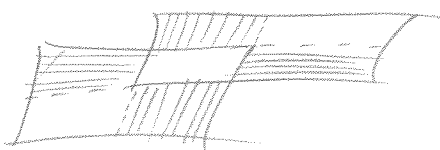
$\boxed{\text{III}}$ $\forall\ \delta>0\ \exists\ \varepsilon>0$ s.t. every ε-p.o. is δ-shadowed by a (unique) ~~true~~ orbit: $\exists!\ x\in M$ s.t. $d(f^n x, x_n)<\delta\ \forall\ n$.

Bowen's construction of a Markov cover $\mathcal{R}$:

- Let $A\subset M$ be (finite) & ε-dense, define $\Sigma=\{\underline{u}\in A^{\mathbb{Z}}\mid (u_n)$ is an ε-p.o.$\}$
- $\pi:\Sigma\to M$ given by shadowing lemma
- $[a]_0=\{\underline{u}\in\Sigma\mid u_0=a\}$, $R_a=\pi[a]_0$
- π is onto ($\mathcal{R}$ covers M) because A is ε-dense

$\boxed{\text{PROBLEM}}$ π is generally ∞-to-1, which is bad news for studying f-inv. meas. on M using σ-inv. meas. on Σ.

Sinai: construct Markov partition from Markov cover:

- Each element of cover is a rectangle: partitioned into W^s  & W^u , any 2 intersect uniquely
- Two rectangles intersect as $\longrightarrow$ 

Take all intersections: this gives a refinement into a Markov partition $\mathcal{R}$:
  - Each $R\in\mathcal{R}$ is closed, has small diameter, and $(x,y\in R \Rightarrow [x,y]\in R)$ ($\Leftrightarrow$ rectangle)
  - If $x\in R_i$, $fx\in R_j$, then $fW^u(x,R_i)\supset W^u(fx,R_j)$ and $fW^s(x,R_i)\subset W^s(fx,R_j)$.

Then π is 1-1 on a residual set, and bdd-to-1 everywhere.

CONCLUSIONS

- $\forall\ \mu\in\mathcal{M}_f(M)\ \exists\ \nu\in\mathcal{M}_\sigma(\Sigma)$ s.t. $\pi_*\nu=\mu$, $h_\nu(\sigma)=h_\mu(f)$
  $\Rightarrow$ If f top trans then $\exists!$ MME, else only finitely many ergodic
- $\#\{x\in M\mid f^n x=x\}\asymp\#\{\underline{u}\in\Sigma\mid f^n\underline{u}=\underline{u}\}\asymp e^{n h_{top}(f)}$
  (when min period $\mid n$)

3 Positive entropy surface diffeos

Drop the Axiom A assumption, but work in $\dim M = 2$ and let f be $C^{1+\beta}$, $\beta > 0$. Then any $\mu \in \mathcal{M}_f$ with $h_\mu(f) > 0$ is hyperbolic, so Pesin theory applies. In particular if $h_{top}(f) > 0$ the following conjectures were made:

$\boxed{\text{CONJ}}$ (Katok, Buzzi) If f has an MME then $\exists\, p \in \mathbb{N}$ s.t.

$$\lim_{\substack{n\to\infty \\ p \mid n}} \frac{\#\{f^n x = x\}}{e^{n h_{top}(f)}} > 0$$

$\boxed{Q}$ finite if C^∞?

$\boxed{\text{CONJ}}$ (Buzzi) f has at most countably many ergodic MMEs

* Both are true – proved by Sarig using ctbl Markov partitions, mimicking Bowen's construction w/ Pesin theory.

Original construction fails because no uniform local product structure or shadowing lemma. Pesin theory gives <u>non-uniform</u> versions of these, requiring a countable Markov partition.

$\boxed{\text{THEOREM}}$ (Sarig) $\forall\ 0 < \chi < h_{top}(f)$ $\exists$ locally cpt top Markov shift Σ_χ & Hölder $\pi : \Sigma_\chi \to M$ s.t.

1. $\pi \circ \sigma = f \circ \pi$ $\quad$ ($\Sigma_\chi^\# = \{\underline{u} \in \Sigma_\chi \mid \underline{u}$ contains $v, w \in \mathcal{V}$ i.o. $\to$ & $\leftarrow$, resp.$\}$)
2. $\pi(\Sigma_\chi^\#)$ is χ-large $\quad$ ($\mu = 1$ $\forall\ \mu \in \mathcal{M}_f^e$, $h_\mu > \chi$)
3. $x \in \pi(\Sigma_\chi^\#) \Rightarrow |\pi^{-1} x| < \infty$. In fact, $\exists\ \varphi : \mathcal{V} \times \mathcal{V} \to \mathbb{N}$ s.t. if $x = \pi(\underline{u})$, $u_i = v$ i.o. $(i \geq 0)$ & w i.o. $(i \leq 0) \Rightarrow |\pi^{-1} x| \leq \varphi(v, w)$
4. $\forall\ \mu \in \mathcal{M}_f(M)$, $h_\mu(f) > \chi$, $\exists\ \hat{\mu} \in \mathcal{M}_\sigma^e(\Sigma_\chi)$ s.t. $\mu = \pi_* \hat{\mu}$, $h_{\hat{\mu}} = h_\mu$.

<u>Rough idea of adaptations needed</u>

For Axiom A, get local product structure & hyperbolic behaviour on an ε-nbhd of every $x \in M$.

For general $f : M \circlearrowleft$, $C^{1+\beta}$, get $Q : M \to [0, \varepsilon]$ s.t. LPS & hyp hold on $Q(x)$-nbhd of x.

- $\{x \mid Q(x) = 0\}$ has entropy 0
- $h_\mu(f) > 0 \Rightarrow \mu\{x \mid Q(x) > 0\} = 1$
  (in fact Q depends on $\chi > 0$ & we need $h_\mu > \chi$)
- $[x, y]$ makes sense if x, y and $Q(x), Q(y)$ are close.
- $Q(f^n x)$ varies with n, but for a.e. orbit it can be chosen to vary slowly ($e^{\pm\varepsilon}$)
- Q is continuous on $\{x \mid Q > \eta\}$ $\forall\, \eta > 0$, but these should be thought of as Cantor sets (no interior) and in particular Q is not at all continuous on any nbhd in M.
- Markov partition will capture $\{Q > 0\}$:

$$\pi(\Sigma_\chi) = \{x \in M \mid Q(x) > 0\} \quad \text{(morally)}$$

KEY FEATURES OF PROOF

Both location and asymptotic data are close

- Replace ε-pseudo-orbits with $\underline{\varepsilon\text{-chains}}$ to allow use of non-uniform shadowing lemma.
- This gives a countable Markov cover, but Sinai's refinement method relied on finiteness (otherwise could get  $\to \dots \to$ pts) so need to show <u>locally finite</u>.

$\boxed{\text{NUH}}$

<u>Ruelle + Oseledets</u>

Let $NUH_\chi(f)$ = set of $x \in M$ s.t. $T_xM = E^s(x) \oplus E^u(x)$ with

① $\lim\limits_{n\to\pm\infty} \frac{1}{n} \log \|Df_x^n(v^s)\| < -\chi$

② $\lim\limits_{n\to\pm\infty} \frac{1}{n} \log \|Df_x^n(v^u)\| > \chi$

(and limits exist)

③ $\lim\limits_{n\to\pm\infty} \frac{1}{n} \log \measuredangle(Df_x^n E^s, Df_x^n E^u) = 0$

Then $\mu(NUH_\chi(f)) = 1 \quad \forall\, \mu \in \mathcal{M}_f^e,\ h_\mu(f) > \chi$

<u>Pesin Theory</u>

- $s_\chi(x) = \left(\sum\limits_{k\geq 0} e^{2k\chi} \|Df_x^k v^s\|^2\right)^{\frac{1}{2}} \qquad v^s \in E^s,\ \|v^s\| = 1$

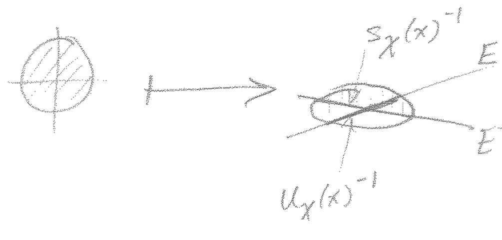
  $u_\chi(x) = \left(\sum\limits_{k\geq 0} e^{2k\chi} \|Df_x^{-k} v^u\|^2\right)^{\frac{1}{2}} \qquad v^u \in E^u,\ \|v^u\| = 1$

- Pesin charts

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "C_\chi(x)"] \arrow[d, "f_\chi"'] \arrow[rr, bend right, "\Psi_x"'] & T_xM \arrow[r, "\exp_x"] & M \arrow[d, "f"] \\
\mathbb{R}^2 \arrow[r, "C_\chi(fx)"] \arrow[rr, bend right, "\Psi_{fx}"'] & T_{f(x)}M \arrow[r, "\exp_{fx}"] & M
\end{tikzcd}$$

Lyapunov charts

$$\mathbb{R}^2 \xrightarrow{C_\chi(x)} T_xM$$



$$\|C_\chi(x)^{-1}\| = \frac{\sqrt{s_\chi(x)^2 + u_\chi(x)^2}}{|\sin \measuredangle(E^s(x), E^u(x))|}$$

Fix $\varepsilon > 0$ small, let

$$Q_\chi(x) = \varepsilon \|C_\chi(x)^{-1}\|^{-1} = \varepsilon \frac{|\sin \measuredangle|}{\sqrt{s^2+u^2}}$$

On ball of radius $Q(x)$, f_x is well-defined with

$$\begin{cases} f_x(0) = 0, \quad Df_x(0) = \begin{bmatrix} A(x) & \\ & B(x) \end{bmatrix} & \begin{array}{l} |A(x)| < e^{-\chi} \\ |B(x)| > e^{\chi} \end{array} \\ \|f_x - Df_x(0)\|_{C^{1+\beta/2}} < \varepsilon \end{cases}$$

NON-UNIFORM PRODUCT STRUCTURE:

$[x,y]$ makes sense if x,y are close relative to $Q(x), Q(y)$

Rough idea now is to replace M with $M \times (0,\varepsilon]$, x with (x,q), where $q < Q(x)$, and proceed as before.

- Destroys chance of $R = \overline{\operatorname{int} R}$ for elements of Markov partition since small Q are ubiquitous.
- Coarse-grain by requiring $q = e^{-\ell\varepsilon}$, $\ell \in \mathbb{N}$.
- To get local finiteness, will need both $fx \sim x'$ and $q \sim q'$ for $(x,q) \sim (x', q')$

Symbols are Ψ_x^η = Pesin chart of size η

- $\Psi_{x_1}^{\eta_1}$ overlaps $\Psi_{x_2}^{\eta_2}$ if $\eta_1 = e^{\pm\varepsilon}\eta_2$, $d(x_1,x_2) < (\eta_1\eta_2)^4$, $\|C_\chi(x_1) - C_\chi(x_2)\| < (\eta_1\eta_2)^4$

asymptotic data are close

- This implies $\Psi_{x_1}^{-1} \circ \Psi_{x_2} \approx Id$.
- If Ψ_{fx}^{η} overlaps $\Psi_y^{\eta'}$, then $f_{xy} = \Psi_y^{-1} \circ f \circ \Psi_x$ has same form as f_x.

$NUH^{\#}(f)$ = set of $x \in NUH(f)$ s.t.

① $\|C(f^k x)^{-1}\|$, $|s(f^k x)|$, $|u(f^k x)|$, $\det C_\chi(f^k x)$ all vary subexponentially

② $\exists\ q_\varepsilon : \{f^k x\} \to (0, \varepsilon]$ s.t. $q_\varepsilon(x) \leq Q(x)$, $q_\varepsilon(fx) = e^{\pm\varepsilon} q_\varepsilon(x)$, and $\overline{\lim_{n\to\pm\infty}}\, q_\varepsilon(f^n x) > 0$.

Then $NUH^{\#}(f)$ is χ-large, and $\exists$ a countable collection $\mathcal{A}$ of Pesin charts Ψ_x^η s.t.

① <u>Sufficiency</u>: $\forall\, x \in NUH^{\#}$, $\eta_n \in (0, Q_\varepsilon(f^n x)] \cap \{e^{-\ell\varepsilon}\}$ with $\eta_{n+1} = e^{\pm\varepsilon}\eta_n$, $\exists\ \{\Psi_{x_n}^{\eta_n}\}_{n\in\mathbb{Z}} \in \mathcal{A}$ s.t.

(a) $\Psi_{x_n}^{\eta_n}$ overlaps $\Psi_{f^n(x)}^{\eta_n}$, $Q_\varepsilon(f^n x) = e^{\pm\varepsilon} Q_\varepsilon(x_n)$

(b) $\Psi_{f^{\pm 1}x_n}^{\eta_{n\pm 1}}$ overlaps $\Psi_{x_{n\pm 1}}^{\eta_{n\pm 1}}$

② <u>Discreteness</u>: $\forall\, t > 0$, $|\{\Psi_x^\eta \in \mathcal{A} \mid \eta > t\}| < \infty$

$\boxed{\text{Rmk}}$ No clear connection between (a), (b)

Define Markov shift via graph:

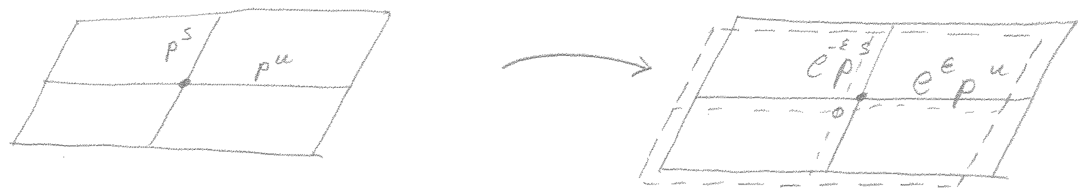
- Vertices are <u>double charts</u> $\Psi_x^{p^u, p^s} = (\Psi_x^{p^u}, \Psi_x^{p^s}) \in \mathcal{A}^2$
- Edges are $\Psi_x^{p^u, p^s} \longrightarrow \Psi_y^{q^u, q^s}$ where
  - $\Psi_y^{q^u \wedge q^s}$ overlaps $\Psi_{fx}^{q^u \wedge q^s}$
  - $\Psi_x^{p^u \wedge p^s}$ overlaps $\Psi_{f^{-1}y}^{p^u \wedge p^s}$
  - $q^u = e^\varepsilon p^u \wedge Q_\varepsilon(y)$, $p^s = e^\varepsilon q^s \wedge Q_\varepsilon(x)$

Paths in this graph are <u>ε-chains</u>. This graph is locally finite. Every $x \in NUH^{\#}(f)$ has an ε-chain s.t. $\Psi_{x_k}^{p_k^u \wedge p_k^s}$ overlaps $\Psi_{f^k x}^{p_k^u \wedge p_k^s}$

IDEA: p^u = size of unstable mfd

p^s = size of stable mfd

$p^u \wedge p^s$ = size of window



Structure of proof

- If $\underline{u}, \underline{v} \in \Sigma$ shadow same orbit (& are regular) then they are close (locations are obvious, asymptotic data requires pf)
- $\mathcal{Z} = \{\pi(\text{1-cylinders of } \Sigma)\}$ is a Markov cover: countable & locally finite. $Z \in \mathcal{Z}$ should be thought of as Cantor set.
- Sinai's refinement gives a ctbl Markov partition
- Use modification of Bowen's argument to show bdd-1 on sets with specified symbols i.o.
- Argue that ergodic measures lift